SHARP CONSTANTS FOR PTOLEMAIC NEGATIVE TYPE: THE p-ANCHOR LADDER, AN IMPOSSIBILITY WINDOW ABOVE $q = 1$, AND THE BERNSTEIN CUBE INEQUALITY BEHIND $q(6)$ AND $q(7)$

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Abstract. Baker, Huh, Kummer and Lorscheid attach to a matroid M a threshold $q(M)$ and conjecture exact values for $q(n) = q(U_{2,n})$. Ercan has recently settled the first two open cases, $q(6) = \log_2 \frac{9}{4}$ and $q(7) = 1$, and his proof has exactly one computer-assisted step: four explicit polynomials H_x, H_y, N, Δ in six real variables satisfy $H_x \geq 3$, $H_y \geq 3$, $N \geq 0$ and $\Delta \geq 0$ on $[0,1]^6$. He certifies this with a stored list of 24,305 exact rational Bernstein coefficients. We give a proof of that inequality in which nothing is stored: the certified object is *built* from the defining equations by exact integer polynomial arithmetic inside the proof, so no coefficient list is trusted and there is no transcription step between the printed formulas and the checked data. We then prove three things the source does not state. First, a five-variable form: H_x, H_y and N see the four variables h, e, f, d only through the three products he, hf, hd, so those three inequalities are equivalent to five-variable ones with certificates of 213, 213 and 226 terms in place of 699, 699 and 782. Second, that the reduction stops exactly at Δ, which does see h separately: writing g, k, l for the three products once they are decoupled from h, the inequality $\Delta \geq 0$ fails on the genuinely larger region where h is free, each of the three constraints $g \leq h$, $k \leq h$, $l \leq h$ is individually necessary, and dropping any one alone makes Δ equal $-305/4$ at an explicit rational point. Third, the exact ranges $\min H_x = \min H_y = 3$, $\min N = 0$, $\max H_x = \max H_y = \max N = 27$, all attained, and a sandwich $[\,3125/16,\ 11807/54\,]$ for $\sup \Delta$.

We then identify the constant that governs the whole family of sign patterns behind these values. For each anchor count p put $\kappa_q^{(p)} = \big(p\,2^q - (p+1)\big)/(p+1)$, so that $\kappa^{(1)}$ and $\kappa^{(2)}$ are the two constants that appear in the literature. We prove that $\kappa_q^{(p)}$ reaches the copositivity threshold $1/(r-1)$ at exactly the exponent at which the complete split graph $CS(p,r)$ stops having q-negative type; that the exponents so produced reproduce $P(3) = 2$, $P(4) = \log_2 3$, $P(5) = \log_2 \frac{9}{4}$, $P(6) = 1$ and the two conjectural values $2\log_2 \frac{4}{3}$ and $\log_2 \frac{5}{3}$; and that the p-anchor partial correlation of $CS(p,2)$ equals $-\kappa_q^{(p)}$ exactly, for every p and every exponent, so that no smaller constant is available. We then prove a negative result about the estimate whose absence is what leaves the next two cases open. The three-anchor partial correlation of $CS(2,3)$ based at an independent vertex is $(3 - 2\cdot 2^q)/\big(2(3-2^q)\big)$; it meets $-\kappa_q^{(3)}$ exactly at $q = 1$; and at $q = \log_2 \frac{9}{4}$ it equals -1 against $-\frac{11}{16}$. So no three-anchor analogue of the four-point correlation estimate can hold above $q = 1$, and any such estimate is confined to $0 < q \leq 1$ — which is where both remaining targets lie. Whether it holds there is open, and so is the conjecture at $n \geq 8$. Finally we locate exactly where the six-variable relaxation stops being available on the way down: at the single point $x = y = 1$, $e = f = d = 0$ one has $\Delta = 64h(h+1)$ while $H_x = H_y = 12$ and $N = 4$ for every h, so $\Delta \geq 0$ fails there at *every* exponent $0 < q < 1$ and the window $h \geq 0$ is sharp. The measured state of the certificate route below $q = 1$ is reported separately and outside the results claimed here. Every theorem stated here is machine-checked in Lean 4.

1. Introduction

1.1. **Where the problem comes from.** Baker, Huh, Kummer and Lorscheid [3] attach to a matroid M the threshold

$$q(M) \;=\; \sup\big\{q > 0 \;:\; \mathrm{Gr}^{\mathrm{w}}_M(\mathbb{T}_q) \subseteq \mathbb{P}\mathrm{L}_M\big\},$$

comparing weak thin Schubert cells over the generalized triangular hyperfield $\mathbb{T}_q$ with the space of Lorentzian polynomials supported on M, and they write $q(n) = q(U_{2,n})$ for the rank-two uniform

matroid. They prove $q(4) = 2$ and $q(5) = \log_2 3$, and conjecture that for every $n \geq 4$

$$q(n) \;=\; \begin{cases} 2\log_2\!\left(\frac{n}{n-2}\right), & n \text{ even}, \\ \log_2\!\left(\frac{n+1}{n-3}\right), & n \text{ odd}. \end{cases} \tag{1}$$

The first two values (1) leaves open are $q(6) = \log_2 \frac{9}{4}$ and $q(7) = 1$.

The bridge to metric geometry is also theirs. A metric space (X, d) is *Ptolemaic* if $d(x,y)d(z,w) \leq d(x,z)d(y,w) + d(x,w)d(y,z)$ for all $x, y, z, w \in X$, and it has *q-negative type*, in Schoenberg's formulation [5], if $\sum_{x,y} z_x z_y\, d(x,y)^q \leq 0$ for every zero-sum coefficient vector z. Writing

$$P(m) \;=\; \sup\{\, q > 0 \;:\; \text{every Ptolemaic metric on exactly } m \text{ points has } q\text{-negative type} \,\},$$

Baker, Huh, Kummer and Lorscheid prove the exact correspondence $q(n) = P(n-1)$ [3, Prop. 7.5].

Ercan [1] proves $P(5) = \log_2 \frac{9}{4}$ and $P(6) = 1$, hence (1) at $n = 6$ and $n = 7$. The five-point half of that argument separates zero-sum coefficient vectors by sign pattern; the $2 + 3$ pattern needs a sharp four-point partial-correlation bound with the constant

$$\kappa_q \;=\; \frac{2^{q+1} - 3}{3},$$

and it is that bound, and only that bound, which is computer-assisted. In the words of the source's introduction: "The only computer-assisted component is an exact rational Bernstein certificate in the proof of the four-point correlation theorem. … All remaining arguments are analytic."

1.2. **The finite statement.** The appendix of [1] reduces the correlation bound on the double-geodesic face to six real variables. For $(x, y, h, e, f, d) \in \mathbb{R}^6$ put

$$\begin{aligned} A_x &= 1 + x^2 + hex, & A_y &= 1 + y^2 + hfy, \\ P &= 2x + he + y\,(xhf - hd), & Q &= 2y + hf + x\,(yhe - hd), \\ H_x &= 4A_xA_y - P^2, & H_y &= 4A_xA_y - Q^2, \\ N &= PQ + 2hd\,A_xA_y, & \Delta &= \left(\tfrac{1+2h}{3}\right)^2 H_xH_y - N^2. \end{aligned} \tag{2}$$

Write $\mathcal{C} = [0,1]^6$ for the closed unit cube in these coordinates.

Theorem 1.1 ([1, Bernstein cube inequality, Thm. A.3])**.** *On $\mathcal{C}$ one has $H_x \geq 3$, $H_y \geq 3$, $N \geq 0$ and $\Delta \geq 0$; consequently*

$$-\frac{1+2h}{3} \;\leq\; -\frac{N}{\sqrt{H_xH_y}} \;\leq\; 0.$$

The chart behind Theorem 1.1 is metric. With $h = h_q = 2^q - 2$, with $x = o^{q/2}$ and $y = (p/o)^{q/2}$ for the two normalised branch lengths $p \leq o \leq 1$ of an inverted three-leaf star, and with e, f, d the three *defect ratios* $\eta_q(o)/h_q$, $\eta_q(p/o)/h_q$, $\eta_q(p)/h_q$ built from $\eta_q(t) = \big((1+t)^q - 1 - t^q\big)/t^{q/2}$, one has $(x, y, h, e, f, d) \in \mathcal{C}$ by the source's sharp two-summand estimate. The constant $(1 + 2h_q)/3$ is then exactly κ_q; at $q = \log_2 \frac{9}{4}$ it is $\frac{1}{2}$, the copositivity threshold on which the five-point argument of [1] runs.

1.3. **What is proved here.** Theorem 1.1 is a statement about four explicit integer polynomials on a box. The source proves it by exhibiting 24,305 nonnegative rational Bernstein coefficients in an ancillary file [2], checked by a Python script. Two things about that route are worth improving on. The stored list is trusted data: whoever checks it must also believe that the file corresponds to the printed equations; this is addressed in Sections 4–5. And the list is large, because the tensor-product Bernstein basis at multidegree $(4, 4, 6, 4, 4, 4)$ has 21,875 elements, of which the certificate for Δ uses almost all; this is addressed, for three of the four polynomials, in Section 6.

A proof with nothing stored (Sections 4–5). We prove Theorem 1.1 in a form where the certified object is constructed inside the proof. The device is homogenisation: writing u_j for t_j and v_j for $1 - t_j$, a polynomial of multidegree $\mathbf{n}$ has a homogenisation $\widehat{F}(u, v)$ whose monomial coefficients

are the Bernstein coefficients up to positive binomial factors, and which satisfies $\widehat{F}(t,1-t)=F(t)$. Homogenisation is multiplicative, so $\widehat{F}$ can be *built* from the defining equations (2) by the same additions and multiplications the source writes down — no basis conversion, no subdivision, no coefficient list. What remains to check is that a term list of explicit integers has no negative entry, and what remains to prove is an identity between the built object and the closed form of (2). Theorem 5.1 is the resulting statement.

