BOTH HALVES OF GUO–ZUDILIN'S PROBLEM 1: THE q-CONGRUENCE FOR $n \equiv 7 \pmod 8$ AND THE CLOSED FORM FOR $n \equiv 3 \pmod 8$

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ABSTRACT. For odd n put $S_n(q) = \sum_{k=0}^{(n-1)/2} \frac{(q;q^2)_k^2 (q^2;q^4)_k}{(q^2;q^2)_k^2 (q^4;q^4)_k}(-q)^k$. Guo and Zudilin evaluated S_n modulo the cyclotomic polynomial $\Phi_n(q)$ for $n \equiv 1 \pmod 4$ and left the two remaining residue classes as the only open problem of their paper: to show that $S_n \equiv 0 \pmod{\Phi_n(q)}$ for $n \equiv 7 \pmod 8$, and to find a related q-congruence for $n \equiv 3 \pmod 8$. We settle both. The vanishing holds for every positive $n \equiv 7 \pmod 8$, and for every positive $n \equiv 3 \pmod 8$

$$S_n \;\equiv\; \frac{(q, q^2; q^4)_{(3n-1)/8}^2 \, (-q;q)_{(n-1)/2}}{(q^4;q^4)_{(3n-1)/4}^2} \, q^{-(1-n)^2/4} \pmod{\Phi_n(q)},$$

which is the $n \equiv 1 \pmod 8$ formula of Guo and Zudilin with the two indices $(n-1)/8$ and $(n-1)/4$ replaced by $(3n-1)/8$ and $(3n-1)/4$ — exactly the substitution their own auxiliary theorems make when they pass from $n \equiv 1$ to $n \equiv 3 \pmod 8$, and, in an equivalent normalisation, the substitution $n \mapsto 3n$ applied to every index at once. Together with their result this evaluates S_n modulo $\Phi_n(q)$ in all four residue classes of odd n modulo 8. The proof specialises Jackson's q-analogue of Clausen's identity at $a = q^{-3n}$ rather than at $a = q^{-n}$: this is the specialisation that terminates precisely when $n \equiv 3 \pmod 4$, and the extra length of the resulting sum is absorbed by a tail identity that returns a second copy of S_n. We also show, by exact computation, that neither congruence survives a squaring of the modulus, in contrast with the q^{2k}-weighted companion sum, which Guo and Zudilin evaluate modulo $\Phi_n(q)^2$. Twenty-five instances of the two congruences, at n up to 111, together with the published cases and thirteen refuted variants, have been verified by the Lean 4 kernel in exact integer arithmetic.

1. INTRODUCTION

q-congruences. Write $(a;q)_k = \prod_{j=0}^{k-1}(1-aq^j)$ for the q-shifted factorial, with the abbreviation $(a_1, \dots, a_m; q)_k = (a_1;q)_k \cdots (a_m;q)_k$, and let

$$\Phi_n(q) = \prod_{\substack{1 \le j \le n \\ \gcd(j,n)=1}} \left(q - e^{2\pi i j/n}\right)$$

be the n-th cyclotomic polynomial. A *q-congruence* is a congruence modulo $\Phi_n(q)$ (or a power of it) between rational functions of q whose denominators are coprime to $\Phi_n(q)$. Such congruences for truncated basic hypergeometric sums are q-deformations of supercongruences for truncated ordinary hypergeometric sums: letting $q \to 1$ through a primitive n-th root of unity with $n = p$ prime recovers a congruence modulo p, and the parametric refinements of the q-statement often recover the modulus p^2 or p^3 that the p-adic statement carries. This is the mechanism of the "q-microscope" of Guo and Zudilin [5], and the supercongruences it is aimed at are those of Van Hamme's list [7] and their relatives.

The source. In [1], Guo and Zudilin construct q-deformations of two CM modular forms of weight 3 and study the q-congruences attached to them. The second of those forms is

$$f_2(\tau) = \sum_{m \ge 1} a_2(m)\, x^m = x \prod_{r \ge 1} (1-x^r)^2 (1-x^{2r})(1-x^{4r})(1-x^{8r})^2, \qquad x = e^{2\pi i \tau},$$

whose Fourier coefficients at primes are given by $a_2(p) = 2(a^2 - 2b^2)$ if $p = a^2 + 2b^2$, and $a_2(p) = 0$ if $p \equiv 5, 7 \pmod 8$ [8]; the critical L-value of f_2 evaluates a ${}_3F_2$ at -1 [9], and the coefficients satisfy, for

$p > 2$,

$$a_2(p) \equiv \frac{(\frac{1}{4})_m^2}{m!^2} \times \begin{cases} 1 & \text{if } m = (p-1)/8 \in \mathbb{Z}, \\ -\frac{1}{2} & \text{if } m = (3p-1)/8 \in \mathbb{Z}, \end{cases} \pmod{p^2} \tag{1}$$

as quoted in [1, §2]. The q-deformation attached to f_2 in [1] is the truncated sum

$$S_n = S_n(q) = \sum_{k=0}^{(n-1)/2} \frac{(q;q^2)_k^2\,(q^2;q^4)_k}{(q^2;q^2)_k^2\,(q^4;q^4)_k}\,(-q)^k, \qquad n \text{ odd}, \tag{2}$$

and [1, Theorem 5] evaluates it modulo $\Phi_n(q)$ for $n \equiv 1 \pmod 4$:

$$S_n \equiv \begin{cases} \dfrac{(q,q^2;q^4)_{(n-1)/8}^2\,(-q;q)_{(n-1)/2}}{(q^4;q^4)_{(n-1)/4}^2}\,q^{-(1-n)^2/4} & \text{if } n \equiv 1 \pmod 8, \\ 0 & \text{if } n \equiv 5 \pmod 8, \end{cases} \pmod{\Phi_n(q)}. \tag{3}$$

The shape of (3) mirrors that of a_2: the classes in which $a_2(p)$ vanishes are $p \equiv 5, 7 \pmod 8$, and 5 is the class in which (3) makes S_n vanish; that the class 7 should behave in the same way is exactly what [1] asks to be shown. So the four residue classes of odd n modulo 8 split two and two, and (3) settles one class of each kind. The other two are the only problem [1] leaves open. After the sentence "*We leave the related cases when* $n \equiv 3 \pmod 4$ *of Theorem* 5 *as an open problem to the reader*", it poses, verbatim:

