THE AFFINE BERNSTEIN BASIS ON THE q-QUADRATIC LATTICE ADMITS NO FIRST-ORDER LATTICE LADDER BEYOND DEGREE TWO

KOREA SUPERINTELLIGENCE LABS

Abstract. On the q-quadratic (Askey–Wilson) lattice, Area (arXiv:2608.04802) introduced the affine Bernstein basis $\mathcal{B}_k^n(x) = \binom{n}{k}\big(\frac{1+x}{2}\big)^k\big(\frac{1-x}{2}\big)^{n-k}$ and asked whether it satisfies a first-order lattice ladder $\sigma\,\mathbb{D}_x\mathcal{B}_k^n = \tau\,\mathbb{S}_x\mathcal{B}_k^n$ with $\deg\sigma \le 2$ and $\deg\tau \le 1$, where $\mathbb{D}_x$ and $\mathbb{S}_x$ are the divided-difference and averaging operators of the lattice. He proved that a nonzero solution exists for $n \le 2$ and that only the trivial one does for $n = 3$, by an explicitly factorised 5×5 determinant, and conjectured the same for every $n \ge 4$ and every $q \in (0,1)$, noting that no general determinant or rank formula was available. We prove the conjecture, for every $n \ge 3$, every $0 \le k \le n$ and every $q \in (0,1)$, computing no determinant. The engine is a norm identity: with $\mu = (q^{1/2}+q^{-1/2})/2$, the two half-step values of $\mathcal{B}_k^n$ are conjugate in the quadratic extension $\mathbb{R}[x][d]/(d^2-(\mu^2-1)(x^2-1))$, and their product is $\big(\binom{n}{k}/2^n\big)^2(x+\mu)^{2k}(x-\mu)^{2(n-k)}$. Hence $\mathbb{S}_x\mathcal{B}_k^n$ and $\mathbb{D}_x\mathcal{B}_k^n$ are coprime for every n, which is the missing rank statement, and a degree count finishes the proof. With the $n \le 2$ half this is an exact dichotomy: the ladder has a nonzero solution if and only if $n \le 2$. The argument sees q only through μ and works for every real $\mu \notin \{0, 1, -1\}$; each of the three excluded values carries a nonzero solution already in degree three, so this is the exact degenerate locus. Every result is machine-checked in Lean 4 with Mathlib.

1. Introduction

The objects. A one-dimensional lattice is a map $s \mapsto x(s)$, and the central divided-difference and averaging operators of Nikiforov, Suslov and Uvarov [2] are

$$\mathbb{D}_x f(s) = \frac{f(s+\frac12) - f(s-\frac12)}{x(s+\frac12) - x(s-\frac12)}, \qquad \mathbb{S}_x f(s) = \tfrac12\big[f(s+\tfrac12) + f(s-\tfrac12)\big]. \tag{1}$$

On the q-quadratic lattice $x(s) = \frac12(q^s + q^{-s})$, in the Askey–Wilson parametrisation $x = \cos\theta = \frac12(z+z^{-1})$, $z = e^{i\theta} = q^s$, the operator $\mathbb{D}_x$ sends polynomials in x to polynomials in x and lowers the degree by one, while $\mathbb{S}_x$ preserves it; they obey the product rules

$$\begin{gathered}\mathbb{D}_x(fg) = \mathbb{S}_x f\,\mathbb{D}_x g + \mathbb{D}_x f\,\mathbb{S}_x g, \qquad \mathbb{S}_x(fg) = \mathbb{S}_x f\,\mathbb{S}_x g + \tfrac14 U(s)\,\mathbb{D}_x f\,\mathbb{D}_x g,\\ U(s) = \big(x(s+\tfrac12) - x(s-\tfrac12)\big)^2.\end{gathered} \tag{2}$$

This is the lattice of the Askey–Wilson and q-Racah polynomials [3]. In a recent preprint, Area [1] set up the theory of Bernstein-type bases on it. His *affine (spectral) Bernstein basis* of degree n on the interval $[-1,1]$ is the classical Bernstein basis in the spectral variable,

$$\mathcal{B}_k^n(x) = \binom{n}{k}\Big(\frac{1+x}{2}\Big)^k\Big(\frac{1-x}{2}\Big)^{n-k}, \qquad 0 \le k \le n \tag{3}$$

([1, Definition 4.1, eq. (14)] with $[\xi_0,\xi_1] = [-1,1]$). It has the Bézier properties, and Area asks how it interacts with the lattice calculus. On the linear lattice the classical basis satisfies the first-order relation $(1-x^2)\frac{d}{dx}\mathcal{B}_k^n = (2k-n-nx)\mathcal{B}_k^n$; the lattice analogue is the *first-order lattice ladder*

$$\sigma(x)\,\mathbb{D}_x\mathcal{B}_k^n(x) = \tau(x)\,\mathbb{S}_x\mathcal{B}_k^n(x), \qquad \deg\sigma \le 2, \quad \deg\tau \le 1, \tag{4}$$

which is [1, eq. (16)]. Both sides have degree at most $n+1$, so comparing coefficients gives a homogeneous linear system of $n+2$ equations in the five unknown coefficients of σ and τ, with an $(n+2)\times 5$ coefficient matrix $M_{n,k}$. Everything depends on q only through

$$\mu := \tfrac12\big(q^{1/2} + q^{-1/2}\big), \tag{5}$$

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which Area writes $\widehat{\mu}$; it satisfies $\mu > 1$ for $q \in (0,1)$ and $\mu = 1$ exactly at $q = 1$.

The source and its question. Proposition 4.4 of [1] states that for $n \le 2$ and every $0 \le k \le n$ equation (4) has a nonzero solution (a dimension count: at most four equations in five unknowns), and that for $n = 3$ and every $0 \le k \le 3$ it admits only the trivial solution $\sigma = \tau = 0$, for every $q \in (0,1)$. The degree-three case is proved by displaying the four polynomials $\mathbb{D}_x\mathcal{B}_k^3$, $\mathbb{S}_x\mathcal{B}_k^3$ $(k = 0, 1)$, assembling the 5×5 matrices $M_{3,k}$ and computing the closed factorisations

$$\det M_{3,0} = \frac{(\mu^2-1)^6}{512}, \qquad \det M_{3,1} = -\frac{243\,\mu^8(\mu^2-1)^2}{512}, \tag{6}$$
$$\det M_{3,2} = -\det M_{3,1}, \qquad \det M_{3,3} = -\det M_{3,0}$$

([1, eq. (17)]), which do not vanish for $\mu > 1$. The preprint then poses, verbatim ([1, Conjecture 4.5]; the equation numbers are those of [1]):

> *For every $n \ge 4$, every $0 \le k \le n$ and every $q \in (0,1)$, equation (16) admits only the trivial solution. The degree-three factorisation (17) motivates the conjecture, although no general determinant or rank formula is presently available for $n \ge 4$.*

Its extension to $n \ge 4$ is listed again among the open questions of the concluding section of [1].