A five-variable form, and its exact limit (Section 6). Three of the four polynomials of (2) involve h,e,f,d only through the three products $g=he$, $k=hf$, $l=hd$ — an observation the source does not make, although it is visible in (2) once one looks for it. Since the substitution $(x,y,h,e,f,d)\mapsto(x,y,he,hf,hd)$ maps $\mathcal{C}$ *onto* $[0,1]^5$, the three inequalities $H_x\ge 3$, $H_y\ge 3$, $N\ge 0$ on $\mathcal{C}$ are equivalent to three inequalities on $[0,1]^5$ (Theorem 6.1), whose certificates have 213, 213 and 226 terms against 699, 699 and 782 — a basis of 243 elements in place of 729, 729 and 972. The fourth polynomial is different: Δ carries the factor $\big((1+2h)/3\big)^2$ and so sees h separately, and on the genuinely larger region where h is decoupled from g,k,l the inequality fails. We locate the obstruction exactly: each of $g\le h$, $k\le h$, $l\le h$ is individually necessary, and dropping any one of them alone already gives $\Delta=-305/4$ at an explicit rational point (Theorem 6.2). So the extra variable of the source's parametrisation is used, and used only, by Δ.

Exact ranges (Section 7). On $\mathcal{C}$,

$$\min H_x=\min H_y=3,\quad \min N=0,\quad \max H_x=\max H_y=\max N=27,$$

each attained at an explicit corner, and $\sup\Delta\in[\,3125/16,\,11807/54\,]$ (Theorem 7.1). The source states no bounds on these polynomials at all; the numbers 3, 27, 1/12, 4/9, 11807/54 occur in its appendix table as extreme Bernstein *coefficients*, never as statements about the functions.

The chart, and what is not claimed (Section 8). We also verify the metric specialisation of the source's projective star formula — $\mathcal{H}_0=x^2H_x$, $\mathcal{H}_1=x^4y^2H_y$, $\mathcal{N}=-x^3yN$, three polynomial identities — and compose them with Theorem 5.1 to obtain the double-geodesic correlation bound of [1] *on the chart* (Theorem 8.2). We do not obtain $q(6)$ or $q(7)$. The star formula itself, which identifies $\mathcal{N}/\sqrt{\mathcal{H}_0\mathcal{H}_1}$ with a partial correlation through a Schoenberg-matrix Schur complement, is not verified here, nor is the parametrisation of the crossing Ptolemy face that produces the chart; and neither is the analytic half of [1] — the sharp two-summand estimate that puts the defect ratios in $[0,1]$, the endpoint reduction of the Ptolemaic geodesic chamber, the copositivity identity, the coefficient transport under metric inversion — nor the correspondence $q(n)=P(n-1)$ of [3]. What is verified is the finite statement to which the computer-assisted step of [1] reduces, together with the three additions above.

Section 9 records the negative controls that make the certificate route non-vacuous.

1.4. **The ladder above the cube, and where it stops.** Theorem 1.1 is one member of a family. The five-point argument of [1] separates zero-sum coefficient vectors by sign pattern, and a pattern with p positive and r negative coefficients is handled by conditioning the Schoenberg matrix on $p-1$ *anchors* and bounding the resulting 2×2 partial correlation from below by a constant depending on p and q; a copositivity criterion then closes the pattern as soon as that constant is at most $1/(r-1)$. For $p=1$ and $p=2$ the two constants appear in [1] as separate theorems, the second of them being the κ_q of Corollary 5.2. The closing section of that paper records that the next two conjectural values, $P(7)=2\log_2\frac{4}{3}$ and $P(8)=\log_2\frac{5}{3}$, reduce to the $3+4$ and $3+5$ patterns, and that these "require an analogue of the four-point correlation estimate for sign patterns with three positive coefficients." The second half of this paper is about that analogue: not a proof of it, but a determination of its constant and of the exponent range in which it can possibly be stated.

The constant (Section 11). Put $\kappa_q^{(p)}=\big(p\,2^q-(p+1)\big)/(p+1)$, which is $h_q/2$ at $p=1$ and κ_q at $p=2$. Theorem 11.1 shows that $\kappa_q^{(p)}$ equals the copositivity threshold $1/(r-1)$ at exactly the exponent at which the complete split graph $CS(p,r)$ — a clique on p vertices joined to an independent set of r vertices, with the two distance values 1 and 2 — stops having q-negative type. The exponents

produced this way (Theorem 11.2) are 2, $\log_2 3$, $\log_2 \frac{9}{4}$, 1, $2\log_2 \frac{4}{3}$, $\log_2 \frac{5}{3}$ on $3, 4, 5, 6, 7, 8$ points: the four known values of P and the two conjectural ones. And the constant is attained, for every p at once: on $CS(p, 2)$, based at a clique vertex and conditioned on the remaining clique vertices, the partial correlation of the two independent vertices is exactly $-\kappa_q^{(p)}$ (Theorem 11.3).

An impossibility window (Section 12). The four-point estimate of [1] is stated on $1 \le q \le \log_2 3$. Its three-anchor analogue cannot be. On the five-point space $CS(2, 3)$, based at an independent vertex and conditioned on the two clique vertices, the partial correlation is exactly $(3-2t)/\big(2(3-t)\big)$ with $t = 2^q$; it crosses $-\kappa_q^{(3)}$ precisely at $t = 2$, that is at $q = 1$; and at $q = \log_2 \frac{9}{4}$ it equals -1 while $-\kappa_q^{(3)} = -\frac{11}{16}$ (Theorem 12.1). So the analogue fails above $q = 1$ — at $q = \log_2 \frac{9}{4}$ by exact evaluation, and on all of $1 < q < \log_2 3$ once the sign of one rational function is read off — and any correct statement of it is confined to $0 < q \le 1$ (Corollary 12.2), which happens to be exactly where the two remaining targets live. This is a proved negative, obtained from an object [1] never forms: it uses its own extremal five-point space, but conditioned on a pair of points rather than on one. It says nothing about whether the analogue holds on $0 < q \le 1$, and we do not claim that it does; the conjecture at $n \ge 8$ remains open.

The supporting layer (Section 10). Section 10 sets up the Schoenberg matrix, the Schur complement in a witness form that never inverts the anchor block, Schoenberg's identity, the p-anchor decomposition of the gap that turns a sign pattern into a copositivity question, and the invariances under metric inversion and under change of base.

The window in h, and the measured state (Section 13). Below $q = 1$ the constant $h_q = 2^q - 2$ turns negative and the chart behind Theorem 1.1 is no longer available. Theorem 13.1 says exactly how it fails: at the single point $x = y = 1$, $e = f = d = 0$ the first three polynomials of (2) are the constants $12, 12, 4$ for every h, while $\Delta = 64h(h+1)$; so $\Delta \ge 0$ is false there for every $h \in (-1, 0)$ — that is at every exponent $0 < q < 1$, not merely at the two of interest — and is true for every $h \ge 0$, with equality at $h = 0$. The window is therefore sharp on both sides, and the failure is a failure of the relaxation and not of the geometry, since the point is the image of no metric. The rest of Section 13 reports the state of the certificate route below $q = 1$, as numerical experiments and as statements of the certificate infrastructure, neither being among the results claimed here: the relaxation needs a coupling that pinches at the balanced configuration, such a coupling exists numerically, the reduction to endpoints used by [1] fails at four anchors and above, and the certificate the whole route would need has now been priced and is out of reach at four leaves.

Section 15 records the measured cost of the development, and Section 16 says precisely what the formal verification covers and where exhaustive computation enters.

Numbering. Sections 1–9 are those of the first version of this paper, changed only in this introduction and by one added paragraph in Section 2; no numbered statement in them was added, removed or reordered, so the numbers of all their statements are stable. Sections 10–13 are new; the three closing sections, previously numbered 10–12, are now Sections 14–16. Credit where the earlier version already placed it: the Lean formalisation [4] of the four-point Ptolemaic snowflake theorem and of $q(5) = \log_2 3$ is not only prior art for the first half of this paper but an *input* to the second, since $P(4) \ge q$ is exactly what makes the diagonal entries of the three-anchor Schur complement nonnegative.

2. Provenance and prior art

The mathematics of Theorem 1.1 is Ercan's. This paper contributes a proof of it that trusts no stored data, together with the statements of Sections 6 and 7, and those of Sections 11 and 12, that the source does not make. The following is what a literature search turned up, recorded so that a reader can judge the claim rather than repeat the search.

A Lean culture already surrounds this conjecture, and the case one step below was already done. Baker, Huh, Kummer and Lorscheid state in a footnote of [3]: "With the help of David Renshaw and Claude Code, we have formalized the proofs of [the four-point Ptolemaic snowflake theorem]

and [the corollary $q(5) = \log_2 3$] in Lean. The Lean code can be found at `https://github.com/icarm/FourPointPtolemaic.git`". That development [4] is prior art in the strict sense: it is a Lean 4 formalisation of the four-point Ptolemaic snowflake theorem and of $q(5) = \log_2 3$, i.e. of the case immediately below the one treated here, and its last commit predates the appearance of [1] by five weeks. It is three Lean files, of which the mathematical one, `Ptolemaic.lean`, is 2186 lines; it invokes the `nlinarith` arithmetic tactic 68 times, 65 of them in that file; it uses no compiled evaluation (`native_decide` does not occur) and no `sorry`; and the string "Bernstein" does not occur in it. It contains nothing about $q(6)$, $q(7)$, or the cube inequality. The honest summary is that $n = 5$ was done and $n = 6, 7$ was not; this paper closes the finite half of $n = 6, 7$ and leaves the analytic half open.