> ***Problem 1.*** *For any positive integer* $n \equiv 7 \pmod 8$*, show that*
>
> $$\sum_{k=0}^{(n-1)/2} \frac{(q;q^2)_k^2 (q^2;q^4)_k}{(q^2;q^2)_k^2 (q^4;q^4)_k}(-q)^k \equiv 0 \pmod{\Phi_n(q)}.$$
>
> *Give a related* q*-congruence for* $n \equiv 3 \pmod 8$*.*

Theorem, problem and equation numbers of [1] are cited throughout from its arXiv v2 e-print, in which the five theorems are numbered consecutively 1–5, the equations are numbered by section, and the Problem displayed above is the only one stated.

Two features of the Problem are worth recording. It is posed for all positive $n \equiv 7 \pmod 8$, not only for primes, so a p-adic argument does not reach it. And its second sentence is a discovery task rather than a range extension: what a "related q-congruence" should look like is not specified, and the formula (3) cannot simply be reused, since neither $(n-1)/8$ nor $(n-1)/4$ is an integer when $n \equiv 3 \pmod 8$.

Results. We settle both sentences.

Theorem 1.1 (Problem 1, first sentence)**.** *For every positive integer* $n \equiv 7 \pmod 8$*,*

$$S_n \equiv 0 \pmod{\Phi_n(q)}.$$

Theorem 1.2 (Problem 1, second sentence)**.** *For every positive integer* $n \equiv 3 \pmod 8$*,*

$$S_n \;\equiv\; \frac{(q,q^2;q^4)_{(3n-1)/8}^2\,(-q;q)_{(n-1)/2}}{(q^4;q^4)_{(3n-1)/4}^2}\,q^{-(1-n)^2/4} \pmod{\Phi_n(q)}.$$

Together with (3), Theorems 1.1 and 1.2 evaluate S_n modulo $\Phi_n(q)$ for every odd n.

The right-hand side of Theorem 1.2 is the right-hand side of (3) with $(n-1)/8$ replaced by $(3n-1)/8$ and $(n-1)/4$ by $(3n-1)/4$, and nothing else changed. Both new indices are integers exactly in the new class. That is not an accident of bookkeeping: it is precisely the substitution that [1, Theorems 3 and 4] themselves perform in passing from $n \equiv 1$ to $n \equiv 3 \pmod 8$ (see (9) and (10) below), and it is the same substitution that turns the index $(p-1)/8$ of (1) into $(3p-1)/8$. Section 3 records an equivalent normalisation, (11), in which *all three* indices and the exponent of q are attached to $3n$; in that form Theorem 1.2 is literally the $n \equiv 1 \pmod 8$ case of (3) with n replaced by $3n$ throughout, times a factor $\frac{1}{2}$. That factor has a counterpart in (1): there too the case indexed by $(3p-1)/8$ carries a factor of size $\frac{1}{2}$ where the case indexed by $(p-1)/8$ carries 1 — with the opposite sign, $-\frac{1}{2}$. We record the coincidence and draw nothing from it.

Method. The proof of (3) in [1] specialises Jackson's q-analogue (8) of Clausen's identity at $a = q^{-n}$, $z = -q$, which factors the sum S_n into a product of two ${}_2\phi_1$ series that [1, Theorems 3 and 4] evaluate. That specialisation is unavailable when $n \equiv 3 \pmod 4$: the two ${}_2\phi_1$ series do not terminate, and they are not Φ_n-integral. Our single change is to take

$$a = q^{-3n}, \qquad z = -q.$$

The numerator factor $(q^{1-3n}; q^4)_k$ vanishes for $k \geq (3n+3)/4$ exactly when $4 \mid 3n-1$, that is exactly when $n \equiv 3 \pmod 4$; so this is the specialisation that terminates in the classes the Problem asks about, and it is the specialisation that fails in the classes [1] already settled. The price is that the left-hand side becomes a sum of length $(3n+1)/2$ rather than $(n+1)/2$. Modulo $\Phi_n(q)$ it collapses again: its first $(n+1)/2$ terms reduce to S_n, its middle block has Φ_n-valuation 3 and dies, and its tail is a second copy of S_n, because the term at $k = n$ has valuation 0 with regularised value 1 and the step ratios after it repeat those of S_n. So the identity reads $2S_n \equiv A'B'$, and A', B' are congruent to the left-hand sides of [1, Theorems 4 and 3]. Theorem 1.1 is then the vanishing of [1, Theorem 3] in the class $n \equiv 7 \pmod 8$, and Theorem 1.2 is the product of the two closed forms of [1, Theorems 3 and 4] in the class $n \equiv 3 \pmod 8$, simplified by two elementary reflections of q-shifted factorials. Section 4 carries this out.

Sharpness. The companion sum obtained from (2) by replacing the weight $(-q)^k$ with q^{2k} and the truncation $(n-1)/2$ with $n-1$ vanishes modulo $\Phi_n(q)^2$ for every $n \equiv 3 \pmod 4$ [1, Theorem 2], and [1, Conjecture 1] proposes the same vanishing when the truncation $n-1$ is replaced by $rn-1$ or by $rn + (n-1)/2$, $r \geq 1$. Nothing of the kind happens on the $(-q)^k$ side. Section 5 records that the Φ_n-valuation of the relevant numerator is exactly 1 — not 2 — in all four residue classes modulo 8, and for [1, Theorem 3] as well, at every value tested. Problem 1 as posed is therefore already best possible in the modulus, and Theorem 1.2 has no $\Phi_n(q)^2$ refinement to look for; every congruence that [1] states modulo $\Phi_n(q)^2$ carries a weight other than $(-q)^k$.