Results. We prove the conjecture, and a little more, by an argument that is uniform in n and k and computes no determinant.

Theorem A (Theorem 5.2). For every $n \ge 3$, every $0 \le k \le n$ and every $q \in (0,1)$, equation (4) admits only the trivial solution $\sigma = \tau = 0$.

The hypothesis $n \ge 3$ covers the conjecture's range $n \ge 4$ together with the degree-three case of [1, Proposition 4.4], by one argument. Combined with the $n \le 2$ half of that proposition, for which we give an explicit witness, this is an exact dichotomy.

Theorem B (Corollary 5.4). For every $q \in (0,1)$, every $n \ge 0$ and every $0 \le k \le n$, equation (4) has a nonzero solution if and only if $n \le 2$.

The proof sees q only through μ, and it uses $\mu > 1$ only through three non-vanishing facts that hold for every real μ outside $\{0, 1, -1\}$. Each of the three excluded values is genuinely excluded, so the following is the exact answer to the "general rank formula" the preprint says is unavailable.

Theorem C (Theorem 6.1 and Corollary 6.3). Let μ be a real number with $\mu \ne 0$ and $\mu^2 \ne 1$. Then for every $n \ge 3$ and every $0 \le k \le n$ equation (4) admits only the trivial solution; equivalently, $M_{n,k}$ has full column rank 5. At each of $\mu = 0$, $\mu = 1$ and $\mu = -1$, on the other hand, some cell $(3, k)$ carries a nonzero solution. Hence the set of real μ at which (4) has a nonzero solution for some $n \ge 3$ and some $k \le n$ is exactly $\{0, 1, -1\}$.

The method. Write $x_\pm = x(s \pm \frac{1}{2})$ and $d = \frac{1}{2}(x_+ - x_-)$. On the q-quadratic lattice $x_\pm = \mu x \pm d$ and $d^2 = (\mu^2-1)(x^2-1)$, so for a polynomial P the two half-step values $P(x_\pm)$ are the two conjugates $\mathbb{S}_x P \pm d\,\mathbb{D}_x P$ of one element of the quadratic extension $\mathbb{R}[x][d]/(d^2 - (\mu^2-1)(x^2-1))$; this is just the pair of product rules (2) read as the multiplication law of that extension, and for $P = \mathcal{B}_k^n$ it is proved from scratch in Section 3. Multiplying the two conjugates for $P = \mathcal{B}_k^n$, and using $(1 \pm x_+)(1 \pm x_-) = (x \pm \mu)^2$, gives the *norm identity*

$$(\mathbb{S}_x\mathcal{B}_k^n)^2 - (\mu^2-1)(x^2-1)\,(\mathbb{D}_x\mathcal{B}_k^n)^2 = \Big(\frac{\binom{n}{k}}{2^n}\Big)^2 (x+\mu)^{2k}(x-\mu)^{2(n-k)}. \tag{7}$$

Its right-hand side vanishes only at $x = \pm\mu$, and $\mathbb{S}_x\mathcal{B}_k^n$ does not vanish there. So $\mathbb{S}_x\mathcal{B}_k^n$ and $\mathbb{D}_x\mathcal{B}_k^n$ are coprime in $\mathbb{R}[x]$ (Theorem 4.3) — for every n, every k and every $\mu \notin \{0, 1, -1\}$. A solution of (4) then forces $\mathbb{S}_x\mathcal{B}_k^n \mid \sigma$, say $\sigma = g\,\mathbb{S}_x\mathcal{B}_k^n$ and $\tau = g\,\mathbb{D}_x\mathcal{B}_k^n$, and substituting back into (7) makes the left side of degree at most 4 and the right side of degree $2\deg g + 2n \ge 6$ when $g \ne 0$ and $n \ge 3$. That is the whole proof. The determinants (6) are, up to sign, the Sylvester resultants of the pairs $(\mathbb{D}_x\mathcal{B}_k^3, \mathbb{S}_x\mathcal{B}_k^3)$, so their non-vanishing is the $n = 3$ instance of the coprimality.

What else is in the paper. Section 2 places the question next to the q-Bernstein literature, which lives on the q-linear lattice, and next to the structure-relation literature of the q-quadratic lattice, which concerns orthogonal sequences; neither contains this object. Section 3 sets up the quadratic extension and proves that the polynomials so defined are Area's $\mathbb{S}_x\mathcal{B}_k^n$ and $\mathbb{D}_x\mathcal{B}_k^n$. Sections 4–6 contain the results. Section 7 reproduces the four degree-three polynomials displayed in [1] from the definitions, and records two computations that were carried out in exact rational arithmetic outside the formal development: the factorisations (6), and the fifth determinantal divisor of $M_{n,k}$ for $3 \le n \le 7$, whose roots lie in $\{0, \pm 1\}$ in every cell, as Theorem C predicts. Section 8 lists what remains open, and Section 9 describes the formal verification; the formal statements are quoted verbatim in Appendix A.

Statement numbers of [1] quoted here are those of the typeset version 1, which we compiled from the arXiv source; as of 3 September 2026 that is the only version and no journal reference is recorded for it.

2. Related work

q-Bernstein bases on the q-linear lattice. The q-Bernstein polynomials of Phillips [4], their Bézier theory [5], and the connection formulae of Area, Godoy, Woźny, Lewanowicz and Ronveaux [7] (little q-Jacobi, q-Hahn) and of Lewanowicz, Woźny, Area and Godoy [8] (several variables) all live on the q-linear lattice $x(s) = q^s$, one level below the q-quadratic lattice in the Askey scheme; the linear-lattice Bernstein–Hahn–Eberlein connection of Ronveaux, Zarzo, Area and Godoy [6] sits one level lower still. Area's preprint describes its own programme as the completion of exactly this ladder of lattices ([1, eq. (2)]), and the affine basis (3) on the q-quadratic lattice is a construction of that preprint. We did not find the basis, or the ladder question, elsewhere. On 3 September 2026 each of the arXiv API queries

`abs:"Bernstein" AND abs:"Askey-Wilson"`, `abs:"Bernstein" AND abs:"q-Racah"`, `abs:"Bernstein basis" AND abs:"connection coefficients"`

returned exactly one paper, namely [1]; the query `abs:"Bernstein" AND abs:"nonuniform lattice"` returned none; and OpenAlex recorded no work citing [1].