On the subject. The literature on $q(n)$ consists of three papers: the two-part work of Baker, Huh, Kummer and Lorscheid (arXiv:2508.02907 and [3], of which the latter is the part relevant here), and [1]. A search of arXiv for "Ptolemaic negative type" returns [1] alone; "triangular hyperfield" together with "negative type" likewise; "triangular hyperfields" returns four papers, of which three are the ones just named. As of 2026-08-29 the source is nine days old, still at version 1, and has no citations recorded on OpenAlex or Semantic Scholar.

On the method. Certifying a polynomial inequality on a box from nonnegative Bernstein coefficients is old, and the real prior art is the PVS formalisation of Muñoz and Narkawicz [6], which represents multivariate Bernstein polynomials, verifies an algorithm bounding the minimum and maximum of a polynomial, and builds proof strategies for polynomial global optimisation on top of it. What is specific here is the combination of that idea with homogenisation, which removes both the basis-conversion layer and the stored coefficient array (Section 4); a search for "Bernstein" together with "Lean 4", and for "Bernstein certificate" together with "formally verified", returned nothing on arXiv.

On the ladder. The bottom two rungs of Section 11 are published: $\kappa^{(1)}$ is the constant in the one-anchor theorem [1, Thm. 3.1] and $\kappa^{(2)}$ is its κ_q, and the exponents $t_{p,r}$ of (11) agree, under $a = p + 1$, $b = r$, with the extremal ratio $ab/\big((a-1)(b-1)\big)$ that appears in the upper-bound argument [3, Prop. 6.1]. The $p = 1$ rung is in the literature outside [1] as well: the metric space $CS(1, r)$ is the complete bipartite graph $K_{1,r}$, and $\log_2 t_{1,r} = \log_2 \frac{2r}{r-1}$ is the case $m = 1$ of the formula $\wp(K_{n,m}) = \log_2 \frac{2nm}{2nm-n-m}$ for the supremal p-negative type of a complete bipartite graph, proved by Sánchez [7, Thm. 3.1 of the arXiv version] and attributed there, in the case $m = 1$, to earlier work of Doust and Weston. What is not in the literature, by the following searches, is the p-indexed family itself, the identity of Theorem 11.1 that selects it, the uniform attainment of Theorem 11.3, and the three-anchor computation of Section 12. A search of arXiv for "complete split graph" together with "negative type" returns [1] alone, as does "Ptolemaic" together with "negative type"; "supremal p-negative type" returns exactly two papers, both on the general theory of negative type for finite metric spaces — Li and Weston [8], on lower bounds for the supremal strict p-negative type, and the paper [7] just cited, which expresses the supremal p-negative type of a finite metric space through its distance matrix and evaluates it on the complete bipartite graphs. Neither contains a split-graph ladder: the word "split" occurs in neither e-print, and neither writes down a p-indexed family of constants. Finally "generalized roundness" together with "split graph" returns nothing. As before, these negatives should be read as "not found", not as "new".

3. Preliminaries

Throughout, $\mathcal{C} = [0, 1]^6$ carries coordinates $\mathbf{t} = (t_1, \dots, t_6) = (x, y, h, e, f, d)$, and A_x, A_y, P, Q, H_x, H_y, N, Δ are the polynomials of (2). All of them lie in $\mathbb{Z}[\mathbf{t}]$ except Δ, which lies in $\frac{1}{9}\mathbb{Z}[\mathbf{t}]$; we therefore work with 9Δ throughout. The monomial counts are 12, 12, 17 and 343 for H_x, H_y, N and 9Δ, so every identity between a certificate and its closed form is a small one.

We also use the five-variable forms obtained by replacing the products he, hf, hd by independent variables g, k, l:

$$\begin{aligned} \tilde{A}_x &= 1 + x^2 + gx, & \tilde{A}_y &= 1 + y^2 + ky, \\ \tilde{P} &= 2x + g + y\,(xk - l), & \tilde{Q} &= 2y + k + x\,(yg - l), \\ \tilde{H}_x &= 4\tilde{A}_x\tilde{A}_y - \tilde{P}^2, & \tilde{H}_y &= 4\tilde{A}_x\tilde{A}_y - \tilde{Q}^2, \\ \tilde{N} &= \tilde{P}\tilde{Q} + 2l\,\tilde{A}_x\tilde{A}_y, & \tilde{\Delta} &= \left(\tfrac{1+2h}{3}\right)^2 \tilde{H}_x\tilde{H}_y - \tilde{N}^2, \end{aligned} \tag{3}$$

so that $\tilde{\Delta}$ retains h as a sixth, now independent, variable. Substituting $(g, k, l) = (he, hf, hd)$ recovers (2):

Lemma 3.1. *As identities of polynomials in x, y, h, e, f, d,*

$$H_x = \tilde{H}_x(x, y, he, hf, hd), \qquad H_y = \tilde{H}_y(x, y, he, hf, hd), \qquad N = \tilde{N}(x, y, he, hf, hd),$$

and $\Delta = \tilde{\Delta}(x, y, h, he, hf, hd)$.

Proof. Expand both sides; the four identities hold coefficientwise. □

The point of Lemma 3.1 is the shape of the substitution, not its verification: H_x, H_y and N factor through the map $(h, e, f, d) \mapsto (he, hf, hd)$ and Δ does not, since the leading factor $\big((1+2h)/3\big)^2$ retains h. This is what Section 6 exploits and what Theorem 6.2 shows to be the exact limit of the exploitation.

4. Certificates without conversion and without storage

4.1. **Homogenisation.** Fix a multidegree $\mathbf{n} = (n_1, \dots, n_6) \in \mathbb{N}^6$ and let $F \in \mathbb{Z}[\mathbf{t}]$ satisfy $\deg_{t_j} F \le n_j$ for each j. Introduce twelve variables $u_1, v_1, \dots, u_6, v_6$ and set

$$\widehat{F}^{\,\mathbf{n}}(u, v) \;=\; \Big(\prod_{j=1}^{6} (u_j + v_j)^{n_j}\Big)\, F\Big(\frac{u_1}{u_1 + v_1}, \dots, \frac{u_6}{u_6 + v_6}\Big). \tag{4}$$

This is a polynomial, homogeneous of degree n_j in each pair (u_j, v_j). Two properties are all we need.

Proposition 4.1. *Let F and $\mathbf{n}$ be as above.*

(i) $\widehat{F}^{\,\mathbf{n}}(\mathbf{t}, \mathbf{1} - \mathbf{t}) = F(\mathbf{t})$ *identically.*

(ii) *Let $B^{\mathbf{n}}_{\mathbf{i}}(\mathbf{t}) = \prod_j \binom{n_j}{i_j} t_j^{i_j}(1 - t_j)^{n_j - i_j}$ be the tensor-product Bernstein basis and let $b_{\mathbf{i}}(F; \mathbf{n})$ be defined by $F = \sum_{\mathbf{i} \le \mathbf{n}} b_{\mathbf{i}}(F; \mathbf{n}) B^{\mathbf{n}}_{\mathbf{i}}$. Then*

$$\widehat{F}^{\,\mathbf{n}}(u, v) \;=\; \sum_{\mathbf{i} \le \mathbf{n}} b_{\mathbf{i}}(F; \mathbf{n}) \prod_{j=1}^{6} \binom{n_j}{i_j} u_j^{i_j} v_j^{n_j - i_j}.$$

(iii) *Homogenisation is multiplicative: $\widehat{(FG)}^{\,\mathbf{m}+\mathbf{n}} = \widehat{F}^{\,\mathbf{m}} \cdot \widehat{G}^{\,\mathbf{n}}$, and raising the multidegree of a summand by $\mathbf{k}$ multiplies its homogenisation by $\prod_j (u_j + v_j)^{k_j}$, which is 1 at $(u, v) = (\mathbf{t}, \mathbf{1} - \mathbf{t})$.*

Proof. (i) is the substitution $u_j + v_j = 1$ in (4). For (ii), substitute $t_j = u_j/(u_j + v_j)$ into $F = \sum_{\mathbf{i}} b_{\mathbf{i}} B^{\mathbf{n}}_{\mathbf{i}}$ and multiply by $\prod_j (u_j + v_j)^{n_j}$: each factor $t_j^{i_j}(1 - t_j)^{n_j - i_j}$ becomes $u_j^{i_j} v_j^{n_j - i_j}/(u_j + v_j)^{n_j}$, and the prefactor clears the denominators exactly. (iii) is immediate from (4), both parts. □

Part (ii) says that "all Bernstein coefficients of F at multidegree $\mathbf{n}$ are nonnegative" and "all monomial coefficients of $\widehat{F}^{\,\mathbf{n}}$ are nonnegative" are the same statement, the binomials being positive. Part (iii) says that $\widehat{F}^{\,\mathbf{n}}$ can be assembled from the homogenisations of the pieces F is built from, using the same arithmetic. Together they replace the basis conversion $b_{\mathbf{i}} = \sum_{\alpha \le \mathbf{i}} c_\alpha \prod_j \binom{i_j}{\alpha_j} / \binom{n_j}{\alpha_j}$ that a stored certificate requires: one never converts, because one never leaves the homogeneous variables.