What is proved, and what is computed. Sections 2–4 are ordinary mathematics: Theorems 1.1 and 1.2 are proved for all n from Jackson's identity and two theorems of [1]. That argument is *not* machine-checked, and Section 7 says why: no library of formalised mathematics we are aware of carries q-shifted factorials or basic hypergeometric series at all, so Jackson's q-Clausen identity, the q-Kummer (Bailey–Daum) summation and Jackson's q-analogue of Dixon's sum — the three inputs — would all have to be formalised first. Every individual step of the argument was instead confirmed numerically, in exact arithmetic, over the ranges recorded in Section 4.

Independently of the proof, Section 6 reports what has been verified by the Lean 4 kernel directly from the definitions: twenty-five instances of the two congruences, fourteen of Theorem 1.1 at $n = 7, 15, \ldots, 111$ and eleven of Theorem 1.2 at $n = 3, 11, \ldots, 83$, of which eleven have n composite; the two published cases (3) and the vanishing of [1, Theorem 3], recomputed through the same code as a control on it; and thirteen refuted claim shapes at 91 parameter pairs, including the sharpness statements of Section 5.

On the status of the Problem. Problem 1 was posed in 2019 and we have not found it addressed since. Two citation indexes, enumerated on 2026-09-03, return 51 citation records for [1] and 42 citing works respectively; between them they name 20 distinct arXiv preprints, and all 20 were retrieved and searched. The summand of (2) occurs in nine of them, on 32 lines, and *never* with the weight $(-q)^k$: thirty-one of those lines carry q^{2k} instead — that is, they are [1, Theorem 2] or a congruence of the same family, five of these papers developing that $\Phi_n(q)^2$ circle further (arXiv:2001.08079, 2109.12034, 2112.12076, 2310.15207, 2608.13818) — and the remaining line carries the q-integer weight $(-1)^k(1-q^{4k+1})/(1-q)$. None of the 20 contains the word "Clausen", and in none of them does a residue class modulo 8 occur. A search of a full-text arXiv corpus of 882 708 identifiers found the summand together with the weight $(-q)$ in exactly one paper, [1] itself; widening the search to any paper that combines $\Phi_n(q)$ with a residue class modulo 8 adds exactly one more, [6], whose congruences indexed by a class modulo 8 concern other summands, none of them (2). The residual risk is stated plainly: the citation records we could not match to an arXiv preprint — about thirty, with duplicates among them —

were not read; their titles are about the Van Hamme (A.2) and (H.2) supercongruences, Dwork-type q-supercongruences, and consequences of the Watson and Singh transformations, none of which is this Problem, but a solution inside one of them cannot be excluded by the searches above.

2. Notation and the results used

Throughout, n is an odd positive integer and q is an indeterminate. We use the basic hypergeometric series

$$ {}_{r+1}\phi_r\left[\begin{matrix}a_1,\dots,a_{r+1}\\ b_1,\dots,b_r\end{matrix};q,z\right]=\sum_{k\ge 0}\frac{(a_1;q)_k\cdots(a_{r+1};q)_k}{(q;q)_k(b_1;q)_k\cdots(b_r;q)_k}\,z^k. $$

The congruence convention. $\Phi_n(q)$ is irreducible over $\mathbb{Q}$, so the localisation $R_n=\mathbb{Q}[q]_{(\Phi_n(q))}$ is a discrete valuation ring; write $v=v_n$ for its valuation and note that its residue field is $\mathbb{Q}(\zeta_n)$, ζ_n a primitive n-th root of unity. For $f,g\in R_n$ we write $f\equiv g \pmod{\Phi_n(q)^m}$ for $v(f-g)\ge m$. An element of R_n is a rational function of q whose denominator is coprime to $\Phi_n(q)$; we call such a function *integral*. Two facts are used constantly and without further comment:

$$ \Phi_n(q)\mid 1-q^m \iff n\mid m, \qquad \text{and} \qquad q^{\,n}\equiv 1 \pmod{\Phi_n(q)}. \tag{4} $$

In particular q is a unit in R_n, so negative powers of q make sense modulo $\Phi_n(q)$, and an exponent of q matters only modulo n. Finally, for $f\in\mathbb{Z}[q]$ one has $\Phi_n(q)\mid f$ in R_n if and only if $\Phi_n(q)\mid f$ in $\mathbb{Z}[q]$, because Φ_n is primitive and irreducible; so the polynomial divisibilities of Sections 5 and 6 are statements about $\mathbb{Z}[q]$.

The sum, with denominators cleared. Put $M=(n-1)/2$ and

$$ D_n=(q^2;q^2)_M^2\,(q^4;q^4)_M, \qquad N_n=\sum_{k=0}^{M}(q;q^2)_k^2\,(q^2;q^4)_k\,(-q)^k\,\frac{D_n}{(q^2;q^2)_k^2\,(q^4;q^4)_k}. \tag{5} $$

Each k-th denominator divides D_n, so $N_n\in\mathbb{Z}[q]$, and $S_n=N_n/D_n$. A direct count gives $\deg D_n=\deg N_n=n^2-1$. By (4), $\Phi_n(q)$ divides no factor $1-q^{2j}$ or $1-q^{4j}$ with $1\le j\le M$, so D_n is a unit in R_n and

$$ S_n\equiv 0 \pmod{\Phi_n(q)} \iff \Phi_n(q)\mid N_n(q) \text{ in } \mathbb{Z}[q]. \tag{6} $$

Likewise, writing $j=(3n-1)/8$, $d=(3n-1)/4$, $e=(n-1)^2/4$ for $n\equiv 3 \pmod 8$ (all three are integers in that class) and

$$ E_n=q^{\,e}\,N_n\,(q^4;q^4)_d^2\;-\;(q;q^4)_j^2\,(q^2;q^4)_j^2\,(-q;q)_M\,D_n\;\in\;\mathbb{Z}[q], \tag{7} $$

the conclusion of Theorem 1.2 is equivalent to $\Phi_n(q)\mid E_n(q)$ in $\mathbb{Z}[q]$, because $d\le n-1$ makes $(q^4;q^4)_d$ a unit as well.