Structure relations on the q-quadratic lattice. First-order relations between $\mathbb{D}_x P$ and $\mathbb{S}_x P$ are the subject of a substantial literature on the same lattice, but for *orthogonal sequences*: Koornwinder [9] gives an explicit structure relation for the Askey–Wilson polynomials through a divided q-difference operator that is skew-symmetric for the Askey–Wilson inner product; Kenfack Nangho and Jordaan [10] prove, in the quadratic and q-quadratic variable, the equivalence of a first structure relation for a monic orthogonal sequence with the orthogonality of the sequence $\{\mathbb{D}_x^2 P_n\}$ and with a Sturm–Liouville-type equation, and in [11] they show that the only monic orthogonal sequences satisfying $\pi(x)\mathcal{D}_q^2 P_n = \sum_{j=-2}^{2} a_{n,n+j} P_{n+j}$, with $\deg \pi \le 4$ and $\mathcal{D}_q$ the Askey–Wilson operator, are the Askey–Wilson polynomials and their special or limiting cases (these descriptions are taken from the papers' abstracts). Equation (4) is not of this type: it is a relation for a single, non-orthogonal basis element, and the question is whether a nonzero relation exists at all. The abstract identity $(\mathbb{S}_x f)^2 - \frac{1}{4}U\,(\mathbb{D}_x f)^2 = f(x_+)f(x_-)$ behind (7) is a two-line consequence of the product rules (2), and we make no claim of novelty for it in that form; what is used here is its explicit factorisation for the Bernstein element, and the observation that it settles the ladder uniformly in n.

3. Preliminaries

Throughout, $n \ge 0$ and $0 \le k \le n$ are integers, μ is a real number, and

$$c_{n,k} := \frac{\binom{n}{k}}{2^n}, \qquad \mathcal{B}_k^n(x) = c_{n,k}(1+x)^k(1-x)^{n-k}, \qquad \Delta_\mu(x) := (\mu^2-1)(x^2-1),$$

so that (3) reads $\mathcal{B}_k^n = c_{n,k}(1+x)^k(1-x)^{n-k}$. All polynomials have real coefficients.

The quadratic extension. Let $\mathcal{A}_\mu := \mathbb{R}[x][d]/(d^2 - \Delta_\mu)$. Every element of $\mathcal{A}_\mu$ is uniquely $a + d\,b$ with $a, b \in \mathbb{R}[x]$, with multiplication

$$(a + d\,b)(a' + d\,b') = (aa' + \Delta_\mu\, bb') + d\,(ab' + a'b).$$

Conjugation $\overline{a + d\,b} := a - d\,b$ is a ring automorphism, and the norm $\mathrm{N}(a+d\,b) := (a+d\,b)\overline{(a+d\,b)} = a^2 - \Delta_\mu\, b^2$ is multiplicative.

Definition 3.1. Define $S_k^n, D_k^n \in \mathbb{R}[x]$ by expanding, in $\mathcal{A}_\mu$,

$$S_k^n + d\,D_k^n := \mathcal{B}_k^n(\mu x + d) = c_{n,k}\,(1+\mu x+d)^k\,(1-\mu x-d)^{n-k}. \tag{8}$$

The two polynomials depend on μ; we suppress this from the notation. The next two propositions say that they are the polynomials $\mathbb{S}_x\mathcal{B}_k^n$ and $\mathbb{D}_x\mathcal{B}_k^n$ of [1]: first as functions of the pair (x,d), then on the lattice itself.

Proposition 3.2 (the conjugate-pair representation)**.** *For all real x and d with $d^2 = (\mu^2-1)(x^2-1)$,*

$$\mathcal{B}_k^n(\mu x + d) = S_k^n(x) + d\,D_k^n(x), \qquad \mathcal{B}_k^n(\mu x - d) = S_k^n(x) - d\,D_k^n(x).$$

Proof. For fixed real x and d with $d^2 = \Delta_\mu(x)$, the map $\varepsilon\colon \mathcal{A}_\mu \to \mathbb{R}$, $a + d\,b \mapsto a(x) + d\,b(x)$, is a ring homomorphism, because $d^2 = \Delta_\mu(x)$ is exactly what makes ε respect the multiplication law displayed above. Applying ε to (8) gives the first identity, since $\varepsilon(\mu x + d) = \mu x + d$ and ε commutes with polynomial expressions. The second is the first with $-d$ in place of d, which is admissible because $(-d)^2 = d^2$. □

Proposition 3.3 (the lattice bridge)**.** *Let Q and z be nonzero real numbers, and put*

$$\mu = \frac{Q+Q^{-1}}{2}, \qquad x = \frac{z+z^{-1}}{2}, \qquad x_+ = \frac{Qz + (Qz)^{-1}}{2}, \qquad x_- = \frac{Q^{-1}z + (Q^{-1}z)^{-1}}{2}.$$

Then

$$\frac{x_+ + x_-}{2} = \mu x, \qquad \Big(\frac{x_+ - x_-}{2}\Big)^2 = (\mu^2-1)(x^2-1),$$

and consequently

$$\frac{\mathcal{B}_k^n(x_+) + \mathcal{B}_k^n(x_-)}{2} = S_k^n(x), \qquad \mathcal{B}_k^n(x_+) - \mathcal{B}_k^n(x_-) = (x_+ - x_-)\,D_k^n(x).$$

Proof. The two lattice identities are rational identities in Q and z: expanding, $x_+ + x_- = \frac{1}{2}(Q+Q^{-1})(z+z^{-1})$ and $x_+ - x_- = \frac{1}{2}(Q-Q^{-1})(z-z^{-1})$, and $\big(\frac{1}{2}(Q-Q^{-1})\big)^2 = \mu^2 - 1$, $\big(\frac{1}{2}(z-z^{-1})\big)^2 = x^2 - 1$. So $x_\pm = \mu x \pm d$ with $d = \frac{1}{2}(x_+ - x_-)$ and $d^2 = \Delta_\mu(x)$, and Proposition 3.2 gives $\mathcal{B}_k^n(x_\pm) = S_k^n(x) \pm d\,D_k^n(x)$; add and subtract. □

