A QUANTUM ISOMORPHIC PAIR OF NON-ISOMORPHIC GRAPHS NEEDS AT LEAST TEN VERTICES

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Abstract. Two graphs are *quantum isomorphic* when the isomorphism game between them has a perfect quantum-commuting strategy. Such a pair need not be isomorphic: Atserias, Mančinska, Roberson, Šámal, Severini and Varvitsiotis turned the Mermin magic square into a quantum isomorphic, non-isomorphic pair of graphs on 24 vertices. How small can such a pair be? Writing N for the least order of one, the published bracket is $6 \leq N \leq 24$, and it is restated in that form by an expert paper of October 2025. We prove $N \geq 10$. The proof exhibits six connected planar graphs on at most six vertices — the bull, the path P_6, one further tree and three unicyclic graphs — whose homomorphism counts take pairwise distinct values on the 288 248 isomorphism classes of graphs on $5, 6, 7, 8$ and 9 vertices, and then applies the forward direction of Mančinska and Roberson's theorem that quantum isomorphism is exactly equality of homomorphism counts from planar graphs. The two halves are logically separate, and we keep them separate. The combinatorial half — that these six graphs separate every pair of non-isomorphic graphs on five to nine vertices — is proved outright here, modulo one imported fact, that `nauty`'s catalogues of small graphs are complete; the planarity of each of the six is certified by an explicit integer straight-line drawing rather than cited. The transfer to quantum isomorphism is the one ingredient we do not prove, and it appears as an explicit hypothesis of the theorem rather than in a remark. Everything below except a handful of elementary steps, each named where it occurs, is machine-checked in Lean 4, in a development that uses neither the axiom of choice nor `sorry`. The bracket becomes $10 \leq N \leq 24$; nothing here says anything about 10 itself.

1. Introduction

The question. Let G and H be finite graphs. The (G, H)-*isomorphism game* is the two-player non-local game in which a referee sends each player a vertex and each player answers with a vertex, the pair of answers being required to reproduce, between the two graphs, the relation (equal, adjacent, distinct and non-adjacent) that holds between the two questions. The game has a perfect classical strategy exactly when $G \cong H$. It may have a perfect *quantum* strategy when it has no perfect classical one, and the graphs are then called *quantum isomorphic*; we write $G \cong_{qc} H$ for the quantum-commuting version of this relation and $G \cong_q H$ for its finite-dimensional refinement, so that $\cong_q$ implies $\cong_{qc}$.

That the relation is strictly weaker than isomorphism was established by Atserias, Mančinska, Roberson, Šámal, Severini and Varvitsiotis [1], who convert a linear binary constraint system that is quantum satisfiable but not satisfiable into a quantum isomorphic, non-isomorphic pair of graphs. Applied to the Mermin magic square this gives a pair on 24 vertices, and that is the smallest such pair on record. In the other direction, Meunier [3] proves that the class $\mathcal{G}_5$ of graphs on at most five vertices is *tractable*, one of the three clauses of that definition being

"(QI) for all G and $H \in \mathcal{F}$, we have that $G \simeq_q H$ if and only if $G = H$",

(here $\mathcal{F}$ is the class in question and $\simeq_q$ is that paper's symbol for quantum isomorphism; its introduction states the same clause as "G is quantum isomorphic to H if and only if G is isomorphic to H"). So a quantum isomorphic, non-isomorphic pair needs at least six vertices. Let

$$N = \min\{\, n \,:\, \text{there are graphs } G \ncong H \text{ on } n \text{ vertices with } G \text{ quantum isomorphic to } H \,\}.$$

The two results above bracket N, and the bracket is where the subject stands. Freslon, Meunier and Pournajafi [4], in a paper last revised on 21 October 2025, put it in as many words:

> "For instance, we still do not know a smallest pair of quantum isomorphic graphs which are not isomorphic: the smallest examples known have 24 vertices [1, Theorem 6.5], and it follows from [3, Theorem B.2] that such graphs need at least 6 vertices."

(Only the two bibliography keys have been changed, to our own labels for the same two papers; the pointers into them are theirs, and the note on attribution closing this introduction says why the first of them deserves a comment.) The same paper narrows the *shape* of a minimal pair — its Corollary 5.7 shows that a non-isomorphic quantum isomorphic pair of minimum order is connected, and is either 2-connected or has every block of one graph carried to itself by the block-level isomorphisms, the same holding for the pair of complements — without moving either end of the bracket.

What is proved here. We move the lower end from 6 to 10.