The three results of [1] **that are used.** Jackson's generalisation of Clausen's identity [3] (see also [4, eq. (3.2)]) is quoted in [1, eq. (3.1)] in the form

$$ {}_2\phi_1\left[\begin{matrix}qa,\,q/a\\ q^4\end{matrix};q^4,z\right]{}_2\phi_1\left[\begin{matrix}qa,\,q/a\\ q^4\end{matrix};q^4,q^2z\right]=\sum_{k\ge 0}\frac{(aq;q^2)_k(q/a;q^2)_k(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k}\,z^k. \tag{8} $$

The two evaluations we use are [1, Theorems 3 and 4], both proved there for every odd n: the first from the q-Kummer (Bailey–Daum) summation, the second from Jackson's terminating q-analogue of Dixon's sum, which [1] quotes as [2, Appendix (II.9)] and [2, Appendix (II.15)] respectively. Writing

$$ T_3(n)=\sum_{k=0}^{n-1}\frac{(q;q^4)_k^2}{(q^4;q^4)_k^2}(-q)^{3k}, \qquad T_4(n)=\sum_{k=0}^{n-1}\frac{(q;q^4)_k^2}{(q^4;q^4)_k^2}(-q)^{k}, $$

[1, Theorem 3] states that, modulo $\Phi_n(q)$,

$$T_3(n) \equiv \begin{cases} \dfrac{(q^5,q^7;q^8)_{(n-1)/8}}{(q^4;q^4)_{(n-1)/4}} & n \equiv 1 \pmod 8, \\ \dfrac{(q^5,q^7;q^8)_{(3n-1)/8}}{(q^4;q^4)_{(3n-1)/4}} & n \equiv 3 \pmod 8, \\ 0 & n \equiv 5,7 \pmod 8, \end{cases} \tag{9}$$

and [1, Theorem 4] states that, modulo $\Phi_n(q)$,

$$T_4(n) \equiv \frac{(q,q^2,-q^3,-q^4;q^4)_{(n-1)/8}}{(q^4;q^4)_{(n-1)/4}}\, q^{(1-n^2)/8} \quad (n \equiv 1 \pmod 8), \tag{10}$$

with $(n-1)$ replaced by $(3n-1)$ in both indices when $n \equiv 3 \pmod 8$, and two further closed forms, which we do not need, when $n \equiv 5,7 \pmod 8$. Note the pattern: in both theorems the class $n \equiv 3 \pmod 8$ is the class $n \equiv 1 \pmod 8$ with every index attached to $3n$. That is the pattern Theorem 1.2 extends to S_n.

3. The two congruences

Theorem 1.1 and Theorem 1.2 were stated in the introduction; this section records two reformulations and one consequence, and Section 4 proves them.

Remark 3.1 (the $3n$ normalisation). For $n \equiv 3 \pmod 8$, Theorem 1.2 is equivalent to

$$S_n \;\equiv\; \frac{1}{2} \cdot \frac{(q,q^2;q^4)_{(3n-1)/8}^2\,(-q;q)_{(3n-1)/2}}{(q^4;q^4)_{(3n-1)/4}^2}\, q^{-(1-3n)^2/4} \pmod{\Phi_n(q)}, \tag{11}$$

which is the $n \equiv 1 \pmod 8$ case of (3) with n replaced by $3n$ in every index and in the exponent, times $\frac{1}{2}$. The two forms agree because

$$(-q;q)_{(3n-1)/2} \equiv 2\,(-q;q)_{(n-1)/2} \quad \text{and} \quad \tfrac{1}{4}(1-3n)^2 \equiv \tfrac{1}{4}(1-n)^2 \pmod n,$$

the first because the missing factors $\prod_{i=(n+1)/2}^{(3n-1)/2}(1+q^i)$ run over a complete residue system modulo n and $\prod_{r=0}^{n-1}(1+\zeta_n^r) = 2$ for odd n, the second because $\frac{1}{4}\big((1-3n)^2-(1-n)^2\big) = n(2n-1)$. The form of Theorem 1.2 is the one to quote, being free of the factor $\frac{1}{2}$; the form (11) is the one the proof produces.

Corollary 3.2. *For every odd positive integer n, S_n is congruent modulo $\Phi_n(q)$ to 0 if $n \equiv 5,7 \pmod 8$, and to $(q,q^2;q^4)_m^2(-q;q)_{(n-1)/2}q^{-(1-n)^2/4}/(q^4;q^4)_{2m}^2$ with $m = (n-1)/8$ if $n \equiv 1 \pmod 8$ and $m = (3n-1)/8$ if $n \equiv 3 \pmod 8$.*

Proof. The classes $n \equiv 1,5 \pmod 8$ are (3), and $2m = (n-1)/4$ resp. $(3n-1)/4$ in the two closed-form cases; the classes $n \equiv 3,7 \pmod 8$ are Theorems 1.2 and 1.1. □

4. Proof of Theorems 1.1 and 1.2

Throughout this section $n \equiv 3 \pmod 4$, so that $(3n-1)/4$ and $(3n-1)/2$ are integers. Put

$$T_k = \frac{(q;q^2)_k^2(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k}(-q)^k, \qquad T'_k = \frac{(q^{1-3n};q^2)_k\,(q^{1+3n};q^2)_k\,(q^2;q^4)_k}{(q^2;q^2)_k^2(q^4;q^4)_k}(-q)^k,$$

so that $S_n = \sum_{k=0}^{(n-1)/2} T_k$, and

$$A' = \sum_{k=0}^{(3n-1)/4} \frac{(q^{1-3n};q^4)_k(q^{1+3n};q^4)_k}{(q^4;q^4)_k^2}(-q)^k, \qquad B' = \sum_{k=0}^{(3n-1)/4} \frac{(q^{1-3n};q^4)_k(q^{1+3n};q^4)_k}{(q^4;q^4)_k^2}(-q)^{3k}.$$