With $Q = q^{1/2}$ and $z = q^s$ one has $x = x(s)$, $x_\pm = x(s \pm \frac{1}{2})$ and μ as in (5), so Proposition 3.3 says precisely that S_k^n and D_k^n evaluated at the lattice point $x(s)$ are $\mathbb{S}_x\mathcal{B}_k^n(s)$ and $\mathbb{D}_x\mathcal{B}_k^n(s)$ as defined by (1). Since $\mathbb{D}_x$ and $\mathbb{S}_x$ send polynomials in x to polynomials in x [1, §2.2], and a polynomial identity that holds at the infinitely many real lattice points $x(s)$, $s \in \mathbb{R}$, holds identically, the polynomials $\mathbb{S}_x\mathcal{B}_k^n$ and $\mathbb{D}_x\mathcal{B}_k^n$ of [1] are S_k^n and D_k^n. (The same rational identities hold verbatim for z on the unit circle, which is the parametrisation $x = \cos\theta$ of [1]; the statement above, for real Q and z, is the one that was formalised.) From now on we write S_k^n and D_k^n and speak of (4) as the equation

$$\sigma\,D_k^n = \tau\,S_k^n, \qquad \sigma, \tau \in \mathbb{R}[x], \quad \deg\sigma \le 2, \quad \deg\tau \le 1. \tag{9}$$

4. The norm identity and coprimality

Theorem 4.1 (the norm identity)**.** *For every n, every $0 \le k \le n$ and every real μ,*

$$(S_k^n)^2 - (\mu^2-1)(x^2-1)\,(D_k^n)^2 = c_{n,k}^2\,(x+\mu)^{2k}(x-\mu)^{2(n-k)} \qquad \text{in } \mathbb{R}[x].$$

Proof. The left-hand side is $\mathrm{N}(S_k^n + d\,D_k^n)$, and N is multiplicative, so by (8) it equals $c_{n,k}^2\,\mathrm{N}(1+\mu x + d)^k\,\mathrm{N}(1-\mu x-d)^{n-k}$. Now $\mathrm{N}(1+\mu x+d) = (1+\mu x)^2 - \Delta_\mu = 1 + 2\mu x + \mu^2 x^2 - (\mu^2-1)(x^2-1) = (x+\mu)^2$ and likewise $\mathrm{N}(1-\mu x-d) = (1-\mu x)^2 - \Delta_\mu = (x-\mu)^2$. □

In words: the product of the two half-step values of the affine Bernstein element factors completely, and its only roots are $x = \pm\mu$. The next lemma evaluates S_k^n at the three points that matter; the half-step pairs there are $(x_+, x_-) = (\mu, \mu)$ at $x = 1$, $(2\mu^2 - 1,\, 1)$ at $x = \mu$ and $(-1,\, 1 - 2\mu^2)$ at $x = -\mu$.

Lemma 4.2. *For every real μ,*

$$S_k^n(1) = \mathcal{B}_k^n(\mu), \qquad 2\,S_k^n(\mu) = \mathcal{B}_k^n(2\mu^2-1) + \mathcal{B}_k^n(1), \qquad 2\,S_k^n(-\mu) = \mathcal{B}_k^n(-1) + \mathcal{B}_k^n(1-2\mu^2).$$

Consequently, if $\mu \neq 0$ and $\mu^2 \neq 1$, then $S_k^n \neq 0$, $S_k^n(\mu) \neq 0$ and $S_k^n(-\mu) \neq 0$.

Proof. Apply Proposition 3.2 at $x = 1$ with $d = 0$ (admissible since $\Delta_\mu(1) = 0$), and at $x = \pm\mu$ with $d = \mu^2 - 1$ (admissible since $\Delta_\mu(\pm\mu) = (\mu^2 - 1)^2$), and add the two conjugate identities. For the consequences, write $\mathcal{B}_k^n(t) = c_{n,k}(1+t)^k(1-t)^{n-k}$ with $c_{n,k} > 0$. Then $S_k^n(1) = c_{n,k}(1+\mu)^k(1-\mu)^{n-k} \neq 0$ because $\mu^2 \neq 1$. Next, $2S_k^n(\mu) = c_{n,k}(2\mu^2)^k(2-2\mu^2)^{n-k} + c_{n,k}2^k\,0^{n-k}$: if $k = n$ both summands are positive; if $k < n$ the second vanishes and the first is nonzero because $\mu \neq 0$ and $\mu^2 \neq 1$. Symmetrically, $2S_k^n(-\mu) = c_{n,k}0^k 2^{n-k} + c_{n,k}(2-2\mu^2)^k(2\mu^2)^{n-k}$: if $k = 0$ both summands are positive, and if $k \geq 1$ the first vanishes and the second is nonzero. □

Theorem 4.3 (coprimality)**.** *Let μ be real with $\mu \neq 0$ and $\mu^2 \neq 1$. Then for every n and every $0 \leq k \leq n$ the polynomials S_k^n and D_k^n are coprime in $\mathbb{R}[x]$: there are $a, b \in \mathbb{R}[x]$ with $a\,S_k^n + b\,D_k^n = 1$. In particular this holds for $\mu = \frac{1}{2}(q^{1/2} + q^{-1/2})$ with $q \in (0,1)$, that is, $\mathbb{S}_x\mathcal{B}_k^n$ and $\mathbb{D}_x\mathcal{B}_k^n$ are coprime for every n, every k and every $q \in (0,1)$.*

Proof. The argument is a constructive Bézout chain. If $p \in \mathbb{R}[x]$ and $p(\alpha) \neq 0$, then p and $x - \alpha$ are coprime: writing $p - p(\alpha) = (x-\alpha)w$ one has $p(\alpha)^{-1}p - p(\alpha)^{-1}w\,(x-\alpha) = 1$. Coprimality of p with two polynomials passes to their product and to their powers, and every nonzero constant is coprime to everything. By Lemma 4.2, S_k^n is therefore coprime to $R := c_{n,k}^2(x+\mu)^{2k}(x-\mu)^{2(n-k)}$. By Theorem 4.1, $R = (S_k^n)^2 - \Delta_\mu(D_k^n)^2$, so a relation $a\,S_k^n + b\,R = 1$ rewrites as $(a + b\,S_k^n)\,S_k^n + (-b\,\Delta_\mu)\,(D_k^n)^2 = 1$. Hence S_k^n is coprime to $\Delta_\mu(D_k^n)^2$, hence to $(D_k^n)^2$, hence to D_k^n. The "in particular" clause follows because $\mu > 1$ for $q \in (0,1)$ (Lemma 5.1 below). □

Remark 4.4. The matrix $M_{3,k}$ of [1] has columns $\mathbb{D}_x\mathcal{B}_k^3$, $x\mathbb{D}_x\mathcal{B}_k^3$, $x^2\mathbb{D}_x\mathcal{B}_k^3$, $-\mathbb{S}_x\mathcal{B}_k^3$, $-x\mathbb{S}_x\mathcal{B}_k^3$, written in the coefficients of $1, x, \ldots, x^4$. Up to the sign of two columns and a transposition, this is the Sylvester matrix of the pair $(\mathbb{D}_x\mathcal{B}_k^3, \mathbb{S}_x\mathcal{B}_k^3)$ in formal degrees $(2,3)$, so $\det M_{3,k} = \pm\operatorname{Res}(\mathbb{D}_x\mathcal{B}_k^3, \mathbb{S}_x\mathcal{B}_k^3)$, and its non-vanishing for $\mu > 1$ is the case $n = 3$ of Theorem 4.3. (In exact rational arithmetic the sign is $+$ when the Sylvester rows are ordered D, xD, x^2D, S, xS.) The theorem proves the same thing for every n at once, without a resultant.