Theorem 1.1. *Let G and H be quantum isomorphic graphs on n vertices with $5 \le n \le 9$. Then G and H are isomorphic. Consequently $N \ge 10$, and the bracket for the smallest non-isomorphic quantum isomorphic pair is*

$$10 \ \le\ N \ \le\ 24.$$

The instrument is Mančinska and Roberson's characterisation [2]: $G \cong_{qc} H$ if and only if $\hom(F, G) = \hom(F, H)$ for every planar graph F. We use only the forward implication, and only at six specific graphs F. In that direction each single planar F with $\hom(F, G) \neq \hom(F, H)$ is a complete certificate that G and H are *not* quantum isomorphic, so the lower half of the bracket becomes a finite computation: fingerprint every graph on n vertices by its homomorphism counts from a fixed finite list of planar graphs, and look for collisions.

The finite computation is the second half of the paper, and its content is that a very small list of patterns suffices.

Theorem 1.2. *There are six connected planar graphs on at most six vertices — two trees, namely the path P_6 and the spider with legs $2, 1, 1, 1$, and four unicyclic graphs, namely the bull and three graphs on six vertices — whose vectors of homomorphism counts take pairwise distinct values on the $288\,248$ isomorphism classes of graphs on $5, 6, 7, 8$ and 9 vertices.*

Lovász's classical theorem [5] says that the homomorphism counts from *all* graphs determine a graph up to isomorphism, and its usual finitary reading is that the patterns on at most n vertices already suffice for targets on n vertices. Theorem 1.2 restricts the pattern class on two axes at once: the patterns are *planar*, so that K_5 and $K_{3,3}$ and every graph containing them are unavailable, and they have at most six vertices against targets of nine. Six of them are enough. Section 5 shows that this is not slack dressed up as a theorem: the thirty connected planar graphs on at most five vertices — five times as many patterns — already fail one step lower, at $n = 8$.

What is not proved here. Three disclaimers, all of which we would rather state than have inferred.

First, Theorem 1.1 is a lower bound and says nothing about $n = 10$. We do not know whether a quantum isomorphic, non-isomorphic pair on ten vertices exists; Section 6 records what a decision at $n = 10$ would cost and why the pattern list of Theorem 1.2 is already known not to suffice there.

Second, the computation is elementary. With `nauty` and a C program, generating the five catalogues and fingerprinting every entry of them against the six patterns takes under a minute of CPU time; reproducing every number reported below, the 129-pattern control included, takes about seven. What makes it a result is not its difficulty but that the transfer legitimising it — [2], 2020 — appears not to have been run in this direction, and that the bracket it moves is restated as $[6, 24]$ by an expert paper of October 2025.

Third, the transfer itself is cited, not proved. We are explicit about where the boundary runs. Three layers, in increasing order of what they assume:

(a) For each $n \in \{5, \ldots, 9\}$, the entries of `nauty`'s catalogue of the graphs on n vertices carry pairwise distinct homomorphism-count vectors from the six patterns, and each of the six patterns admits a crossing-free straight-line drawing with integer coordinates, hence is planar. This layer is finite, self-contained, and verified by machine; it cites nothing.
(b) Adding that `nauty`'s catalogues are complete (every graph on n vertices is isomorphic to a listed one) and that homomorphism counts are isomorphism invariants, one gets Theorem 1.2 in the form: any two graphs on $n \leq 9$ (and ≥ 5) vertices with equal fingerprints are isomorphic.
(c) Adding the forward direction of [2] one gets Theorem 1.1 for $5 \leq n \leq 9$; adding [3] for $n \leq 5$ one gets $N \geq 10$.

In the formal development the ingredient of layer (c) is not asserted, not axiomatised and not hidden in a comment: it is a hypothesis in the type of the theorem, quantified over an arbitrary relation, so that a reader of the statement cannot fail to see it and so that the theorem applies verbatim to any relation contained in $\cong_{qc}$ — in particular to $\cong_q$. The same is done for the two ingredients of layer (b).

A note on attribution. In the sentence quoted above, [4] attributes the 24-vertex pair to "Theorem 6.5" of a work whose bibliography key is named after Mančinska and the year 2019. The attribution is right: the key resolves, in that paper's own bibliography, to Atserias, Mančinska, Roberson, Šámal, Severini and Varvitsiotis [1]. But the key's *name* invites the reading "Mančinska–Roberson 2019", and [2] has no Theorem 6.5 at all: in the arXiv v2 numbering used here, its 6.5 is a corollary on non-crossing partitions inside §6, "A Tale of Two Tensor Categories", and the planar characterisation is Theorem 7.16. The pointer "6.5" is into the published version of [1], which is behind a paywall and which we have not read; in the arXiv v3 that we did read, the separation theorem is Theorem 6.4 and the 24-vertex pair is named in the paragraph immediately after it. We therefore cite [1] for the 24-vertex pair and [2] for the planar characterisation, state in the bibliography which version each of our theorem numbers refers to, and say all this here because the misreading is easy and, once made, is invisible downstream.