(a) The specialisation. Take $a = q^{-3n}$ and $z = -q$ in (8), so that $aq = q^{1-3n}$ and $q/a = q^{1+3n}$. Both sides terminate. On the right, $(q^{1-3n};q^2)_k$ contains the factor $1 - q^{1-3n+2j}$ with $j = (3n-1)/2$, which is 0; hence the sum stops at $k = (3n-1)/2$. On the left, $(q^{1-3n};q^4)_k$ contains the factor $1 - q^{1-3n+4j}$ with $j = (3n-1)/4$, which is 0 — and this requires $4 \mid 3n-1$, i.e. $n \equiv 3 \pmod 4$; hence each ${}_2\phi_1$ stops at $k = (3n-1)/4$. Therefore

$$\sum_{k=0}^{(3n-1)/2} T'_k = A'\,B'. \tag{12}$$

This is the one point at which our argument differs from [1], whose proof of (3) takes $a = q^{-n}$: for $n \equiv 3 \pmod 4$ that choice makes $(q^{1-n};q^4)_k$ never vanish, the two ${}_2\phi_1$ series do not terminate, and they are not integral. *Numerical confirmation.* Identity (12) was verified as an exact identity of rational functions by evaluating both sides in exact rational arithmetic at $q = 3/2,\ 5/3,\ 7/4,\ 11/5,\ 4/7$ for each $n = 3, 7, 11, \dots, 31$; the same test with $a = q^{-5n}$ fails, as it must.

(b) The head. For $0 \le k \le (n-1)/2$ one has $T'_k \equiv T_k$. Indeed $q^{3n} \equiv 1$ gives $(q^{1\pm3n};q^2)_k \equiv (q;q^2)_k$, and by (4) every denominator factor $1 - q^{2j}$, $1 - q^{4j}$ with $1 \le j \le (n-1)/2$ is a unit. Hence $\sum_{k=0}^{(n-1)/2} T'_k \equiv S_n$.

(c) The middle block vanishes. For $(n+1)/2 \le k \le n-1$ we claim $v(T'_k) = 3$, so $T'_k \equiv 0$. Count the factors divisible by Φ_n, using (4). In $(q^{1-3n};q^2)_k$ the factor $1 - q^{1-3n+2j}$ is divisible exactly when $n \mid 1+2j$; for $0 \le j \le k-1 \le n-2$ the odd number $1+2j$ lies in $[1, 2n-3]$, so this happens only at $j = (n-1)/2$, which occurs precisely because $k \ge (n+1)/2$. The same count applies to $(q^{1+3n};q^2)_k$, and to $(q^2;q^4)_k$, where $n \mid 2+4j$ is equivalent to $n \mid 1+2j$ since n is odd. The numerator therefore has valuation 3. In the denominator, $n \mid 2j$ or $n \mid 4j$ with $1 \le j \le k \le n-1$ is impossible, so the denominator is a unit.

(d) The tail is a second copy of the head. Let $n \le k \le (3n-1)/2$ and write $k = n + i$ with $0 \le i \le (n-1)/2$. We claim

$$T'_{n+i} \equiv T'_n\,T_i, \qquad \text{and} \qquad T'_n \equiv 1. \tag{13}$$

For the first part, compare step ratios. For $1 \le i' \le (n-1)/2$,

$$\frac{T'_{n+i'}}{T'_{n+i'-1}} = \frac{(1-q^{2i'-n-1})(1-q^{2i'+5n-1})(1-q^{4i'+4n-2})}{(1-q^{2n+2i'})^2(1-q^{4n+4i'})}\,(-q),$$

and reducing every exponent modulo n turns this into

$$\frac{(1-q^{2i'-1})^2(1-q^{4i'-2})}{(1-q^{2i'})^2(1-q^{4i'})}\,(-q) = \frac{T_{i'}}{T_{i'-1}}.$$

Every factor here is a unit: $2i'-1$ runs through $1, 3, \dots, n-2$ and $4i'-2 = 2(2i'-1)$ with n odd, while $2i'$ and $4i'$ are nonzero modulo n because $i' \le (n-1)/2$. Multiplying the ratios from $i' = 1$ to $i' = i$ gives $T'_{n+i}/T'_n \equiv T_i/T_0 = T_i$.

For the second part, count valuations at $k = n$ as in (c). Each of $(q^{1-3n};q^2)_n$, $(q^{1+3n};q^2)_n$ and $(q^2;q^4)_n$ contributes exactly one Φ_n-divisible factor, namely at $j = (n-1)/2$, so the numerator has valuation 3; the denominator factors $(q^2;q^2)_n^2$ and $(q^4;q^4)_n$ contribute 2 and 1, at $j = n$. Hence $v(T'_n) = 0$. Cancelling the six singular factors,

$$\frac{(1-q^{-2n})(1-q^{4n})(1-q^{2n})}{(1-q^{2n})^2(1-q^{4n})} = \frac{1-q^{-2n}}{1-q^{2n}} = -q^{-2n},$$

while the surviving factors contribute $\Pi^3/(\Pi^2\Pi) = 1$ with $\Pi = \prod_{r=1}^{n-1}(1-q^r)$, because each of $j \mapsto 1+2j$, $j \mapsto 2+4j$, $j \mapsto 2j$ and $j \mapsto 4j$ runs over every nonzero residue modulo n exactly once as j omits the singular index. With the weight $(-q)^n \equiv -1$ this gives $T'_n \equiv (-q^{-2n}) \cdot 1 \cdot (-q^n) = q^{-n} \equiv 1$.

Numerical confirmation. The value $T'_n = 1$ was recomputed exactly in $\mathbb{Z}[q]$ at $n = 3, 7, 11, \dots, 31$ by cancelling Φ_n^3 from numerator and denominator before reducing; numerator and denominator multiplicity were 3 each, as predicted. This step is where the argument turns, and it is the step a reader should check first.

(e) The left-hand side is $2S_n$. By (b), (c) and (d), the left-hand side of (12) is congruent to $S_n + 0 + T'_n \sum_{i=0}^{(n-1)/2} T_i = 2S_n$, so

$$2\,S_n \equiv A'\,B' \pmod{\Phi_n(q)}. \tag{14}$$

Numerical confirmation. Congruence (14) was checked directly for every $n \equiv 3 \pmod 4$ with $n \le 51$.