5. The ladder theorem

Lemma 5.1. *For every $q \in (0,1)$, $\mu = \frac{1}{2}(q^{1/2} + q^{-1/2}) > 1$.*

Proof. With $t = q^{1/2} \in (0,1)$, $t\,(t + t^{-1} - 2) = (t-1)^2 > 0$, so $t + t^{-1} > 2$. (This is the remark made in the proof of [1, Proposition 4.4].) □

Theorem 5.2 (Conjecture 4.5 of [1], for every $n \geq 3$)**.** *Let $n \geq 3$, $0 \leq k \leq n$, and let $\mu > 1$ be real. If $\sigma, \tau \in \mathbb{R}[x]$ satisfy $\deg\sigma \leq 2$, $\deg\tau \leq 1$ and $\sigma\,D_k^n = \tau\,S_k^n$, then $\sigma = \tau = 0$. In the variable of* [1]*: for every $n \geq 3$, every $0 \leq k \leq n$ and every $q \in (0,1)$, equation* (4) *admits only the trivial solution.*

Proof. By Lemma 4.2, $S_k^n \neq 0$, and by Theorem 4.3, S_k^n and D_k^n are coprime. From $S_k^n \mid \sigma\,D_k^n$ we get $S_k^n \mid \sigma$, say $\sigma = g\,S_k^n$ with $g \in \mathbb{R}[x]$; substituting and cancelling the nonzero factor S_k^n in $g\,S_k^n\,D_k^n = \tau\,S_k^n$ gives $\tau = g\,D_k^n$. Suppose $g \neq 0$. By Theorem 4.1,

$$\sigma^2 - \Delta_\mu\,\tau^2 = g^2\big((S_k^n)^2 - \Delta_\mu(D_k^n)^2\big) = g^2\,c_{n,k}^2(x+\mu)^{2k}(x-\mu)^{2(n-k)},$$

whose right-hand side has degree exactly $2\deg g + 2n \geq 2n \geq 6$, since $c_{n,k} \neq 0$. But $\deg\sigma^2 \leq 4$ and $\deg(\Delta_\mu\tau^2) \leq 2 + 2 = 4$, so the left-hand side has degree at most 4. This contradiction shows $g = 0$, hence $\sigma = \tau = 0$. The q-form follows from Lemma 5.1. □

Nothing in the proof depends on n or k beyond the degree count, and no determinant, rank or resultant is computed. The source's $n = 3$ case, proved there through (6), is included.

Proposition 5.3 (sharpness in n)**.** *Let $n \leq 2$, $0 \leq k \leq n$ and $\mu > 1$. Then $(\sigma, \tau) = (S_k^n, D_k^n)$ is a nonzero solution of* (9)*: $S_k^n \neq 0$, $\deg S_k^n \leq 2$, $\deg D_k^n \leq 1$ and $S_k^n\,D_k^n = D_k^n\,S_k^n$.*

Proof. $S_k^n \neq 0$ by Lemma 4.2, and the equation holds trivially. For the degrees, give x and d weight one; then Δ_μ has weight two, consistent with $d^2 = \Delta_\mu$, each factor in (8) has weight at most one, and the product has weight at most n, so $\deg S_k^n \leq n \leq 2$ and $\deg D_k^n \leq n - 1 \leq 1$. (The formal proof simply expands the six cases $0 \leq k \leq n \leq 2$.) □

Corollary 5.4 (the dichotomy)**.** *For every $q \in (0,1)$ and every $0 \leq k \leq n$, equation* (4) *has a nonzero solution if and only if $n \leq 2$.*

Proof. Proposition 5.3 and Theorem 5.2, with Lemma 5.1. □

[1, Proposition 4.4] has the "if" direction by a dimension count and the "only if" direction at $n = 3$ only; the witness in Proposition 5.3 replaces the count and Theorem 5.2 supplies every $n \geq 3$.

6. The exact degenerate locus

The proof of Theorem 5.2 used $\mu > 1$ only through Lemma 4.2, which needs just $\mu \neq 0$ and $\mu^2 \neq 1$. So the theorem holds on that larger set, and the three excluded points are each genuinely excluded.

Theorem 6.1. *Let μ be real with $\mu \neq 0$ and $\mu^2 \neq 1$, let $n \geq 3$ and $0 \leq k \leq n$. If $\sigma, \tau \in \mathbb{R}[x]$ satisfy $\deg \sigma \leq 2$, $\deg \tau \leq 1$ and $\sigma\, D_k^n = \tau\, S_k^n$, then $\sigma = \tau = 0$. Equivalently, the $(n+2) \times 5$ coefficient matrix $M_{n,k}$ of (9) has full column rank 5 for every such μ.*

Proof. Word for word the proof of Theorem 5.2, with Theorem 4.3 and Lemma 4.2 in their general form. The matrix reformulation is the definition of $M_{n,k}$: its kernel is the space of solutions of (9). □

Proposition 6.2 (controls)**.** (i) (Non-vacuity.) *For $n \geq 3$, $0 \leq k \leq n$ and $\mu > 1$, D_k^n is not the zero polynomial; so (9) is a genuine constraint.*

(ii) (The degree bounds are load-bearing.) *For every n, $0 \leq k \leq n$ and $\mu > 1$, the pair (S_k^n, D_k^n) satisfies $\sigma\, D_k^n = \tau\, S_k^n$ with $\sigma \neq 0$; without the bounds $\deg \sigma \leq 2$, $\deg \tau \leq 1$ there is always a nonzero solution.*

(iii) (The excluded values of μ are excluded.) *With the degree bounds in force, the following are nonzero solutions of (9) in degree three:*

$$\begin{aligned} \mu = 1: &\quad (1 - x^2)\, D_0^3 = (-3 - 3x)\, S_0^3, \\ \mu = -1: &\quad (1 - x^2)\, D_0^3 = (-3 + 3x)\, S_0^3, \\ \mu = 0: &\quad 1 \cdot D_1^3 = (-1) \cdot S_1^3. \end{aligned}$$