4.2. **What the proofs actually use.** It is worth isolating the logical content, because it is smaller than the discussion above. The proofs of Sections 5–7 use only the following.

Lemma 4.2 (Certificate soundness)**.** *Let $G \in \mathbb{Z}[u_1, v_1, \dots, u_6, v_6]$ have all monomial coefficients nonnegative and satisfy $G(\mathbf{t}, \mathbf{1}-\mathbf{t}) = F(\mathbf{t})$ in $\mathbb{Z}[\mathbf{t}]$. Then $F \geq 0$ on $\mathcal{C}$.*

Proof. At a point of $\mathcal{C}$ all twelve numbers t_j, $1-t_j$ are nonnegative, so every monomial of G evaluates to a nonnegative real; a nonnegative combination of these is nonnegative, and it equals $F(\mathbf{t})$. □

So a certificate for $F \geq 0$ on $\mathcal{C}$ is a representation

$$F(\mathbf{t}) \;=\; \sum_{\mathbf{i} \leq \mathbf{n}} c_{\mathbf{i}} \prod_{j=1}^{6} t_j^{\,i_j} (1-t_j)^{\,n_j - i_j}, \qquad c_{\mathbf{i}} \in \mathbb{Z}_{\geq 0}, \tag{5}$$

and Proposition 4.1 says that such a representation exists exactly when the Bernstein coefficients of F at $\mathbf{n}$ are nonnegative, with $c_{\mathbf{i}} = b_{\mathbf{i}} \prod_j \binom{n_j}{i_j}$. Only (5) enters the proofs; the identification of $c_{\mathbf{i}}$ with a scaled Bernstein coefficient is what tells us where to look, not part of what is checked. This is the sense in which no data is trusted: the exhibited $c_{\mathbf{i}}$ are produced by exact integer arithmetic from (2) and the identity (5) is proved, not asserted.

5. The cube inequality

Theorem 5.1. *For every $(x, y, h, e, f, d) \in \mathcal{C}$,*

$$H_x \;\geq\; 3, \qquad H_y \;\geq\; 3, \qquad N \;\geq\; 0, \qquad \Delta \;\geq\; 0.$$

Proof sketch. Four certificates of the shape (5) are exhibited, for $H_x - 3$ and $H_y - 3$ at multidegree $(2,2,2,2,2,2)$, for N at $(2,2,3,2,2,2)$, and for 9Δ at $(4,4,6,4,4,4)$. Each is obtained by evaluating the homogenised versions of the eight equations (2) in the twelve variables u_j, v_j, in the order the source writes them: $\hat{A}_x$, $\hat{A}_y$, $\hat{P}$, $\hat{Q}$, then $\hat{H}_x = 4\hat{A}_x\hat{A}_y - \hat{P}^2$ and so on, using Proposition 4.1(iii) to keep the multidegrees aligned. The resulting term lists have 699, 699, 782 and 21,460 entries, against the 729, 729, 972 and 21,875 elements of the corresponding Bernstein bases; the missing slots are exactly the coefficients that vanish — already, for N and 9Δ, whose Bernstein expansions have 190 and 415 zero coefficients, and after the constant 3 is subtracted, for H_x and H_y, whose least Bernstein coefficient is 3. Lemma 4.2 then gives the four inequalities, provided that (a) each term list has no negative coefficient, and (b) each list, evaluated at $u_j = t_j$, $v_j = 1 - t_j$, is the intended polynomial, namely $H_x - 3$, $H_y - 3$, N and 9Δ.

Claim (b) is an identity between two explicit polynomials, of 12, 12, 17 and 343 monomials on the closed-form side, and is proved by expansion. Claim (a) is the only part that rests on exhaustive computation: it is four Boolean evaluations, one per polynomial, each testing that every entry of a list of 699, 699, 782 or 21,460 explicit integers is nonnegative. No search over $\mathcal{C}$ is performed, and no coefficient is read from a file. □

Corollary 5.2. *For every $(x, y, h, e, f, d) \in \mathcal{C}$,*

$$-\frac{1+2h}{3} \;\leq\; -\frac{N}{\sqrt{H_x H_y}} \;\leq\; 0.$$

Proof. By Theorem 5.1, $H_x H_y \geq 9 > 0$, so $R = \sqrt{H_x H_y} > 0$ and $R^2 = H_x H_y$. Writing $K = (1+2h)/3 > 0$, the inequality $\Delta \geq 0$ reads $N^2 \leq (KR)^2$, whence $N \leq KR$ since $KR > 0$; dividing by R gives the left bound. The right bound is $N \geq 0$. □

Corollary 5.2 is the display the source draws from its Theorem A.3, and the constant is its κ_q once h is specialised to h_q:

Proposition 5.3. *If $1 \le q \le \log_2 3$ then $h_q = 2^q - 2$ lies in $[0,1]$. At $q = \log_2 \frac{9}{4}$ one has $h_q = \frac{1}{4}$, so the constant of Corollary 5.2 is exactly $-\frac{1}{2}$: for every (x,y,e,f,d) with $x,y,e,f,d \in [0,1]$ and $h = \frac{1}{4}$,*

$$-\tfrac{1}{2} \;\le\; -\frac{N}{\sqrt{H_x H_y}}.$$

Proof. Monotonicity of $q \mapsto 2^q$ gives $2 \le 2^q \le 3$ on $[1, \log_2 3]$, and $2^{\log_2(9/4)} = \frac{9}{4}$. The last display is Corollary 5.2 at $h = \frac{1}{4}$, since $(1 + 2 \cdot \frac{1}{4})/3 = \frac{1}{2}$. □

The value $-\frac{1}{2}$ is the reason the appendix of [1] exists: it is the copositivity threshold for three nonnegative coordinates on which the five-point argument of that paper runs, and $q = \log_2 \frac{9}{4}$ is exactly $P(5)$.

6. A five-variable form, and exactly where it stops

By Lemma 3.1, three of the four polynomials of (2) see (h,e,f,d) only through (he, hf, hd). This makes three of the four cells of Theorem 1.1 statements in five variables.

Theorem 6.1. *For every $(x,y,g,k,l) \in [0,1]^5$,*

$$\tilde{H}_x \;\ge\; 3, \qquad \tilde{H}_y \;\ge\; 3, \qquad \tilde{N} \;\ge\; 0,$$

with $\tilde{H}_x, \tilde{H}_y, \tilde{N}$ as in (3). Consequently, by Lemma 3.1, the first three assertions of Theorem 5.1 hold on $\mathcal{C}$.

Proof sketch. Exactly the method of Section 5, in five variables. Certificates of the shape (5) are built for $\tilde{H}_x - 3$, $\tilde{H}_y - 3$ and $\tilde{N}$ at multidegree $(2,2,2,2,2)$, from the five-variable equations (3); the identity between each built object and its closed form is proved by expansion, and the exhaustive part is three Boolean evaluations testing that the term lists, of 213, 213 and 226 entries, have no negative coefficient. For the consequence: on $\mathcal{C}$ one has $he, hf, hd \in [0,1]$, so $(x, y, he, hf, hd) \in [0,1]^5$, and Lemma 3.1 transports the three inequalities. □

The gain is in size, not in generality, and it is worth being precise about which. The substitution $\sigma(x,y,h,e,f,d) = (x,y,he,hf,hd)$ maps $\mathcal{C}$ *onto* $[0,1]^5$: given $(x,y,g,k,l) \in [0,1]^5$, take $h = 1$, $e = g$, $f = k$, $d = l$. So Theorem 6.1 and the first three assertions of Theorem 5.1 are equivalent, each implying the other through Lemma 3.1. What changes is the certificate: the tensor-product Bernstein basis at multidegree $(2,2,2,2,2)$ has 243 elements against the 729, 729 and 972 of the source's six-variable expansions, and $213 + 213 + 226 = 652$ stored terms replace $699 + 699 + 782 = 2180$. Three of the four cells of Theorem 1.1 are therefore certified at under a third of the coefficient count.

The fourth cell is genuinely different, because Δ retains h outside the three products, and there the decoupling really does enlarge the region. Replacing the constrained set $\{(x,y,h,g,k,l) : g,k,l \le h\}$ — which is what σ produces when h is carried along — by the full cube $[0,1]^6$ destroys the inequality, and each of the three constraints is needed on its own.

Theorem 6.2. *The inequality $\tilde{\Delta} \ge 0$ is false on the cube $[0,1]^6$ in the variables (x,y,h,g,k,l): at the point $(1,1,0,0,0,1)$ one has $\tilde{\Delta} = -56$. Moreover each of the three couplings $g \le h$, $k \le h$, $l \le h$ is individually necessary: dropping only one of them gives, at the three points*

$$(1,1,\tfrac{1}{2},1,\tfrac{1}{2},\tfrac{1}{2}), \qquad (1,1,\tfrac{1}{2},\tfrac{1}{2},1,\tfrac{1}{2}), \qquad (1,1,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},1),$$

respectively (each of which satisfies the other two couplings), the common value

$$\tilde{\Delta} \;=\; -\frac{305}{4}.$$

Proof. Four exact evaluations of the rational function (3) at rational points. At $(1,1,\frac{1}{2},1,\frac{1}{2},\frac{1}{2})$, say, $h = \frac{1}{2}$ and $k = l = \frac{1}{2} \le h$ while $g = 1 > h$, so precisely the coupling $g \le h$ fails. □

Theorem 6.2 says that the six-variable parametrisation of [1] is not an artefact of the chart: it carries information, and all of that information is used by Δ and by nothing else. It is also the cell of highest degree in h: that degree is 6, against 2, 2 and 3 for H_x, H_y and N, and it is where 21,875 of the 24,305 Bernstein coefficients of the source's certificate go. In particular the honest relaxed statement for Δ is one over the region $\{(x,y,h,g,k,l) \in [0,1]^6 : g,k,l \le h\}$, which is not a box, so it lies outside the reach of the instrument used here; reparametrising $g = hg'$, $k = hk'$, $l = hl'$ puts it back on a box and returns exactly the source's six variables.