(f) Identifying A' and B'. For $k \le (3n-1)/4 \le n-1$ the denominators $(q^4;q^4)_k^2$ are units and $(q^{1\pm 3n};q^4)_k \equiv (q;q^4)_k$, so

$$A' \equiv \sum_{k=0}^{(3n-1)/4} \frac{(q;q^4)_k^2}{(q^4;q^4)_k^2}(-q)^k, \qquad B' \equiv \sum_{k=0}^{(3n-1)/4} \frac{(q;q^4)_k^2}{(q^4;q^4)_k^2}(-q)^{3k}.$$

Extending each truncation from $(3n-1)/4$ to $n-1$ changes nothing modulo $\Phi_n(q)^2$: for $k > (3n-1)/4$ the factor $1-q^{1+4j}$ at $j = (3n-1)/4$, where $1+4j = 3n$, occurs twice in $(q;q^4)_k^2$. Hence

$$A' \equiv T_4(n), \qquad B' \equiv T_3(n) \pmod{\Phi_n(q)}. \tag{15}$$

(g) Proof of Theorem 1.1. Let $n \equiv 7 \pmod 8$. By (9) and (15), $B' \equiv 0$. The sum A' is integral, all its denominators being units by (f). So $2S_n \equiv A'B' \equiv 0$ by (14), and 2 is a unit in R_n; therefore $S_n \equiv 0 \pmod{\Phi_n(q)}$. □

(h) Proof of Theorem 1.2. Let $n \equiv 3 \pmod 8$ and put $j = (3n-1)/8$, $d = (3n-1)/4 = 2j$. By (9), (10) and (15),

$$A' \equiv \frac{(q, q^2, -q^3, -q^4; q^4)_j}{(q^4;q^4)_d}\, q^{(1-n^2)/8}, \qquad B' \equiv \frac{(q^5, q^7; q^8)_j}{(q^4;q^4)_d}.$$

We simplify the product with two elementary identities.

Reflection. Since $q^{3n} = q^{8j+1} \equiv 1$, we have $q^{5+8i} \equiv q^{4-8(j-i)}$ and $q^{7+8i} \equiv q^{6-8(j-i)}$, so writing $m = j - i$ and using $1 - q^{-t} = -q^{-t}(1-q^t)$,

$$(q^5;q^8)_j \equiv (-1)^j q^{-4j^2}(q^4;q^8)_j, \qquad (q^7;q^8)_j \equiv (-1)^j q^{2j-4j^2}(q^2;q^8)_j,$$

whence

$$(q^5, q^7; q^8)_j \equiv q^{\,j(8j+4)}\,(q^2, q^4; q^8)_j \pmod{\Phi_n(q)}, \tag{16}$$

the exponents $2j - 8j^2$ and $j(8j+4)$ differing by $2j(8j+1) = 6jn$, a multiple of n.

Splitting. Directly from the definition, $(q^2;q^8)_j = (q;q^4)_j(-q;q^4)_j$ and $(q^4;q^8)_j = (q^2;q^4)_j(-q^2;q^4)_j$, and

$$(-q;q^4)_j(-q^2;q^4)_j(-q^3;q^4)_j(-q^4;q^4)_j = (-q;q)_{4j} = (-q;q)_{(3n-1)/2}. \tag{17}$$

Multiplying A' and B' and applying (16), then (17),

$$A'B' \equiv \frac{(q, q^2; q^4)_j^2\,(-q;q)_{(3n-1)/2}}{(q^4;q^4)_d^2}\, q^{\,\varepsilon}, \qquad \varepsilon = \frac{1-n^2}{8} + j(8j+4) = \frac{(n+1)(4n-1)}{4}.$$

Now $\varepsilon \equiv -(n-1)^2/4 \pmod n$, since $\varepsilon + \frac14(n-1)^2 = \frac14 n(5n+1)$ and $4 \mid 5n+1$ for $n \equiv 3 \pmod 4$. With $(-q;q)_{(3n-1)/2} \equiv 2(-q;q)_{(n-1)/2}$ (Remark 3.1), (14) becomes

$$2\,S_n \equiv 2\,\frac{(q, q^2; q^4)_j^2\,(-q;q)_{(n-1)/2}}{(q^4;q^4)_d^2}\, q^{-(1-n)^2/4},$$

and dividing by the unit 2 gives Theorem 1.2. □

Numerical confirmation of (h). The identities (16) and (17) and the exponent congruence $\varepsilon \equiv -(n-1)^2/4 \pmod n$ were checked in exact arithmetic for every $n \equiv 3 \pmod 8$ with $n \le 107$. The

same computation with $3n$ replaced by n throughout, and step (e) replaced by "$S_n \equiv AB$, no tail", reproduces the $n \equiv 1 \pmod 8$ case of (3) including its exponent identity $(n-1)/4 \equiv -(n-1)^2/4 \pmod n$; that reproduction was run for $n = 9, 17, \ldots, 57$ as a control on the whole derivation.

Remark 4.1 (what rests on computation here). Nothing in Section 4 rests on computation logically: (a)–(h) are complete proofs from (8), (9) and (10). The numerical confirmations reported above are independent checks of each step, run before the step was written down, in exact rational or exact integer arithmetic over the stated finite ranges. They are not part of the proof, and they are not machine-checked; see Section 7.

5. Sharpness: the modulus cannot be squared

Replacing the weight $(-q)^k$ in (2) by q^{2k} and the truncation $(n-1)/2$ by $n-1$ produces the sum of [1, Theorem 2], which vanishes modulo $\Phi_n(q)^2$ for every $n \equiv 3 \pmod 4$; [1, Conjecture 1] proposes the same vanishing with the truncation $n-1$ replaced by $rn-1$ or $rn+(n-1)/2$, for every positive integer r, and several later papers develop this circle of $\Phi_n(q)^2$ congruences. The $(-q)^k$ side behaves differently: every congruence in the present circle of ideas is exact at the first power of $\Phi_n(q)$. We record this as a finite computation. Let $T_3^{\mathrm{num}}(n)$ denote the numerator of $T_3(n)$ over the common denominator $(q^4;q^4)_{n-1}^2$.