Organisation. Section 2 fixes notation and states the two cited theorems. Section 3 gives the six patterns and the separation theorem. Section 4 deduces the bound. Section 5 collects the controls that fix the meaning of the computation and show the pattern list is not oversized. Section 6 records the price of $n = 10$. Section 7 describes the machine verification.

2. Preliminaries

Graphs and homomorphism counts. All graphs here are finite, simple and undirected. For graphs F and G, a *homomorphism* $\varphi\colon F \to G$ is a map $V(F) \to V(G)$ with $\varphi(u)\varphi(v) \in E(G)$ whenever $uv \in E(F)$; it is not required to be injective. We write $\hom(F, G)$ for the number of homomorphisms from F to G. Two standing conventions are worth naming because everything below depends on them and because they are the ones [2] uses: homomorphisms are counted as maps, not up to anything, and they are not required to be injective. Thus $\hom(K_1, G) = |V(G)|$, $\hom(K_2, G) = 2|E(G)|$, $\hom(P_3, G) = \sum_v \deg(v)^2$ and $\hom(K_3, G) = 6 \cdot \#\{\text{triangles of } G\}$, where P_3 is the path with three vertices.

The graphs of [2] may carry loops, at most one per vertex, and need not be connected. This costs us nothing: the patterns we use are loopless connected simple graphs, hence graphs in their sense, and we use only the implication that quantum isomorphism forces equality of homomorphism counts. (The reductions "connected patterns suffice" and "loopless patterns suffice", which [2] also proves, are needed for the converse implication, which we do not use.)

Plane drawings. A graph is *planar* when it can be drawn in the plane without crossings. We use a finitary certificate of planarity, and this is deliberate: no criterion — not Kuratowski's, not Wagner's, not Euler's formula — is invoked anywhere below.

Definition 2.1. A *certified plane drawing* of a graph F on the vertex set $\{0, \dots, k-1\}$ is a pair of integer vectors $(x_0, \dots, x_{k-1})$, $(y_0, \dots, y_{k-1})$ such that

(1) the points $p_i = (x_i, y_i)$ are pairwise distinct;
(2) no three of the p_i are collinear; and
(3) for any two edges $ab, cd \in E(F)$ with $\{a, b\} \cap \{c, d\} = \varnothing$, the segments $p_a p_b$ and $p_c p_d$ do not cross properly.

Conditions (1)–(3) together guarantee that drawing every edge as the straight segment between its endpoints produces a drawing of F in the plane in which no two edges meet except at a shared endpoint: (2) rules out a vertex lying in the interior of an edge, two edges with a common endpoint overlapping in a segment, and two disjoint edges meeting at a single point that is an endpoint of one of them; (3) rules out the remaining possibility, a proper crossing. So a graph with a certified plane drawing is planar, by the definition of planarity. We never need the converse — that every planar graph admits such a drawing — and do not claim it: the drawings used below were found, not postulated.

Each of the three conditions is a finite conjunction of inequalities between integers. Condition (3) is checked through the orientation predicate $\mathrm{cross}(a, b, c) = (x_b - x_a)(y_c - y_a) - (y_b - y_a)(x_c - x_a)$: the segments $p_a p_b$ and $p_c p_d$ cross properly exactly when c and d lie strictly on opposite sides of the line $p_a p_b$ and a and b lie strictly on opposite sides of the line $p_c p_d$.

Quantum isomorphism, and the two cited theorems. We do not reproduce the operator-algebraic definitions; the reader is referred to [1, 2, 3]. What we need from the literature is exactly two statements, and both are used as black boxes.

Theorem 2.2 (Mančinska–Roberson [2, Theorem 7.16])**.** *Let G and H be graphs. Then $G \cong_{qc} H$ — that is, there is a perfect quantum strategy for the (G, H)-isomorphism game — if and only if G and H "admit the same number of homomorphisms from any planar graph".*

Theorem 2.3 (Meunier [3, Theorem B.2], with the definition of *tractable*)**.** *Quantum isomorphic graphs on at most five vertices are isomorphic.*

The cited theorem reads, in full, "The class $\mathcal{G}_5$ is tractable", and the content we use is the clause (QI) of *tractable* quoted in Section 1; anyone checking the number 6 has to read both.

Only the "only if" half of Theorem 2.2 is used below, and it is used only at six particular planar graphs. We isolate that half as a property of a relation, because that is the form in which it enters our theorems.

Definition 2.4. A relation $\approx$ between graphs is *planar-transparent* if $G \approx H$ implies $\hom(F, G) = \hom(F, H)$ for every planar graph F.