Remark 6.3. As a consistency check outside the formal development, an exhaustive evaluation over the rational grid $\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}^6$ in the six variables (x,y,h,e,f,d) found no violation of any of the four inequalities of Theorem 5.1, as it must not.

7. Exact ranges

The same instrument applied to $27 - F$ bounds the four polynomials from above, and for three of them the bound is attained.

Theorem 7.1. *On $\mathcal{C}$,*

$$3 \le H_x \le 27, \qquad 3 \le H_y \le 27, \qquad 0 \le N \le 27, \qquad 0 \le \Delta \le \frac{11807}{54},$$

and the first six of these bounds are attained:

$$\begin{array}{lll} H_x(0,0,1,1,0,0) = 3, & H_y(0,0,1,0,1,0) = 3, & N(0,0,0,0,0,0) = 0, \\ H_x(1,1,1,1,1,1) = 27, & H_y(1,1,1,1,1,1) = 27, & N(1,1,1,1,1,1) = 27. \end{array}$$

Hence $\min H_x = \min H_y = 3$, $\min N = 0$ *and* $\max H_x = \max H_y = \max N = 27$ *on* $\mathcal{C}$. *Finally* $\Delta(1,1,1,\frac{1}{2},\frac{1}{2},\frac{1}{2}) = 3125/16$, *so*

$$\frac{3125}{16} \;\le\; \sup_{\mathcal{C}} \Delta \;\le\; \frac{11807}{54}.$$

Proof sketch. The lower bounds are Theorem 5.1. For the upper bounds, four further certificates of the shape (5) are built, for $27 - H_x$, $27 - H_y$, $27 - N$ and $11807 - 6 \cdot 9\Delta = 11807 - 54\Delta$, and Lemma 4.2 applies; the exhaustive part is again four Boolean nonnegativity tests on explicit integer term lists. The seven displayed values are exact rational evaluations of (2). □

Two remarks on sharpness. For H_x, H_y and N the certified bounds are exact on both sides, and that is exactly the content of the six displayed evaluations: a bound that is attained is a minimum or a maximum. For Δ no such pair is available. The certified upper bound $11807/54 = 218.648\ldots$ is the largest Bernstein coefficient of Δ at multidegree $(4,4,6,4,4,4)$, whereas the largest value we have found anywhere on $\mathcal{C}$ is $3125/16 = 195.3125$, at the interior point $(1,1,1,\frac{1}{2},\frac{1}{2},\frac{1}{2})$, to which a multi-start pattern search over $\mathcal{C}$ converged from every seed. The ratio is 1.119, so the certificate is about twelve percent loose there. We claim only the sandwich; that $3125/16$ is the true supremum is not proved. Remark 13.5 narrows the upper end to $1761/9$ by a different instrument, and says why no finite subdivision reaches $3125/16$ either.

The source states none of these bounds. Its appendix table records the extreme Bernstein *coefficients* 3, 27, 1/12, 4/9, 11807/54, and a coefficient is not by itself a statement about the polynomial: for Δ, as just seen, the largest coefficient is an upper bound that no point of the cube has been found to attain. What Theorem 7.1 adds is the passage from coefficients to functions, together with the attaining points.

8. The star formula and the metric specialisation

The appendix of [1] reaches (2) through a projective star formula. For $X, Y, Z \in \mathbb{R}$ let

$$\mathcal{H}(X,Y,Z) \;=\; 2XY + 2YZ + 2ZX - X^2 - Y^2 - Z^2$$

be the Heron polynomial. Given a three-leaf star with centre B_0, leaves A_0, P, O and positive branch lengths a, b, c, inverted at A_0, the source writes the partial correlation of A_0 and B_0 based at P and conditioned on O as $\chi_q = \mathcal{N}/\sqrt{\mathcal{H}_0\mathcal{H}_1}$. Here $A = a^q$, $B = b^q$ and $C = c^q$ are the branch powers, $U = (a+b)^q$, $V = (a+c)^q$ and $W = (b+c)^q$ the pair-sum powers, and

$$\mathcal{H}_0 = \mathcal{H}(U, V, W), \quad \mathcal{H}_1 = \mathcal{H}(AW, BV, CU), \tag{6}$$

$$\mathcal{N} = AW(U+V-W) + BV(U+W-V) + CU(V+W-U) - 2UVW.$$

On the chart of the appendix — $a = 1$, $p = b$, $o = c$, $x = o^{q/2}$, $y = (p/o)^{q/2}$, so that $A = 1$, $B = x^2y^2$, $C = x^2$ and

$$U = 1 + x^2y^2 + hd\,xy, \qquad V = A_x, \qquad W = x^2A_y,$$

— these three quantities collapse onto (2).

Theorem 8.1 (after [1, Lem. A.4])**.** *With U, V, W as displayed and $\mathcal{H}_0, \mathcal{H}_1, \mathcal{N}$ as in* (6)*, one has the polynomial identities*

$$\mathcal{H}_0 = x^2H_x, \qquad \mathcal{H}_1 = x^4y^2H_y, \qquad \mathcal{N} = -x^3y\,N.$$

Proof. Substitute and expand; each is an identity in $\mathbb{Z}[x, y, h, e, f, d]$. No hypothesis on the variables is used. □

Theorem 8.2. *Let $x, y \in (0, 1]$ and $h, e, f, d \in [0, 1]$. Then $\mathcal{H}_0\mathcal{H}_1 > 0$ and*

$$-\frac{1+2h}{3} \;\le\; \frac{\mathcal{N}}{\sqrt{\mathcal{H}_0\mathcal{H}_1}} \;\le\; 0.$$

Proof. By Theorem 8.1, $\mathcal{H}_0\mathcal{H}_1 = (x^3y)^2H_xH_y$, and $H_xH_y \ge 9$ by Theorem 5.1, so $\mathcal{H}_0\mathcal{H}_1 > 0$ and $\sqrt{\mathcal{H}_0\mathcal{H}_1} = x^3y\sqrt{H_xH_y}$ since $x^3y > 0$. Then $\mathcal{N}/\sqrt{\mathcal{H}_0\mathcal{H}_1} = -N/\sqrt{H_xH_y}$ and Corollary 5.2 applies. □

Theorem 8.2 is the source's double-geodesic correlation bound [1, Thm. A.6] — its lower half exactly, and its upper half $\chi_q < 0$ in the non-strict form $\chi_q \le 0$ — *as a statement about the chart.* Turning it into a statement about four-point Ptolemaic metrics needs three further inputs of [1], none of them verified here: the parametrisation of the crossing Ptolemy face by an inverted three-leaf star [1, Prop. A.1]; the star formula [1, Thm. A.2] itself, which identifies $\mathcal{N}/\sqrt{\mathcal{H}_0\mathcal{H}_1}$ with the partial correlation χ_q by a Schur complement of the Schoenberg matrix; and the fact that the three defect ratios $\eta_q(o)/h_q$, $\eta_q(p/o)/h_q$, $\eta_q(p)/h_q$ lie in $[0, 1]$, i.e. the source's sharp two-summand estimate $0 \le \eta_q(t) \le h_q$ for $1 \le q \le 2$. The last of these is the analytic one. It is a self-contained real-analysis argument — the source proves it by showing that $F_x(q) = \log\big(\sinh(qx)/\sinh x\big) - (q-1)\log(2\cosh x)$ vanishes at $q = 1, 2$ and is concave — and it is the natural next step.

9. Sharpness and negative controls

A certificate route can be vacuous in two ways: the nonnegativity test could accept anything, or the hypotheses could be decorative. The following rules both out.

Theorem 9.1. (i) The test discriminates. *The nonnegativity test rejects a term list with a single negative coefficient prepended to a valid certificate; and it rejects the certificates that the same construction produces for the four overclaims $H_x \ge 4$, $N \ge 1$, $9\Delta \ge 1$, and $\tilde{\Delta} \ge 0$ with g, k, l free.*

(ii) The constants are attained, so none of the four inequalities can be tightened. *$H_x = 3$ at $(0, 0, 1, 1, 0, 0)$; $N = 0$ at the origin; $H_x = 27$ at $(1, 1, 1, 1, 1, 1)$; and Δ vanishes identically on the face $x = y = e = f = d = 1$, for every h. A second vanishing point is $(1, 0, 1, 1, 1, 1)$.*

(iii) The box hypothesis is load-bearing, one variable at a time. *For each of the six variables there is a point at which that variable alone leaves* $[0,1]$*, the other five staying inside, and one of the four inequalities fails:*

$$N(-1,0,\tfrac12,0,\tfrac12,0)=-\tfrac12,\quad N(0,-1,\tfrac12,\tfrac12,0,0)=-\tfrac12,\quad N(0,0,-1,0,0,\tfrac12)=-1,$$
$$H_x(0,0,1,2,0,0)=0,\qquad H_y(0,0,1,0,2,0)=0,\qquad N(0,0,\tfrac12,0,0,-1)=-1,$$

witnessing that $0\le x$*,* $0\le y$*,* $0\le h$*,* $e\le 1$*,* $f\le 1$ *and* $0\le d$ *cannot be dropped. A seventh point,* $\Delta(0,0,1,1,1,2)=-16$*, shows that the box hypothesis is load-bearing for* Δ *as well; there it is* $d\le 1$ *that is dropped.*

(iv) The hypothesis is satisfiable and the conclusion is not constant. *Both* $(0,0,0,0,0,0)$ *and* $(1,1,1,1,1,1)$ *lie in* $\mathcal{C}$*, and* $H_x(0,0,0,0,0,0)=4\neq 27=H_x(1,1,1,1,1,1)$.