Proof. (i) If $D_k^n = 0$ then $(\sigma, \tau) = (1, 0)$ would be a nonzero solution, contradicting Theorem 5.2. (ii) is Proposition 5.3 without the degree bookkeeping. (iii) From the explicit polynomials of Proposition 7.1 below: at $\mu = 1$, $8D_0^3 = -3(1-x)^2$ and $8S_0^3 = (1-x)^3$; at $\mu = -1$, $8D_0^3 = -3(1+x)^2$ and $8S_0^3 = (1+x)^3$; at $\mu = 0$, $\frac{8}{3}D_1^3 = -x^2$ and $\frac{8}{3}S_1^3 = x^2$. The three identities are then one-line checks, and $1 - x^2$ and 1 are nonzero of degree ≤ 2 while $-3 \mp 3x$ and -1 have degree ≤ 1. □

Corollary 6.3 (the exact locus)**.** *For every $n \geq 3$ and $0 \leq k \leq n$, the set of real μ at which (9) has a nonzero solution — equivalently, at which $\operatorname{rank} M_{n,k} < 5$ — is contained in $\{0, 1, -1\}$; and the union of these sets over all $n \geq 3$ and $0 \leq k \leq n$ is exactly $\{0, 1, -1\}$, each point being realised already by a cell with $n = 3$. In the variable of [1], $q \in (0, 1)$ is the special case $\mu > 1$.*

Proof. Theorem 6.1 and Proposition 6.2(iii). □

Remark 6.4. The containment is not an equality cell by cell: the factorisations (6) show that $M_{3,0}$ is singular only at $\mu = \pm 1$ and $M_{3,1}$ only at $\mu \in \{0, \pm 1\}$, and the computation of Remark 7.3 shows that most cells with $n \leq 7$ are regular at $\mu = 0$. The point $\mu = 1$ is $q = 1$, where $\mathbb{D}_x \to \frac{d}{dx}$ and $\mathbb{S}_x \to \mathrm{Id}$ and the classical ladder $(1 - x^2)\frac{d}{dx}\mathcal{B}_k^n = (2k - n - nx)\mathcal{B}_k^n$ displayed in [1, Proposition 4.4] is restored; the witness at $\mu = 1$ above is its case $(n, k) = (3, 0)$. The points $\mu = 0$ and $\mu = -1$ are not of the form (5) for any $q > 0$; $\mu = 0$ is the degeneracy that the factor μ^8 in $\det M_{3,1}$ records, and at $\mu = -1$ the lattice reflects, $S_0^3 = (1 + x)^3/8$.

7. The degree-three data of the source, and two computations

The proof of [1, Proposition 4.4] displays four polynomials. They follow from Definition 3.1 by expanding (8), and the expansion was carried out inside the formal development, so the definitions used here reproduce the printed data coefficient by coefficient.

Proposition 7.1. *For every real μ, as identities in $\mathbb{R}[x]$,*

$$\begin{aligned} 8\, D_0^3 &= (\mu^2 - 4) + 6\mu x - (4\mu^2 - 1)x^2, \\ 8\, S_0^3 &= (4 - 3\mu^2) + 3\mu(\mu^2 - 2)x + 3(2\mu^2 - 1)x^2 - \mu(4\mu^2 - 3)x^3, \\ \tfrac{8}{3}\, D_1^3 &= -\mu^2 - 2\mu x + (4\mu^2 - 1)x^2, \\ \tfrac{8}{3}\, S_1^3 &= \mu^2 - \mu(3\mu^2 - 2)x - (2\mu^2 - 1)x^2 + \mu(4\mu^2 - 3)x^3. \end{aligned}$$

These are the four displays in the proof of [1, Proposition 4.4]*, with* $c_{3,0} = \frac{1}{8}$ *and* $c_{3,1} = \frac{3}{8}$.

Proof. Expand $c_{3,k}(1 + \mu x + d)^k(1 - \mu x - d)^{3-k}$ in $\mathcal{A}_\mu$ using $d^2 = (\mu^2 - 1)(x^2 - 1)$ and read off the two components. □

The two remarks that follow report computations carried out in exact rational arithmetic (fraction-free Bareiss elimination over $\mathbb{Q}[\mu]$, in a short program of our own, under twenty seconds in total). They are *not* part of the formal development and nothing above depends on them; they are recorded because the first reproduces the printed data of [1] and the second is the $n \geq 4$ data the preprint says is unavailable.

Remark 7.2 (the factorisations (6); computation outside the formal development). Assembling $M_{3,k}$ from the polynomials of Proposition 7.1 in the column order of [1], the entries of $8M_{3,0}$ agree entry by entry with the matrix displayed in the proof of [1, Proposition 4.4], and the four determinants are exactly (6). We attempted to carry this determinant into the formal development as well, by a 5×5 cofactor expansion over $\mathbb{R}[\mu]$; the elaboration did not finish in fifteen minutes at a heartbeat budget of 4,000,000, with resident memory near 0.7 GB throughout, so the obstacle is elaboration time rather than memory. The attempt is parked, for two reasons: the load-bearing consequence, that the determinants do not vanish for $\mu > 1$, is the case $n = 3$ of Theorem 4.3 and costs nothing (Remark 4.4); and a revival should go through the resultant rather than through cofactors.

Remark 7.3 (determinantal divisors for $3 \leq n \leq 7$; computation outside the formal development). For $n \geq 4$ the matrix $M_{n,k}$ is not square, and the right invariant is its fifth determinantal divisor $\delta_{n,k} \in \mathbb{Q}[\mu]$, the greatest common divisor of its $\binom{n+2}{5}$ maximal minors, whose roots are exactly the μ at which $\operatorname{rank} M_{n,k} < 5$. Computed for $3 \leq n \leq 7$ and every $0 \leq k \leq n$, it is in every cell of the form $\delta_{n,k} = c\,\mu^{a(n,k)}(\mu^2 - 1)^{b(n,k)}$ with c a nonzero rational, with $b(n,k) = 6$ for $k \in \{0, n\}$ and $b(n,k) = 2$ otherwise, and with

n	$a(n,k),\ k = 0, \ldots, n$
3	0, 8, 8, 0
4	0, 5, 9, 5, 0
5	0, 0, 8, 8, 0, 0
6	0, 0, 5, 9, 5, 0, 0
7	0, 0, 0, 8, 8, 0, 0, 0

The row $n = 3$ is (6) and served as the control. The roots are $\{0, \pm 1\}$ or $\{\pm 1\}$ in every cell, as Theorem 6.1 requires for every n; the table is finite evidence for the pattern of the exponents, not for the theorem, which is proved. The integer sequence $a(n,k)$ read row by row is not in the OEIS (searched 3 September 2026).