Theorem 2.2 says that $\cong_{qc}$ is planar-transparent. Any relation contained in a planar-transparent one is planar-transparent, so $\cong_q$ is too. The relation of Theorem 2.3 needs a separate word, since the two cited theorems have to meet somewhere. In [3] a *quantum isomorphism* from G to H is a magic unitary P over a unital C^*-algebra — a matrix of self-adjoint projections summing to 1 along every row and along every column — with $PA_G = A_H P$ for the two adjacency matrices. Nothing is asked of the algebra beyond unitality: no state, no trace, no finite dimension, no ambient Hilbert space. That is the relation to which [3] applies Theorem 2.2 directly, in its corollary "Two quantum isomorphic planar graphs are isomorphic", so on its author's reading it *is* $\cong_{qc}$; and the characterisation of $\cong_{qc}$ by C^*-algebras in [1, Theorem 5.13] is an array of projections of the same shape, over an algebra asked in addition to carry a faithful tracial state. We take this identification from the two sources and do not re-prove it. It is used once, in Corollary 4.3, and only in the direction that a pair on at most five vertices related by $\cong_{qc}$ is one of Meunier's pairs.

Remark 2.5. A planar-transparent relation preserves the number of vertices, since K_1 is planar and $\hom(K_1, G) = |V(G)|$; likewise the number of edges, since K_2 is planar. Meunier [3] makes exactly this observation, deriving both preservation statements from [2]; in the arXiv v2 the observation carries no number of its own, being the paragraph that follows the proof of Lemma 2.22, "Two quantum isomorphic graphs have the same number of edges". It is why one may speak of "the number of vertices of a quantum isomorphic pair" at all.

Catalogues. For $n \geq 1$ let $\mathcal{C}_n$ be the list of graphs on n vertices produced by `nauty-geng` [6]: one graph from each isomorphism class. The lengths are

$$|\mathcal{C}_5| = 34, \quad |\mathcal{C}_6| = 156, \quad |\mathcal{C}_7| = 1044, \quad |\mathcal{C}_8| = 12\,346, \quad |\mathcal{C}_9| = 274\,668,$$

which is OEIS A000088 [7] at $n = 5, \ldots, 9$, and $\sum_{n=5}^{9} |\mathcal{C}_n| = 288\,248$. The sum runs from 5 only: the eighteen graphs on at most four vertices lie outside the range swept here and fall under Theorem 2.3 instead.

That these lists are *complete* — that every graph on n vertices is isomorphic to a listed one — is `nauty`'s claim and the single mathematical fact about data that we import. That the listed graphs are pairwise *non*-isomorphic is not imported: it is a consequence of Theorem 3.2 below, since homomorphism counts are isomorphism invariants.

3. Six planar graphs that separate the small catalogues

The patterns. Write $\mathcal{Q} = \{F_1, \ldots, F_6\}$ for the six graphs of Table 1, on vertex sets $\{0, \ldots, k-1\}$. Two are trees and four are unicyclic; no member of $\mathcal{Q}$ has two independent cycles.

	k	edges	description
F_1	5	$01, 02, 12, 13, 24$	the bull: triangle 012 with a pendant at 1 and at 2
F_2	6	$01, 02, 23, 24, 25$	the spider with legs $2, 1, 1, 1$ (a subdivided $K_{1,4}$)
F_3	6	$01, 02, 13, 23, 14, 25$	the 4-cycle 0132 with a pendant at each of $1, 2$
F_4	6	$01, 02, 23, 34, 45$	the path P_6, namely 1 0 2 3 4 5
F_5	6	$01, 02, 23, 24, 35, 45$	the 4-cycle 2354 with the path 2 0 1 attached
F_6	6	$01, 02, 23, 24, 34, 45$	the triangle 234 with the path 2 0 1 and a pendant at 4

TABLE 1. The six patterns. In F_3 the two pendant vertices hang from opposite vertices of the 4-cycle. F_1, F_2, F_4 have five edges and F_3, F_5, F_6 have six.

For a graph G on n vertices put

$$\Phi(G) \;=\; \big(\mathrm{hom}(F_1, G),\ \mathrm{hom}(F_2, G),\ \mathrm{hom}(F_3, G),\ \mathrm{hom}(F_4, G),\ \mathrm{hom}(F_5, G),\ \mathrm{hom}(F_6, G)\big) \in \mathbb{N}^6,$$

the *fingerprint* of G. Since a homomorphism composed with an isomorphism is again a homomorphism, Φ is an isomorphism invariant.

Lemma 3.1. *Each of $F_1, \ldots, F_6$ carries a certified plane drawing in the sense of Definition 2.1, and is therefore planar.*

Proof. Six explicit pairs of integer vectors, each on a 4×4 grid; conditions (1)–(3) of Definition 2.1 are finite integer computations and were checked for all six. For instance F_1 is drawn at $(1,0), (3,3), (1,1), (0,3), (0,1)$ and F_4 at $(3,1), (2,2), (0,3), (3,0), (1,0), (0,1)$. Lemma 3.4 certifies a larger list containing these. □

The separation.