Theorem 5.1. *The Φ_n-valuation of each of the following polynomials is exactly* 1*, at each of the listed n; equivalently, $\Phi_n(q)^2$ divides none of them.*

1. N_n*, for* $n = 7, 15, 23, 31, 39, 47, 55$ *($n \equiv 7 \pmod 8$, Theorem 1.1);*
2. N_n*, for* $n = 5, 13, 21, 29, 37$ *($n \equiv 5 \pmod 8$, the vanishing case of* (3)*);*
3. E_n*, for* $n = 3, 11, 19, 27, 35$ *($n \equiv 3 \pmod 8$, Theorem 1.2);*
4. E_n *with the indices* $(n-1)/8$*,* $(n-1)/4$*, for* $n = 9, 17, 25, 33$ *($n \equiv 1 \pmod 8$, the closed-form case of* (3)*);*
5. $T_3^{\mathrm{num}}(n)$*, for* $n = 5, 7, 13, 15, 21, 23$ *([1, Theorem 3], its vanishing classes).*

Proof. That Φ_n divides each of the five polynomials at each listed n is Theorem 6.3 for (1), Theorem 6.4 for (3), and Proposition 6.2 for (2), (4) and (5). That Φ_n^2 divides none of them is twenty-seven further tests, each an exact Euclidean division of an explicit integer polynomial by $\Phi_n(q)^2$ with nonzero remainder. All of these were carried out by the Lean 4 kernel; see Section 6. □

Theorem 5.1 is a negative result about all four residue classes at once, and it settles the natural next question in the negative: Problem 1 as posed is already best possible in its modulus, and Theorem 1.2 admits no $\Phi_n(q)^2$ refinement: every congruence that [1] states modulo $\Phi_n(q)^2$ carries a weight other than $(-q)^k$. What the computation does *not* exclude is a refinement with a parameter: [1, Theorem 2] is proved there through a one-parameter family valid modulo $(1-aq^n)(a-q^n)$, and whether Theorem 1.2 has such a parametric form is left open.

6. Instances verified by exact computation

This section reports what has been checked directly from the definitions, by exact integer arithmetic inside the Lean 4 kernel, independently of Section 4. All the statements below are about explicit polynomials in $\mathbb{Z}[q]$ built by the recursions (5) and (7); by (6) and Proposition 6.5, they say exactly what the corresponding q-congruences say.

We begin with the two controls that make the rest meaningful.

Proposition 6.1 (the arithmetic engine)**.** *The coefficient arrays computed for $\Phi_1, \Phi_2, \Phi_7, \Phi_{12}, \Phi_{15}$ agree with their textbook values; Φ_{105} has degree* 48*, exactly two coefficients equal to -2 and every other coefficient in $\{0, 1, -1\}$; Φ_n is monic for the twelve values $n = 3, 5, 7, 9, 11, 15, 19, 23, 27, 31, 35, 39$; $\deg N_n = n^2 - 1$ for the fourteen values $n = 3, 5, 7, 9, 11, 15, 19, 23, 27, 31, 35, 39, 43, 47$; and replacing the exponent $e = (n-1)^2/4$ in* (7) *by $e + n$ leaves the divisibility $\Phi_n \mid E_n$ intact for $n = 3, 11, 19, 27$.*

Φ_{105} is the classical first cyclotomic polynomial with a coefficient outside $\{0, \pm 1\}$, and is included for that reason. The last clause of Proposition 6.1 pins the one place where a polynomial divisibility and a q-congruence could drift apart: the exponent of q is only defined modulo n (see (4)), and the corresponding negative control in Proposition 6.6(4) shows that a shift by 1, unlike a shift by n, destroys the congruence.

Proposition 6.2 (the published cases, recomputed)**.** *Through the same arithmetic layer as Theorems 6.3 and 6.4 below, the vanishing case through literally the same predicate and the closed-form case through the same one with its two indices changed:* (3) *holds at* $n = 5, 13, 21, 29, 37, 45$ *(the vanishing case) and at* $n = 9, 17, 25, 33, 41$ *(the closed-form case); and* $T_3(n) \equiv 0 \pmod{\Phi_n(q)}$ *at* $n = 5, 7, 13, 15, 21, 23, 29, 31$ *while* $T_3(n) \not\equiv 0 \pmod{\Phi_n(q)}$ *at* $n = 3, 9, 11, 17, 19, 25, 27$*, in accordance with* (9)*.*

Proposition 6.2 is a control on the computation, not a new result: everything in it is proved in [1]. It is reported because an implementation that could not reproduce the published half of (3) would say nothing about the unpublished half, and because the vanishing of T_3 is the exact input that step (g) of Section 4 consumes. The same principle was applied before any of the statements of this paper was formulated: all five theorems of [1] were recomputed from their statements there, in two independent arithmetics — exact arithmetic in $\mathbb{Z}[q]$ reduced modulo $\Phi_n(q)$, and arithmetic in $\mathbb{F}_p[q]/(\Phi_n(q))$ with $p = 2^{61} - 1$ — at every odd $n \le 57$ for which they are stated.

Theorem 6.3 (instances of Theorem 1.1)**.** $\Phi_n(q)$ *divides* $N_n(q)$ *in* $\mathbb{Z}[q]$ *for each of the fourteen integers*

$$n = 7,\ 15,\ 23,\ 31,\ 39,\ 47,\ 55,\ 63,\ 71,\ 79,\ 87,\ 95,\ 103,\ 111.$$

All fourteen are $\equiv 7 \pmod 8$ *and exactly seven of them —* 15*,* 39*,* 55*,* 63*,* 87*,* 95*,* 111 *— are composite. The largest instance has* $\deg N_{111} = 12\,320$ *and coefficients of up to* 31 *decimal digits.*

Theorem 6.4 (instances of Theorem 1.2)**.** $\Phi_n(q)$ *divides* $E_n(q)$ *in* $\mathbb{Z}[q]$ *for each of the eleven integers*

$$n = 3,\ 11,\ 19,\ 27,\ 35,\ 43,\ 51,\ 59,\ 67,\ 75,\ 83.$$

All eleven are $\equiv 3 \pmod 8$ *and exactly four of them —* 27*,* 35*,* 51*,* 75 *— are composite. The largest instance has* $\deg E_{83} = 24\,193$*.*

Proposition 6.5 (non-vacuity)**.** $\Phi_n(q)$ *divides neither* D_n*, for the fourteen values*

$$n = 3,\ 5,\ 7,\ 9,\ 11,\ 15,\ 19,\ 23,\ 27,\ 31,\ 35,\ 39,\ 43,\ 47,$$

nor $(q^4; q^4)^2_{(3n-1)/4}$*, for the twelve values* $n = 3, 11, 19, 27, 35, 43$ *and* $n = 7, 15, 23, 31, 39, 47$*.*

Proposition 6.5 is what turns Theorems 6.3 and 6.4 into q-congruences: a divisibility whose denominator were also divisible by $\Phi_n(q)$ would be a statement about $0/0$. The elementary argument of Section 2 gives it for all n; it is computed here, on every n used, rather than assumed.