8. Open questions

Question 8.1. Is $\delta_{n,k} = c\,\mu^{a(n,k)}(\mu^2 - 1)^{b(n,k)}$ for every $n \geq 3$, with $b(n,k) = 6$ for $k \in \{0, n\}$ and 2 otherwise, and what is $a(n,k)$? The b half should come from a multiplicity count at $x = \pm\mu$ in Theorem 4.1; the a half is the $\mu = 0$ degeneracy, about which the norm identity says nothing.

Question 8.2. The proof used q only through μ, that is, only through $x_\pm = \mu x \pm d$ with d^2 a quadratic polynomial in x. The quadratic (Wilson/Racah) lattice $x(s) = c_2 s^2 + c_1 s + c_0$ of [1, eq. (7)] has the same half-step structure. Does the same argument settle the ladder for the affine Bernstein basis there?

Conjecture 4.7 of [1], that for generic Askey–Wilson parameters and every $n \geq 2$ the affine connection coefficients $A_k^{\mathrm{sp}}(n,m)$ are not of the form $\kappa_k P_m(\xi_k)$ with $\{P_m\}$ orthogonal for a quasi-definite functional supported on pairwise distinct nodes ξ_k, concerns a different question and is not touched here.

9. Verification

Every theorem, proposition, lemma and corollary of Sections 3–7 is formalised and machine-checked in Lean 4 [12] with Mathlib [13], in four modules of about 860 lines. The polynomials S_k^n and D_k^n are defined as the two components of the pair $\mathcal{B}_k^n(\mu x + d)$ computed in the quadratic extension, exactly as in Definition 3.1; Propositions 3.2 and 3.3 are the formal justification that these are the operators of [1], and the four identities of Proposition 7.1 are the formal check against its printed data. The Bézout chain of Theorem 4.3 is constructive, with explicit cofactors, and the $n \leq 2$ cases of Proposition 5.3 are checked by expanding all six cells. The twenty-two formal statements quoted in Appendix A were each queried with `#print axioms`; every

one depends on exactly `propext`, `Classical.choice` and `Quot.sound`, the standard axioms of Mathlib, and on nothing else: there is no `sorry`, no `native_decide`, no external solver and no axiom of the development's own. Corollaries 5.4 and 6.3 are restatements combining the formal theorems listed under Theorems 5.2 and 6.1 and Propositions 5.3 and 6.2; they are not separate formal statements. Not formalised, and written out above as ordinary mathematics, are: the identification remark after Proposition 3.3 (a polynomial identity holding at infinitely many points holds identically); Remark 4.4; and the two computations of Remarks 7.2 and 7.3, which are labelled as such where they occur and on which nothing depends. The repository is private; repository reference to be added at submission.

Appendix A. The formal statements

Everything is in the namespace `LeanProblemSpec.QbernsteinLadder`, with `open Polynomial`, so `ℝ[X]` is the ring of real polynomials, `X` its variable and `C` the constant embedding. The definitions are

```
def Dl (μ : ℝ) : ℝ[X] := (C μ ^ 2 - 1) * (X ^ 2 - 1)
def qmul (μ : ℝ) (u v : ℝ[X] × ℝ[X]) : ℝ[X] × ℝ[X] :=
  (u.1 * v.1 + Dl μ * (u.2 * v.2), u.1 * v.2 + u.2 * v.1)
def qpow (μ : ℝ) (u : ℝ[X] × ℝ[X]) : ℕ → ℝ[X] × ℝ[X]
  | 0 => (1, 0)
  | m + 1 => qmul μ (qpow μ u m) u
def uPos (μ : ℝ) : ℝ[X] × ℝ[X] := (1 + C μ * X, 1)
def uNeg (μ : ℝ) : ℝ[X] × ℝ[X] := (1 - C μ * X, -1)
def bcoef (n k : ℕ) : ℝ := (n.choose k : ℝ) / 2 ^ n
def bern (n k : ℕ) : ℝ[X] := C (bcoef n k) * ((1 + X) ^ k * (1 - X) ^ (n - k))
def SDpair (n k : ℕ) (μ : ℝ) : ℝ[X] × ℝ[X] :=
  let p := qmul μ (qpow μ (uPos μ) k) (qpow μ (uNeg μ) (n - k))
  (C (bcoef n k) * p.1, C (bcoef n k) * p.2)
def Sp (n k : ℕ) (μ : ℝ) : ℝ[X] := (SDpair n k μ).1
def Dp (n k : ℕ) (μ : ℝ) : ℝ[X] := (SDpair n k μ).2
def muOf (q : ℝ) : ℝ := (Real.sqrt q + (Real.sqrt q)⁻¹) / 2
```

so that `Dl μ` is Δ_μ, `bern n k` is $\mathcal{B}^n_k$, `Sp n k μ` is S^n_k, `Dp n k μ` is D^n_k and `muOf q` is (5). The statements below are quoted verbatim from the source files (proofs omitted); in the last three, `μ : ℝ` is a section variable.

Proposition 3.2.

```
theorem eval_Sp_add_mul_Dp (n k : ℕ) (hk : k ≤ n) (μ x d : ℝ)
    (hd : d ^ 2 = (μ ^ 2 - 1) * (x ^ 2 - 1)) :
    (Sp n k μ).eval x + d * (Dp n k μ).eval x = (bern n k).eval (μ * x + d)
```

Proposition 3.3.

```
theorem lattice_half_step (Q z : ℝ) (hQ : Q ≠ 0) (hz : z ≠ 0) :
    ((Q * z + (Q * z)⁻¹) / 2 + (Q⁻¹ * z + (Q⁻¹ * z)⁻¹) / 2) / 2
      = ((Q + Q⁻¹) / 2) * ((z + z⁻¹) / 2)

theorem sq_half_gap (Q z : ℝ) (hQ : Q ≠ 0) (hz : z ≠ 0) :
    (((Q * z + (Q * z)⁻¹) / 2 - (Q⁻¹ * z + (Q⁻¹ * z)⁻¹) / 2) / 2) ^ 2
      = (((Q + Q⁻¹) / 2) ^ 2 - 1) * (((z + z⁻¹) / 2) ^ 2 - 1)

theorem bridge_S (n k : ℕ) (hk : k ≤ n) (Q z : ℝ) (hQ : Q ≠ 0) (hz : z ≠ 0) :
    ((bern n k).eval ((Q * z + (Q * z)⁻¹) / 2)
        + (bern n k).eval ((Q⁻¹ * z + (Q⁻¹ * z)⁻¹) / 2)) / 2
      = (Sp n k ((Q + Q⁻¹) / 2)).eval ((z + z⁻¹) / 2)

theorem bridge_D (n k : ℕ) (hk : k ≤ n) (Q z : ℝ) (hQ : Q ≠ 0) (hz : z ≠ 0) :
    (bern n k).eval ((Q * z + (Q * z)⁻¹) / 2)
        - (bern n k).eval ((Q⁻¹ * z + (Q⁻¹ * z)⁻¹) / 2)
      = ((Q * z + (Q * z)⁻¹) / 2 - (Q⁻¹ * z + (Q⁻¹ * z)⁻¹) / 2)
          * (Dp n k ((Q + Q⁻¹) / 2)).eval ((z + z⁻¹) / 2)
```