Theorem 3.2. *For each $n \in \{5, 6, 7, 8, 9\}$ the fingerprint map Φ is injective on $\mathcal{C}_n$: distinct entries of $\mathcal{C}_n$ have distinct fingerprints.*

Proof, and what was computed. Exhaustive computation, one n at a time, over the whole of $\mathcal{C}_n$: the fingerprint of every one of the 34, 156, 1044, 12 346 and 274 668 entries is computed and the $|\mathcal{C}_n|$

fingerprints are checked to be pairwise distinct. Nothing is sampled and nothing is pruned by an invariant; the search is the whole catalogue.

Two points about how the check is organised, since "pairwise distinct" over a quarter of a million vectors is not done by $\binom{274668}{2}$ comparisons. First, homomorphisms are counted by the obvious backtracking search: the images of the pattern vertices $0, 1, \ldots, k-1$ are assigned in order, and at each level the admissible images are those adjacent to the images of all already-assigned neighbours. Second, distinctness is decided through a hash table that is treated as an untrusted *hint*. The table is keyed by a 64-bit hash of the fingerprint vector, and the check performed is that every entry of $\mathcal{C}_n$ recovers its own index when its own hash is looked up. A corrupt table can only make this check fail: if every index is recovered then the lookup map is injective on the hashes occurring, so distinct entries have distinct hashes and hence distinct fingerprint vectors. (Only that direction is used; a hash collision could cost a false negative, never a false positive.) So no property of the table construction has to be proved, and none is.

The counts themselves are small — a pattern on six vertices has at most $9^6 = 531\,441$ maps into a nine-vertex target — and the arithmetic is exact throughout. The heaviest case, $n = 9$, took 1 313 seconds of CPU time inside the proof assistant, and checking the single module that holds all the sweeps peaks at about 1.6 GB of resident memory; an independent implementation in C, sharing no code and no data path with it, computes the same fingerprints over the same catalogue in 37 seconds per pass and agrees on every verdict. □

Remark 3.3. How $\mathcal{Q}$ was found plays no role in the proof. For the record: the full matrix of homomorphism counts from all 129 connected planar graphs on at most six vertices into all of $\mathcal{C}_9$ was computed, and a greedy search repeatedly added the pattern maximising the number of refined classes; it stopped at six. The result is then verified directly, so neither the greedy nor the completeness of the 129-element list is trusted for Theorem 3.2. Optimality at one n does not imply optimality at the next: a greedy run on $\mathcal{C}_8$ alone stops at *three* patterns, and those three leave 90 collisions on $\mathcal{C}_9$, in the sense of Remark 5.3.

Lemma 3.4. *There are* 129 *connected planar graphs on at most six vertices up to isomorphism, namely* $1+1+2+6+20+99$ *on* $1, \ldots, 6$ *vertices, and each of them carries a certified plane drawing in the sense of Definition 2.1.*

Proof. The list is `nauty-geng -c` filtered by `nauty-planarg` [6]; its length agrees with OEIS A003094 [8] term by term over $n = 1, \ldots, 6$, including the drop from 21 connected graphs on five vertices to 20 planar ones, which is K_5. The drawings were found by random search over integer grids from 4×4 to 40×40 and are then certified, one by one, by the finite check of Definition 2.1. That every one of the 129 admits such a drawing is an independent confirmation of the planarity classification, since a non-planar graph has no crossing-free drawing at all; the mathematics — that these particular graphs are planar — is classical, and the contribution here is only the certificate. □

Corollary 3.5. *Let* $5 \le n \le 9$ *and let* G, H *be graphs on* n *vertices with* $\Phi(G) = \Phi(H)$*. Then* $G \cong H$*.*

Proof. By completeness of $\mathcal{C}_n$ choose i, j with $G \cong \mathcal{C}_n[i]$ and $H \cong \mathcal{C}_n[j]$. Fingerprints are isomorphism invariants, so $\Phi(\mathcal{C}_n[i]) = \Phi(G) = \Phi(H) = \Phi(\mathcal{C}_n[j])$, whence $i = j$ by Theorem 3.2, and $G \cong \mathcal{C}_n[i] = \mathcal{C}_n[j] \cong H$. □

Corollary 3.5 is Theorem 1.2 of the introduction. It uses two facts beyond Theorem 3.2, and both are named in the formal development as hypotheses rather than assumed silently: completeness of the catalogue, and invariance of homomorphism counts under isomorphism. The first is `nauty`'s claim; the second is elementary (composing a homomorphism with an isomorphism is a bijection between the two homomorphism sets) and is one of the few steps of this paper written out as ordinary mathematics rather than checked by machine; Section 7 lists them all.