Proof sketch. Part (i) is five Boolean evaluations, each returning *false*; the certificates involved are built by the construction of Section 4 applied to H_x-4, $N-1$, $9\Delta-1$, the relaxed $\tilde\Delta$, and to a deliberately corrupted list. Parts (ii)–(iv) are exact rational evaluations of (2), except for the vanishing of Δ on the face $x=y=e=f=d=1$, which is a polynomial identity in h proved by expansion. The six points of (iii) were found by an exhaustive scan over the grid $\{-2,-1,-\frac12,0,\frac12,1,2,3\}^6$ with five of the six coordinates confined to $[0,1]$. □

The vanishing of Δ on the whole face $x=y=e=f=d=1$ has a consequence worth stating, although it is an inference from part (ii) and not itself a machine-checked statement: at such a point $N^2=\big((1+2h)/3\big)^2H_xH_y$ with $N\ge 0$ and $H_xH_y\ge 9$, so $-N/\sqrt{H_xH_y}$ equals $-(1+2h)/3$ exactly. The constant of Corollary 5.2 is therefore attained, for every $h\in[0,1]$, and cannot be lowered.

Part (ii) also recovers, on the chart, the equality analysis of [1, Cor. A.5]: that corollary asserts that $\Delta=0$ forces $x=y=e=f=d=1$ when $0<h<1$, and $e=f=d=1$ when $h=1$. We verify the two vanishing points and the vanishing of the whole face, not the converse implications, which is why part (ii) is phrased as sharpness rather than as an equality classification.

10. Schoenberg matrices, Schur complements, partial correlations

The rest of the paper is about the family of estimates of which Theorem 8.2 is one member, and about the constant that runs through it. Nothing in Sections 10–12 uses a certificate; every proof there is an algebraic identity or an exact rational evaluation.

10.1. **The Schoenberg matrix.** Let (X,d) be a finite metric space, let $q>0$, and write $P(x,y)=d(x,y)^q$ for the *powered* distances. For a base point A the *Schoenberg matrix based at* A is

$$G^A_{XY} \;=\; \tfrac12\big(P(A,X)+P(A,Y)-P(X,Y)\big),\qquad X,Y\neq A. \tag{7}$$

Carrying the powered distances rather than d and q separately is a deliberate choice: every operation below — inversion, change of base, conditioning — is then a rational operation on the data, and no property of $t\mapsto t^q$ is ever needed.

Lemma 10.1 (Schoenberg's identity)**.** *Let* P *be a real-valued function on a finite set with a distinguished point* A *such that* $P(A,A)=0$ *and* $P(Y,A)=P(A,Y)$ *for all* Y*. Then for every coefficient vector* z *with* $\sum_x z_x=0$*,*

$$-\tfrac12\sum_{x,y} z_x z_y\,P(x,y) \;=\; \sum_{X,Y\neq A} z_X z_Y\,G^A_{XY}.$$

Proof. Split the double sum into the four blocks determined by A, use $P(A,A)=0$ on the first, the symmetry hypothesis to merge the two cross blocks, and expand G^A by (7); the resulting expression is $-\frac12\sum_{X,Y\neq A} z_Xz_YP(X,Y)$ plus $\big(\sum_{X\neq A} z_XP(A,X)\big)\big(\sum_{Y\neq A} z_Y\big)$, and the zero-sum hypothesis turns the second factor into $-z_A$, which is exactly the cross contribution. □

Thus q-negative type of a finite metric space is nonnegativity of the quadratic forms G^A on zero-sum vectors, and the *sign pattern* of z organises the analysis: if z has p positive entries, carried by A and by $p-1$ further points $B_1,\dots,B_{p-1}$ which we call *anchors*, and r negative entries carried by $U_1,\dots,U_r$, then G^A blocks as

$$G^A \;=\; \begin{pmatrix} D & g^{\mathsf T} \\ g & H \end{pmatrix}, \tag{8}$$

with D of size $(p-1)\times(p-1)$ on the anchors, H of size $r\times r$ on the U_i, and g the $r\times(p-1)$ cross block.

10.2. **Schur complements in witness form.** We shall need the Schur complement of (8) onto the U-block without ever assuming D invertible, because the boundary conventions of [1] allow D to degenerate. Accordingly:

Definition 10.2. Let D, B, C be real matrices of sizes $\alpha\times\alpha$, $\alpha\times\beta$, $\beta\times\beta$. A pair (W,S) is a *Schur witness* for $\left(\begin{smallmatrix} D & B \\ B^{\mathsf T} & C \end{smallmatrix}\right)$ onto β if

$$DW = B \qquad \text{and} \qquad S = C - B^{\mathsf T}W.$$

When D is invertible, $W = D^{-1}B$ and $S = C - B^{\mathsf T}D^{-1}B$ is the usual Schur complement. For β of size 2 we write

$$\chi(S) \;=\; \frac{S_{12}}{\sqrt{S_{11}S_{22}}}$$

for the associated *partial correlation*.

Theorem 10.3 (the p-anchor gap decomposition)**.** *Let D be symmetric and let W satisfy $DW = g^{\mathsf T}$. Then for all vectors a and c,*

$$a^{\mathsf T}Da - 2\,(ga)\cdot c + c^{\mathsf T}Hc \;=\; c^{\mathsf T}Sc + (a-Wc)^{\mathsf T}D\,(a-Wc), \qquad S = H - gW.$$

Proof. Completing the square. Writing $u = a - Wc$, the right-hand side expands to $c^{\mathsf T}Hc - c^{\mathsf T}gWc + a^{\mathsf T}Da - 2\,(Da)\cdot(Wc) + (Wc)^{\mathsf T}D(Wc)$, and $D(Wc) = g^{\mathsf T}c$ turns the last two terms into $-2\,(ga)\cdot c + c^{\mathsf T}gWc$. Symmetry of D is used once, to replace $a^{\mathsf T}D(Wc)$ by $(Da)\cdot(Wc)$. □

For a single anchor Theorem 10.3 is the gap decomposition of [1]. Its role is structural: the left-hand side is, by Lemma 10.1, the generalised-roundness gap of the $p+r$ pattern; the square term is nonnegative as soon as $D \succeq 0$, which for $p = 3$ holds for every $q \le 2$, since D is then the Schoenberg matrix of the three points $\{A, B_1, B_2\}$, the snowflake $d^{q/2}$ is again a metric, and every three-point metric embeds isometrically in a Euclidean space; and what is left is *copositivity of the $r\times r$ matrix S*. The copositivity criterion of [1] then asks for two inputs: $S_{ii} \ge 0$, which is q-negative type of the $(p+1)$-point set $\{A, B_1, \dots, B_{p-1}, U_i\}$; and a lower bound $S_{ij} \ge -\theta\sqrt{S_{ii}S_{jj}}$ with $\theta \le 1/(r-1)$, which is a statement about $p+2$ points. For $p = 3$ the first input is $P(4) \ge q$, that is, the four-point theorem of [3] formalised in [4]; the second is what Section 11 is about.

10.3. **Two invariances.** Both of the following are used to normalise a configuration before computing; they are recorded here because the four-point versions in [1] are stated for four points only.

Proposition 10.4 (inversion covariance)**.** *Let $\hat d$ be the metric inversion of d at A, so that $\hat d(X,Y) = d(X,Y)/\big(d(A,X)d(A,Y)\big)$ and $\hat d(A,X) = 1/d(A,X)$, and let $\hat G^A$ be its Schoenberg matrix based at the same point A. Then, whenever $P(A,X) \ne 0$ for all $X \ne A$,*

$$\hat G^A_{XY} \;=\; \frac{G^A_{XY}}{P(A,X)P(A,Y)}, \qquad \textit{that is} \qquad \hat G^A = \Lambda G^A \Lambda, \quad \Lambda = \operatorname{diag}\big(P(A,X)^{-1}\big).$$

Moreover, if (W,S) is a Schur witness for $\left(\begin{smallmatrix} D & B \\ B^{\mathsf T} & C \end{smallmatrix}\right)$ onto β and u, v are vectors with u nowhere zero, then $\big(\operatorname{diag}(u)^{-1}W\operatorname{diag}(v),\ \operatorname{diag}(v)S\operatorname{diag}(v)\big)$ is a Schur witness for the congruence of the block matrix by $\operatorname{diag}(u)\oplus\operatorname{diag}(v)$; and if $v > 0$ then χ is unchanged. Consequently the p-anchor partial

correlation is invariant under inversion at the base, for every anchor count and every conditioning set.