Proposition 6.6 (negative controls)**.** *Each of the following is false at each listed value of* n*; thirteen claim shapes at* 91 *parameter pairs in total, counting the twenty-seven of Theorem 5.1 and the twenty-six of Proposition 6.5.*

(1) (modulus squared) *the five statements of Theorem 5.1;*
(2) (wrong class) $S_n \equiv 0 \pmod{\Phi_n(q)}$ *for* $n = 9, 17, 25, 33, 41$ *(*$n \equiv 1 \pmod 8$*) and for* $n = 3, 11, 19, 27, 35, 43$ *(*$n \equiv 3 \pmod 8$*); and* $T_3(n) \equiv 0 \pmod{\Phi_n(q)}$ *for* $n = 3, 9, 11, 17, 19, 25, 27$*;*
(3) (wrong Pochhammer index) *Theorem 1.2 with* $(3n-1)/8$ *replaced by* $(3n-1)/8 - 1$*, and with* $(3n-1)/8 + 1$*, at* $n = 3, 11, 19, 27, 35$*;*
(4) (wrong denominator index, wrong exponent) *Theorem 1.2 with* $(3n-1)/4$ *replaced by the index* $(n-1)/4$ *of* (3)*, and with the exponent* e *replaced by* $e + 1$*, at* $n = 3, 11, 19, 27, 35$*;*
(5) (non-vacuity) *the two statements of Proposition 6.5.*

Item (3) is what pins the index of Theorem 1.2: neither neighbour of $(3n-1)/8$ works. Item (4) pins the two places where the formula could have been copied from (3) without the substitution, and

— read together with the last clause of Proposition 6.1 — shows that the exponent control tests the exponent rather than the periodicity $q^n \equiv 1$.

Outside the formal development, the same computations were carried out in an independent implementation, which extends the range of Theorem 6.3 to $n = 111$ (as above) and that of Theorem 6.4 to $n = 107$; a second arithmetic in $\mathbb{F}_p[q]/(\Phi_n(q))$, $p = 2^{61} - 1$, sharing no code with the exact integer path, agrees at every value. Those runs are not machine-checked and are reported only as corroboration.

7. Verification

The statements of Sections 5 and 6 — Theorem 5.1, Theorems 6.3 and 6.4, and Propositions 6.1, 6.2, 6.5 and 6.6 — are verified in Lean 4 [10]: 26 theorems in two files above a file of definitions, compiling in about ten minutes of wall time with a peak resident set of 1.56 GB. The definitions file imports nothing at all — not Mathlib [11], not Batteries, not the standard library beyond Lean's core — and the two theorem files add only a small module defining an annotation attribute used for bookkeeping. The reason for working without a library is that `Polynomial.cyclotomic` is noncomputable in Mathlib (it is defined through the axiom of choice), so no decision procedure can evaluate a divisibility phrased with it. The layer used instead represents a polynomial as a bare array of arbitrary-precision integers, low degree first, and implements addition, multiplication, Euclidean division by a monic divisor, the q-shifted factorials, Φ_n by the recursion $\Phi_n = (q^n - 1)/\prod_{d|n,\, d<n} \Phi_d$, and the polynomials N_n, D_n, E_n of (5) and (7). Each theorem is closed by a single compiled evaluation (`native_decide`), so `#print axioms` reports, for each, propositional extensionality, the soundness of quotient types, and the one evaluation axiom generated for that declaration — and, for the two purely combinatorial statements about the lists of n in Theorems 6.3 and 6.4, that evaluation axiom alone. No `sorryAx`, no choice, no axiom stated by hand appears anywhere. Two consequences must be stated plainly. First, what the kernel checks is a divisibility between the explicit integer coefficient arrays just described; the array produced for Φ_n is *not* proved equal to the library's cyclotomic polynomial, a bridge we price at about a day's work and have not built, and it is instead pinned by Proposition 6.1. Second, the general proof of Theorems 1.1 and 1.2 in Section 4 is not formalised, and formalising it is a programme rather than a check: there is no formalised theory of q-shifted factorials or basic hypergeometric series to build on, so Jackson's q-Clausen identity (8), the q-Kummer (Bailey–Daum) summation and Jackson's q-analogue of Dixon's sum would have to come first. Repository reference to be added at submission.

8. What is left open

Three questions remain. *(i)* Does Theorem 1.2 have a parametric refinement, valid modulo $(1 - aq^n)(a - q^n)$, in the style of the proof of [1, Theorem 2]? By Theorem 5.1 such a refinement cannot specialise to a $\Phi_n(q)^2$ congruence at $a = 1$, but it would be the natural route to r-fold extensions in the style of [1, Conjecture 1]. *(ii)* The proof of Theorem 1.1 uses only that A' is integral, and never the closed forms of [1, Theorem 4] in the classes $n \equiv 5, 7 \pmod 8$. If B' could be evaluated modulo $\Phi_n(q)^2$ rather than $\Phi_n(q)$, the argument would upgrade; but Theorem 5.1(5) finds valuation exactly 1 for $T_3^{\mathrm{num}}(n)$ at every value tested, which closes that particular route and leaves the parametric one. *(iii)* A formal proof of Theorems 1.1 and 1.2 for all n requires a formalised theory of basic hypergeometric series; a first, much smaller step would be to prove the array-to-library bridge for Φ_n when n is prime, which would cover seven of the fourteen values of Theorem 6.3.

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