4. The lower bound

Theorem 4.1. *Let $\approx$ be a planar-transparent relation between graphs, in the sense of Definition 2.4. Let $5 \leq n \leq 9$ and let G, H be graphs on n vertices with $G \approx H$. Then $G \cong H$.*

Proof. Each $F_i \in \mathcal{Q}$ is planar by Lemma 3.1, so $G \approx H$ gives $\hom(F_i, G) = \hom(F_i, H)$ for $i = 1, \ldots, 6$, that is $\Phi(G) = \Phi(H)$. Apply Corollary 3.5. □

Corollary 4.2. *No two non-isomorphic graphs on n vertices with $5 \leq n \leq 9$ are quantum isomorphic — in the quantum-commuting sense $\cong_{qc}$, hence also in the finite-dimensional sense $\cong_q$ and in any other sense whose relation is contained in $\cong_{qc}$.*

Proof. $\cong_{qc}$ is planar-transparent by Theorem 2.2; apply Theorem 4.1. □

Corollary 4.3. *$N \geq 10$, and therefore $10 \leq N \leq 24$.*

Proof. A non-isomorphic quantum isomorphic pair has both graphs on the same number n of vertices by Remark 2.5. The range $5 \leq n \leq 9$ is excluded by Corollary 4.2 and the range $n \leq 5$ by Theorem 2.3, so $n \geq 10$. That the two exclusions concern the same relation is the identification recorded after Definition 2.4, taken from the sources; both then hold a fortiori for every smaller relation. The upper bound is the 24-vertex pair of [1], which is a pair for the finite-dimensional relation and hence for $\cong_{qc}$ as well. □

Remark 4.4 (what rests on what). Corollary 4.3 is the composite of four inputs, and it is worth separating them once more. Theorem 3.2 and Lemmas 3.1, 3.4 are proved here outright and depend on no citation. Corollary 3.5 adds the completeness of `nauty`'s catalogues and the invariance of homomorphism counts. Theorem 4.1 is stated *conditionally*, for an arbitrary planar-transparent relation, so it too depends on no citation; the citation [2] enters as an ingredient exactly once, at Corollary 4.2, to say that quantum isomorphism is such a relation. The range $n \leq 4$, which our catalogues do not cover, is [3]'s and is used only in Corollary 4.3. In particular the combinatorial statement — six planar graphs on at most six vertices separate every pair of non-isomorphic graphs on 5 to 9 vertices — stands whatever happens to the operator-algebraic literature, and the quantum reading of it is exactly as strong as [2, Theorem 7.16].

Remark 4.5 (what the bound does not say). Nothing above concerns $n = 10$. A quantum isomorphic, non-isomorphic pair on ten vertices is not excluded, and the six patterns of $\mathcal{Q}$ would not exclude it even if the sweep were affordable: see Section 6. Nor does the argument produce any structural information about a minimal pair; for that see [4, Corollary 5.7], which is independent of the bound and complementary to it.

5. Controls

A computation that certifies a negative — "no collision occurs" — fails safe in the wrong direction: a homomorphism counter that systematically over-counts, or a separation predicate that is vacuously true, would produce exactly the same verdict. The following two propositions are the controls that close that gap. They are stated as results because they were run as results, not because either is new; both are routine.

Proposition 5.1 (the engine counts homomorphisms)**.** *Over all* $1\,044$ *graphs G on seven vertices:* $\hom(K_1, G) = 7$*;* $\hom(K_2, G) = 2|E(G)|$*;* $\hom(P_3, G) = \sum_v \deg(v)^2$*; and* $\hom(K_3, G) = 6 \cdot \#\{\text{triangles of } G\}$*. Moreover the search used in Theorem 3.2 and a transparent search that tests each candidate image against each already-assigned neighbour separately agree on all* $129 \times 1044 = 134\,676$ *pairs consisting of a connected planar pattern on at most six vertices and a graph on seven vertices. Finally the catalogue lengths are* $34, 156, 1044, 12\,346, 274\,668$*.*

Proof, and what was computed. Exhaustive over $\mathcal{C}_7$ in each case. The edge, degree and triangle counts on the right-hand sides are recomputed from the adjacency data by a route sharing no code with the homomorphism search, which is the point: a control one has not computed independently is not a control. Two of the four identities, those for K_1 and K_2, occur in the literature in exactly this form — in [3], in the observation cited in Remark 2.5 — which makes them a check on the conventions, loopless graphs and homomorphisms not required injective, and not only on the arithmetic. □

Proposition 5.2 (the pattern list is not oversized, and the drawing check is not vacuous)**.** *Each of the following holds.*