Proof. The first display is a two-line computation with $\hat{P}(A,X) = 1/P(A,X)$ and $\hat{P}(X,Y) = P(X,Y)/\big(P(A,X)P(A,Y)\big)$. The Schur-witness statement is verified entrywise: $DW = B$ scales by $u_i v_k$ in the (i,k) entry and $C - B^{\mathsf{T}}W$ by $v_k v_l$, the factors u_j cancelling in the sum over the anchor index. For the last claim, χ scales by $v_1 v_2$ in the numerator and by $\sqrt{v_1^2 v_2^2} = v_1 v_2$ in the denominator. The four-point case of the first display is the inversion lemma of [1]. □

Proposition 10.5 (base change)**.** *Let P satisfy $P(B,B) = 0$. Then for* every *pair X, Y, including $X = A$,*

$$G^B_{XY} \;=\; G^A_{XY} - G^A_{BX} - G^A_{BY} + G^A_{BB}.$$

In matrix form this is a congruence $G^{(B)} = EG^{(A)}E^{\mathsf{T}}$ by a matrix E that is block-lower unitriangular on the anchor block. Moreover a Schur complement onto β is unchanged by every such congruence: if $D^{\mathsf{T}} = D$, if (W,S) is a Schur witness onto β, and if M^{T} has a right inverse, then

$$\Big(\ \big(M^{\mathsf{T}}\big)^{-1}\big(Q^{\mathsf{T}} + W\big),\ S\ \Big)$$

is a Schur witness for $\big(MDM^{\mathsf{T}},\ M(DQ^{\mathsf{T}} + B),\ QDQ^{\mathsf{T}} + QB + B^{\mathsf{T}}Q^{\mathsf{T}} + C\big)$ — with the same *S.*

Proof. The identity is one expansion of (7) using $P(B,B) = 0$. For the second part, substitute and use $DW = B$ and $D^{\mathsf{T}} = D$; the right inverse of M^{T} cancels in both the witness equation and the complement, and every term produced by Q cancels against a term of $QDQ^{\mathsf{T}} + QB + B^{\mathsf{T}}Q^{\mathsf{T}}$. □

Geometrically Proposition 10.5 says that the Schur complement sees only the span of the anchors and the two remaining vectors modulo that span, so it does not depend on which of $A, B_1, \dots, B_{p-1}$ is taken as the base. We prove the two halves of that statement; the composite over a concrete point set, which requires matching the index bookkeeping of E with a chosen ordering of the points, is not carried out here.

11. The p-anchor ladder and its sharp constant

11.1. **The constant.** Throughout this section $t = 2^q$. Define, for $p > 0$,

$$\kappa_q^{(p)} \;=\; \frac{p\,2^q - (p+1)}{p+1}. \tag{9}$$

The first three members are

$$\kappa_q^{(1)} = \frac{2^q - 2}{2} = \frac{h_q}{2}, \qquad \kappa_q^{(2)} = \frac{2^{q+1} - 3}{3} = \kappa_q, \qquad \kappa_q^{(3)} = \frac{3\cdot 2^q - 4}{4},$$

so that $\kappa^{(1)}$ is the quantity bounding the one-anchor patterns in [1], $\kappa^{(2)}$ is exactly the constant κ_q of Corollary 5.2 and of that paper's two-anchor estimate, and $\kappa^{(3)}$ is the constant an estimate for three anchors would have to carry. The source treats the one-anchor and two-anchor cases as separate theorems with separate constants and writes no p-indexed family.

11.2. **The complete split graphs.** For integers $p \geq 1$, $r \geq 2$ let $CS(p,r)$ be the metric space on $p + r$ points obtained from the complete split graph — a clique on p vertices joined completely to an independent set of r vertices — by giving every clique-clique and clique-independent pair distance 1 and every independent-independent pair distance 2. This is [1, Def. 5.3], and these are the configurations that paper identifies as attaining its bounds. That every $CS(p,r)$ is Ptolemaic is asserted and proved there in the unnumbered paragraph immediately following that definition, by a case check on the three products of opposite distances in a four-point restriction; it is quoted here and is not part of the formal development. Testing q-negative type of $CS(p,r)$ on the balanced

zero-sum vector, with mass $1/p$ on each clique vertex and $-1/r$ on each independent vertex, gives the *roundness gap*

$$\gamma_{p,r}(t) \;=\; 1 - \frac{p-1}{2p} - \frac{t\,(r-1)}{2r}, \tag{10}$$

whose vanishing locus we abbreviate by

$$t_{p,r} \;=\; \frac{r\,(p+1)}{p\,(r-1)}. \tag{11}$$

Theorem 11.1 (the ladder identity)**.** *Let $p > 0$ and $r > 1$ be real. Then for every t,*

(i) $\gamma_{p,r}(t) = 0$ *if and only if* $t\,p(r-1) = r(p+1)$*, if and only if* $t = t_{p,r}$*;*
(ii) $\kappa_q^{(p)} = \dfrac{1}{r-1}$ *if and only if* $\gamma_{p,r}(t) = 0$*;*
(iii) *consequently* $\kappa^{(p)}$ *evaluated at* $t = t_{p,r}$ *equals* $\dfrac{1}{r-1}$.

Proof. Clearing denominators in (10) gives $\gamma_{p,r}(t) = \big(r(p+1) - t\,p(r-1)\big)/(2pr)$ with $2pr > 0$, which is (i); and (9) with $p+1 > 0$, $r-1 > 0$ turns $\kappa_q^{(p)} = 1/(r-1)$ into the same linear equation, which is (ii). Part (iii) is (i) and (ii) composed. □

Part (ii) is the reason (9) is the right constant and not merely a constant. The copositivity criterion recalled after Theorem 10.3 needs $\theta \le 1/(r-1)$ for r negative coordinates; Theorem 11.1 says that $\kappa^{(p)}$ reaches that threshold at *exactly* the exponent at which the $p+r$ pattern genuinely fails on $CS(p,r)$. An estimate $\chi \ge -\kappa_q^{(p)}$ is therefore neither wasteful nor insufficient: it proves the $p+r$ pattern up to precisely the exponent at which the $p+r$ pattern stops being true.

Theorem 11.2 (the table)**.** *The following hold exactly:*

$$\begin{array}{lll}
t_{1,2} = 4, & & \\
t_{2,2} = t_{1,3} = 3, & & \\
t_{2,3} = \tfrac{9}{4}, & \kappa^{(2)} \text{ there} = \tfrac{1}{2}, & \\
t_{2,4} = t_{3,3} = 2, & & \\
t_{3,4} = \tfrac{16}{9}, & \kappa^{(3)} \text{ there} = \tfrac{1}{3} = \tfrac{1}{4-1}, & \\
t_{3,5} = t_{4,4} = \tfrac{5}{3}, & \kappa^{(3)} \text{ there} = \tfrac{1}{4} = \tfrac{1}{5-1}. &
\end{array}$$

Proof. Six evaluations of (9) and (11) at rational arguments. □

The six lines of Theorem 11.2 are arithmetic; what makes them worth stating is the column of exponents they produce. Reading $q = \log_2 t$ and recalling that $CS(p,r)$ has $p+r$ points:

3 points	$\log_2 4 = 2$	$= P(3)$,	
4 points	$\log_2 3$	$= P(4)$	[3],
5 points	$\log_2 \frac{9}{4}$	$= P(5)$	[1],
6 points	$\log_2 2 = 1$	$= P(6)$	[1],
7 points	$\log_2 \frac{16}{9} = 2\log_2 \frac{4}{3}$	$= P(7)?$	conjectural,
8 points	$\log_2 \frac{5}{3}$	$= P(8)?$	conjectural.

Every published value of P appears, and so do the two values that (1) leaves open at $n = 8$ and $n = 9$; numerically the last two are $2\log_2 \frac{4}{3} = 0.8300750\ldots$ and $\log_2 \frac{5}{3} = 0.7369656\ldots$, both below 1. We claim no more than the identities: that the minimum over $p + r = m$ of $\log_2 t_{p,r}$ *is* $P(m)$ is exactly the conjecture, and is not proved here.

11.3. **The constant is attained, uniformly in p.** The following computation is what forces (9): it says that whatever the p-anchor estimate turns out to be, its constant cannot be smaller than $\kappa_q^{(p)}$, at any exponent.

Theorem 11.3. *Fix $p \geq 1$ and let $t = 2^q$. In $CS(p,2)$ take the base A and the $p-1$ anchors $B_1, \dots, B_{p-1}$ to be clique vertices and the correlated pair U, V to be the two independent vertices. Then the blocks* (8) *of the Schoenberg matrix are*

$$D_{ij} = \begin{cases} 1, & i = j \\ \frac{1}{2}, & i \neq j \end{cases} \qquad g_{ji} = \tfrac{1}{2} \qquad H = \begin{pmatrix} 1 & \frac{2-t}{2} \\ \frac{2-t}{2} & 1 \end{pmatrix},$$

the constant matrix $W \equiv 1/p$ is a Schur witness, the resulting Schur complement is

$$S = \frac{1}{2p} \begin{pmatrix} p+1 & p+1-tp \\ p+1-tp & p+1 \end{pmatrix},$$

and

$$\chi(S) \;=\; -\,\kappa_q^{(p)}.$$

Proof. The blocks are (7) read off the distances of $CS(p,2)$: all clique and cross distances are 1, and the two independent vertices are at distance 2, so their powered distance is t. That $W \equiv 1/p$ is a witness is the row-sum identity $1 \cdot \frac{1}{p} + (p-2) \cdot \frac{1}{2} \cdot \frac{1}{p} = \frac{1}{2}$ for each anchor row; the displayed S is then $H - gW$ entry by entry. Finally $S_{11} = S_{22} > 0$, so $\sqrt{S_{11}S_{22}} = S_{11}$ and $\chi(S) = S_{12}/S_{11} = (p+1-tp)/(p+1)$, which is $-\kappa_q^{(p)}$ by (9). □

At $p = 1$ Theorem 11.3 is the equality case of the sharp two-summand estimate [1, Lem. 2.1] at equal arguments, and at $p = 2$, at the exponent $q = \log_2 \frac{9}{4}$, it is the equality case of that paper's boundary corollary [1, Cor. A.7], whose equality configuration is exactly the four-point restriction $CS(2,2)$ of $CS(2,3)$; the content added here is the statement for every p and every exponent at once, and in particular the case $p = 3$, which is the sharpness input that any three-anchor estimate must match.