Theorem 4.1.

```
theorem norm_SD (n k : ℕ) (hk : k ≤ n) (μ : ℝ) :
    (Sp n k μ) ^ 2 - Dl μ * (Dp n k μ) ^ 2
      = C (bcoef n k) ^ 2 * ((X + C μ) ^ (2 * k) * (X - C μ) ^ (2 * (n - k)))
```

Theorem 4.3.

```
theorem coprime_Sp_Dp (n k : ℕ) (hk : k ≤ n) (μ : ℝ) (hμ : 1 < μ) :
    IsCoprime (Sp n k μ) (Dp n k μ)

theorem coprime_Sp_Dp_gen (n k : ℕ) (hk : k ≤ n) (h0 : μ ≠ 0) (hsq : μ ^ 2 ≠ 1) :
    IsCoprime (Sp n k μ) (Dp n k μ)
```

Lemma 5.1 and Theorem 5.2.

```
theorem one_lt_muOf (q : ℝ) (h0 : 0 < q) (h1 : q < 1) : 1 < muOf q

theorem ladder_trivial (n k : ℕ) (hk : k ≤ n) (hn : 3 ≤ n) (μ : ℝ) (hμ : 1 < μ)
    (σ τ : ℝ[X]) (hσ : σ.degree ≤ 2) (hτ : τ.degree ≤ 1)
    (h : σ * Dp n k μ = τ * Sp n k μ) : σ = 0 ∧ τ = 0

theorem conj_ladder_q (n k : ℕ) (hk : k ≤ n) (hn : 3 ≤ n) (q : ℝ) (hq0 : 0 < q) (hq1 : q < 1)
    (σ τ : ℝ[X]) (hσ : σ.degree ≤ 2) (hτ : τ.degree ≤ 1)
    (h : σ * Dp n k (muOf q) = τ * Sp n k (muOf q)) : σ = 0 ∧ τ = 0
```

Proposition 5.3.

```
theorem ladder_nontrivial_of_le_two (n k : ℕ) (hk : k ≤ n) (hn : n ≤ 2) (μ : ℝ) (hμ : 1 < μ) :
    ∃ σ τ : ℝ[X], σ ≠ 0 ∧ σ.degree ≤ 2 ∧ τ.degree ≤ 1 ∧ σ * Dp n k μ = τ * Sp n k μ
```

Theorem 6.1.

```
theorem ladder_trivial_gen (n k : ℕ) (hk : k ≤ n) (hn : 3 ≤ n) (h0 : μ ≠ 0) (hsq : μ ^ 2 ≠ 1)
    (σ τ : ℝ[X]) (hσ : σ.degree ≤ 2) (hτ : τ.degree ≤ 1)
    (h : σ * Dp n k μ = τ * Sp n k μ) : σ = 0 ∧ τ = 0
```

Proposition 6.2.

```
theorem Dp_ne_zero (n k : ℕ) (hk : k ≤ n) (hn : 3 ≤ n) (μ : ℝ) (hμ : 1 < μ) :
    Dp n k μ ≠ 0

theorem ladder_nontrivial_without_degree_bounds (n k : ℕ) (hk : k ≤ n) (μ : ℝ) (hμ : 1 < μ) :
    ∃ σ τ : ℝ[X], σ ≠ 0 ∧ σ * Dp n k μ = τ * Sp n k μ

theorem ladder_nontrivial_at_mu_one :
    (1 - X ^ 2 : ℝ[X]) * Dp 3 0 1 = (-3 - 3 * X : ℝ[X]) * Sp 3 0 1
      ∧ (1 - X ^ 2 : ℝ[X]) ≠ 0 ∧ (1 - X ^ 2 : ℝ[X]).degree ≤ 2
      ∧ (-3 - 3 * X : ℝ[X]).degree ≤ 1

theorem ladder_nontrivial_at_mu_zero :
    (1 : ℝ[X]) * Dp 3 1 0 = (-1 : ℝ[X]) * Sp 3 1 0
      ∧ (1 : ℝ[X]) ≠ 0 ∧ (1 : ℝ[X]).degree ≤ 2 ∧ (-1 : ℝ[X]).degree ≤ 1

theorem ladder_nontrivial_at_mu_neg_one :
    (1 - X ^ 2 : ℝ[X]) * Dp 3 0 (-1) = (-3 + 3 * X : ℝ[X]) * Sp 3 0 (-1)
      ∧ (1 - X ^ 2 : ℝ[X]) ≠ 0 ∧ (1 - X ^ 2 : ℝ[X]).degree ≤ 2
      ∧ (-3 + 3 * X : ℝ[X]).degree ≤ 1
```

Proposition 7.1.

```
theorem source_D30 (μ : ℝ) :
    Dp 3 0 μ = C (bcoef 3 0) * ((C μ ^ 2 - 4) + 6 * C μ * X - (4 * C μ ^ 2 - 1) * X ^ 2)

theorem source_S30 (μ : ℝ) :
    Sp 3 0 μ = C (bcoef 3 0) * ((4 - 3 * C μ ^ 2) + 3 * C μ * (C μ ^ 2 - 2) * X
      + 3 * (2 * C μ ^ 2 - 1) * X ^ 2 - C μ * (4 * C μ ^ 2 - 3) * X ^ 3)

theorem source_D31 (μ : ℝ) :
    Dp 3 1 μ = C (bcoef 3 1) * (-(C μ ^ 2) - 2 * C μ * X + (4 * C μ ^ 2 - 1) * X ^ 2)

theorem source_S31 (μ : ℝ) :
    Sp 3 1 μ = C (bcoef 3 1) * (C μ ^ 2 - C μ * (3 * C μ ^ 2 - 2) * X
      - (2 * C μ ^ 2 - 1) * X ^ 2 + C μ * (4 * C μ ^ 2 - 3) * X ^ 3)
```

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