(1) *The four connected graphs on at most three vertices — the single vertex, the edge, the path on three vertices and the triangle — fail to separate $\mathcal{C}_5$, and fail on $\mathcal{C}_6$ and on $\mathcal{C}_7$ as well.*
(2) *The thirty connected planar graphs on at most five vertices separate $\mathcal{C}_7$, and do* not *separate $\mathcal{C}_8$.*
(3) *K_4 drawn as a square with both diagonals fails the check of Definition 2.1; the same K_4 with one vertex placed inside the triangle of the other three passes it.*

Item (2) is the sharpest of these. Patterns on at most five vertices are genuinely insufficient, and they fail exactly one step below where Theorem 3.2 needs them, so the six-element list of Table 1 — which does contain patterns on six vertices — is doing work that no smaller standard set does. Item (3) is the control on Definition 2.1: the check is not true of arbitrary coordinates, so Lemma 3.4 is not vacuous. Item (1) is the control on the separation predicate at the opposite end.

One asymmetry should be stated, since it is invisible in the statements. The hash-table argument in the proof of Theorem 3.2 is sound in one direction only: a positive verdict proves separation outright, whereas the negative verdicts behind items (1) and (2) say that two catalogue entries share a 64-bit fingerprint hash, which is failure to separate unless that hash itself collides. Nothing in this paper rests on that direction. What the two items are for is to show that the separation predicate is not identically true, and that they do establish; item (3), by contrast, is a direct decision and carries no such caveat.

Remark 5.3 (collision counts). The following table refines Proposition 5.2 by measuring *how far* each pattern set is from separating. The entry for a set and an n is its number of *collisions*, by which we mean throughout $|\mathcal{C}_n|$ minus the number of distinct count vectors occurring: a value c means the $|\mathcal{C}_n|$ entries fall into $|\mathcal{C}_n| - c$ classes, and $c = 0$ means separation. These *counts* were produced by an auxiliary program in C and lie outside the formal development, which records only whether the count is zero; they are reported here because they show how the method degrades.

pattern set	#	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$
connected, ≤ 3 vertices	4	2	44	654	11 072	270 922
connected planar, ≤ 4 vertices	10	0	0	10	1 049	70 089
connected planar, ≤ 5 vertices	30	0	0	0	4	252
connected planar, ≤ 6 vertices	129	0	0	0	0	0
$\mathcal{Q}$ of Table 1	6	0	0	0	0	0

Pattern *count* is a poor proxy for pattern *cost*: the thirty patterns on at most five vertices take 23.8 seconds over $\mathcal{C}_9$ against 36.9 seconds for the six patterns of $\mathcal{Q}$ — five times as many patterns in two thirds of the time — because a pattern on five vertices has far fewer homomorphisms into a nine-vertex target than one on six.

6. The frontier at ten vertices

This section records measurements rather than theorems; none of it is part of the formal development, and nothing above depends on it.

There are 12 005 168 graphs on ten vertices [7]. Generating one sixty-fourth of that catalogue with `nauty-geng` yields 252 694 graphs, and one fingerprinting pass over that slice with the six patterns of $\mathcal{Q}$ costs 51.73 seconds in C, so a full pass costs about 41 CPU-minutes. Inside the proof assistant the same computation was measured at 16.6 times the C time at $n = 8$ and 17.8 times at $n = 9$, and the check makes two passes, so a machine-checked sweep at $n = 10$ would cost on the order of 24 CPU-hours together with a census string of upwards of a hundred megabytes.

That price would not, however, buy the next value of the bound, for a reason that is not about cost: *the six patterns of $\mathcal{Q}$ do not separate at $n = 10$.* The single 1/64 slice above already carries 72 collisions, in the sense of Remark 5.3 and counted by the same program. A larger, hence more expensive, pattern set would be needed first, and since collisions can straddle slice boundaries a chunked run must emit a full fingerprint per graph and sort globally rather than checking each chunk in isolation. One direction is half-measured and looks favourable — weighting the greedy selection by the cost of each pattern rather than by pattern count, which reached 274 628 of the 274 668 classes at $n = 9$ using 21 cheap patterns — but whether a cost-aware set that actually separates stays cheap is not known. At $n = 11$ there are 1 018 997 864 graphs [7], 85 times as many as at $n = 10$, and the approach of this paper does not reach that far.

7. Verification

Formalised and machine-checked in Lean 4 [9] are: Theorem 3.2; Lemmas 3.1 and 3.4; Theorem 4.1 and Corollary 4.2, in the conditional form of Definition 2.4; and Propositions 5.1 and 5.2, as stated. Theorems 1.1 and 1.2 of the introduction are restatements, of Corollaries 4.2 and 4.3 and of Corollary 3.5 respectively. Not formalised, and written out above as ordinary mathematics, are: Corollary 3.5, whose formal counterpart carries the invariance of homomorphism counts under isomorphism as a named hypothesis and concludes that the two graphs have the same catalogue representative, the elementary step from there to $G \cong H$ being left to the reader; Corollary 4.3, which combines our range with [3]'s; Remark 2.5; and the measurements of Remarks 3.3 and 5.3 and of Section 6, which are labelled as measurements where they occur and on which nothing else depends. Remarks 4.4 and 4.5 are commentary, and Theorems 2.2 and 2.3 are the two citations, which are not reproved.

The conditional form deserves one sentence, since it is the whole of our bookkeeping about what is cited. In the formal statement of Theorem 4.1 the relation $\approx$ is universally quantified and the hypothesis is literally "$G \approx H$ implies $\hom(F, G) = \hom(F, H)$ for every F carrying a certified plane drawing", supplied as an argument of the theorem, so that Theorem 2.2 is neither axiomatised nor asserted anywhere in the development; it is discharged, on paper, only at Corollary 4.2. A companion lemma exhibits a reflexive relation satisfying that hypothesis, so the theorem is not vacuous through an unsatisfiable hypothesis.

The modules import no mathematical library at all: a graph is a natural number read as a bitmask over unordered pairs in the `graph6` bit order, so the catalogue files are a mechanical transliteration of `nauty`'s output and every object in sight is a natural number or an array of them. The finite evaluations are carried out by Lean's compiled evaluator (`native_decide`), so each depends on `propext`, `Quot.sound` and one evaluation axiom of its own, and that evaluator is the only trusted step. The audit that matters is which evaluations each result rests on: the $n = 9$ case of Theorem 4.1 rests on exactly two, the planarity of the six patterns and the separation of $\mathcal{C}_9$, and each smaller case on the corresponding pair; everything else in its proof, the whole hint-table argument included, is kernel reasoning. There is no `sorry` anywhere in the development, it declares no axiom of its own, and it uses no form of the axiom of choice. An independently written C program reproduces every collision count and every separation verdict reported above. The repository is private; repository reference to be added at submission.

References

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arXiv:1611.09837. The theorem numbers cited above are those of the arXiv v3, the version we read: the 24-vertex pair is Theorem 6.4 together with the paragraph following it, in which the Mermin magic square is named, and the C^*-algebraic characterisation of $\cong_{qc}$ is Theorem 5.13. [4] points instead at "Theorem 6.5"; we have not seen the published version and make no claim about its numbering.

[2] L. Mančinska and D. E. Roberson, *Quantum isomorphism is equivalent to equality of homomorphism counts from planar graphs*, in: 2020 IEEE 61st Annual Symposium on Foundations of Computer Science (FOCS), IEEE, 2020, pp. 661–672; `doi:10.1109/FOCS46700.2020.00067`; arXiv:1910.06958. Theorem numbers above are those of the arXiv v2, where the main theorem is Theorem 7.16 and where 6.5 is a corollary on non-crossing partitions.

[3] P. Meunier, *Quantum properties of $\mathcal{F}$-cographs*, arXiv:2312.01516v2 (math.OA), revised 8 May 2024; no journal version is recorded for it in Crossref or in the arXiv metadata. Numbering is that of the typeset v2: Theorem B.2 is "The class $\mathcal{G}_5$ is tractable", in Appendix B, "Graphs on at most 5 vertices"; the clause of *tractable* that we use is (QI); the corollary quoted in Section 2 is Corollary A.15.

[4] A. Freslon, P. Meunier and P. Pournajafi, *Block structures of graphs and quantum isomorphism*, arXiv:2502.19343v2 (math.CO), revised 21 October 2025; no journal version is recorded for it in Crossref or in the arXiv metadata. The bracket sentence quoted in Section 1 closes the opening paragraph of the introduction of v2; the corollary on minimal pairs is Corollary 5.7 of the typeset v2.

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[6] B. D. McKay and A. Piperno, *Practical graph isomorphism, II*, J. Symbolic Comput. **60** (2014), 94–112; `doi:10.1016/j.jsc.2013.09.003`. The programs used here are `nauty-geng` and `nauty-planarg`.

[7] OEIS Foundation Inc., *Sequence A000088: number of simple graphs on n unlabeled nodes*, The On-Line Encyclopedia of Integer Sequences, `https://oeis.org/A000088`.

[8] OEIS Foundation Inc., *Sequence A003094: number of unlabeled connected planar simple graphs with n nodes*, The On-Line Encyclopedia of Integer Sequences, `https://oeis.org/A003094`.

[9] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, in: A. Platzer and G. Sutcliffe (eds.), Automated Deduction — CADE 28, Lecture Notes in Computer Science **12699**, Springer, Cham, 2021, pp. 625–635; `doi:10.1007/978-3-030-79876-5_37`.