12. No three-anchor analogue above $q = 1$

The closing section of [1] — Section 10, *Further remarks*, of v1 — states that the conjectural values $P(7) = 2\log_2 \frac{4}{3}$ and $P(8) = \log_2 \frac{5}{3}$ reduce to the $3+4$ and $3+5$ sign patterns, and that "[t]hese cases require an analogue of the four-point correlation estimate for sign patterns with three positive coefficients." By Theorems 11.1 and 11.3 the constant of such an analogue is forced, and the estimate one is asking about is

$$\chi_q(U, V \mid A; B_1, B_2) \;\geq\; -\kappa_q^{(3)} \qquad \text{for all five points } A, B_1, B_2, U, V \text{ of a Ptolemaic metric.} \tag{12}$$

The exponent range matters: the four-point estimate of [1] is proved on $1 \leq q \leq \log_2 3$, and both targets lie *below* 1. The next theorem says that this is not an accident of method. It computes (12) on a five-point configuration that the source itself uses — but conditioned on a pair of points on which the source never conditions.

Theorem 12.1. *In $CS(2,3)$ — clique $\{B_1, B_2\}$ at distance 1, independent part $\{A, U, V\}$ pairwise at distance 2, all cross distances 1 — take the base A in the* independent *part, the two clique vertices as anchors, and the remaining two independent vertices as the correlated pair. With $t = 2^q$ the blocks are*

$$D = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}, \qquad g_{ji} = \tfrac{t}{2}, \qquad H = \begin{pmatrix} t & \frac{t}{2} \\ \frac{t}{2} & t \end{pmatrix},$$

the constant matrix $W \equiv t/3$ is a Schur witness, and

$$S = \begin{pmatrix} t - \frac{t^2}{3} & \frac{t}{2} - \frac{t^2}{3} \\ \frac{t}{2} - \frac{t^2}{3} & t - \frac{t^2}{3} \end{pmatrix}, \qquad \chi(S) \;=\; \frac{3-2t}{2\,(3-t)} \qquad (0 < t < 3).$$

Moreover, on $0 < t < 3$,

$$\chi(S) = -\kappa_q^{(3)} \iff t = 1 \text{ or } t = 2,$$

at $t = 2$ *both this configuration and* $CS(3,2)$ *give the common value* $-\frac{1}{2} = -\kappa_1^{(3)}$, *and at* $t = \frac{9}{4}$

$$\chi(S) = -1 \qquad \text{while} \qquad -\kappa_q^{(3)} = -\tfrac{11}{16}.$$

Proof. The blocks are (7) read off the distances, the powered distance between two independent vertices being t; the witness equation is the row sum $1 \cdot \frac{t}{3} + \frac{1}{2} \cdot \frac{t}{3} = \frac{t}{2}$, and $S = H - gW$. For the partial correlation, $S_{12} = t(3-2t)/6$ and $S_{11} = S_{22} = t(3-t)/3 > 0$ on $0 < t < 3$, so $\chi = S_{12}/S_{11}$ is the displayed rational function. Setting it equal to $-\kappa_q^{(3)} = -(3t-4)/4$ and clearing the two positive denominators gives $6(t-1)(t-2) = 0$. The two remaining assertions are evaluations at $t = 2$ (using Theorem 11.3 with $p = 3$ for the second value) and at $t = \frac{9}{4}$. □

Corollary 12.2. *The estimate* (12) *is false at* $q = \log_2 \frac{9}{4}$*: the five-point configuration* $CS(2,3)$ *refutes it there. Since* $2^q > 1$ *for* $q > 0$, *the only exponent in* $0 < q < \log_2 3$ *at which the two sides of* (12) *agree on that configuration is* $q = 1$. *Hence no three-anchor analogue of the four-point correlation estimate can hold on a range of exponents extending above* $q = 1$, *and the largest range on which such an estimate can possibly hold is* $0 < q \le 1$.

Three comments, of decreasing formality.

First, an inference not part of the formal development: $\chi_q(CS(2,3)) + \kappa_q^{(3)}$ is a continuous function of t on $(0,3)$ vanishing only at $t = 1$ and $t = 2$, so it has constant sign on $(2,3)$, and Theorem 12.1 evaluates that sign as negative at $t = \frac{9}{4}$; so (12) in fact fails at every q with $1 < q < \log_2 3$. We state as proved only the two evaluations and the characterisation of the crossing.

Second, the window found here is exactly the window the remaining cases need: both $2\log_2 \frac{4}{3}$ and $\log_2 \frac{5}{3}$ are less than 1. So Corollary 12.2 constrains the statement of the missing estimate rather than obstructing the programme — but it does say that the missing estimate cannot be an extension of the four-point one to the same exponent range, which is how the four-point estimate of [1] is stated. That (12) *holds* on $0 < q \le 1$ is not proved here and is not claimed.

Third, $CS(2,3)$ is the configuration on which [1] proves its own two-anchor bound to be sharp; what is new is the three-anchor conditioning, an object that paper never forms. The value has a compact form worth recording, an algebraic consequence of Theorem 12.1 and (9) rather than a separate verified statement:

$$-\chi_q\big(CS(2,3)\big) \;=\; \frac{\kappa_q^{(2)}}{1 - \kappa_q^{(2)}}.$$

Equivalently, it is the value of the partial-correlation recursion $(a - bc)/\sqrt{(1-b^2)(1-c^2)}$ at $a = -\kappa_q^{(2)}$ and $b = c = \kappa_q^{(2)}$: the three-anchor value on this configuration is what one step of that recursion returns when it is fed the sharp two-anchor constant.

Theorem 12.3 (negative controls for Sections 10–12)**.**

(i) The constants do not transfer. $\kappa^{(3)}$ *at* $t = \frac{16}{9}$ *is* $\frac{1}{3}$ *and not* $\frac{1}{4}$*; at* $t = \frac{5}{3}$ *it is* $\frac{1}{4}$ *and not* $\frac{1}{3}$*; and* $t_{3,4} \neq t_{3,5}$.

(ii) The source's own constant does not reach the threshold. *At the seven-point exponent,* $\kappa^{(2)} = \frac{5}{27} \neq \frac{1}{3}$. *Since* $\frac{1}{3}$ *is the copositivity threshold for four negative coordinates, the* $3 + 4$ *pattern cannot be run through the two-anchor estimate of* [1]*: a genuinely three-anchor estimate is needed.*

(iii) The bound is attained, so no smaller constant is possible. *On* $CS(3,2)$ *at* $t = \frac{16}{9}$ *the three-anchor partial correlation is exactly* $-\frac{1}{3}$.

(iv) The refutation is specific to $q > 1$. *At* $t = \frac{3}{2}$ *the configuration of Theorem 12.1 gives* $\chi = 0$ *against* $-\kappa^{(3)} = -\frac{1}{8}$, *and at* $t = \frac{5}{2}$ *it gives* -2 *against* $-\frac{7}{8}$.

(v) Every hypothesis is load-bearing. *The positivity of* v *in Proposition 10.4 cannot be dropped: for* $v = (1,-1)$ *the partial correlation changes sign. The hypothesis* $P(B,B) = 0$ *in Proposition 10.5 cannot be dropped: for the constant* $P \equiv 1$ *the two sides are* $\frac{1}{2}$ *and* 0. *The zero-sum*

hypothesis in Lemma 10.1 cannot be dropped: on two points at distance 1 *with* $z = (1,1)$ *the two sides are* -1 *and* $+1$*. The symmetry of* D *in Theorem 10.3 cannot be dropped: for* $D = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $g = 0$, $H = 0$, $W = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, $a = (1,1)$, $c = (1)$ *the witness equation* $DW = g^{\mathsf{T}}$ *still holds and the two sides are* 1 *and* 0*.*

Proof. Exact rational evaluations of (9), (11), (7) and of the matrices of Theorems 11.3 and 12.1. □

13. Below $q = 1$: the measured state of the certificate route

The estimate (12) lives, by Corollary 12.2, at exponents $q < 1$, where $h_q = 2^q - 2$ is *negative*. The chart of Section 5 is not available there, and this section records what is known about repairing it. One statement here is a theorem of this paper, of the same standing as those above: Theorem 13.1, which locates exactly where the relaxation of Theorem 1.1 stops holding in h. Everything after it is labelled, and is of one of two kinds: a numerical experiment, reported with its protocol so that it can be repeated; or a statement verified in the same Lean development as Sections 5 and 10–12 but belonging to the certificate infrastructure, quoted with that caveat.

13.1. **Why the cube itself has to change.** Below $q = 1$ the defect ratios still lie in $[0,1]$: there $h_q < 0$ and $h_q \le \eta_q(z) \le 0$, so the quotient $e = \eta_q(z)/h_q$ is again a number between 0 and 1. What fails is the inequality, not the range. The mechanism is that the relaxation permits all branch lengths equal ($x = y = 1$) *together with* all defect ratios zero ($e = f = d = 0$), which no metric does: $e = 0$ forces a ratio of branch lengths tending to infinity and $x = 1$ forces it to be 1.

Theorem 13.1 (the exact validity window of the relaxation in h)**.** *At that one point the four polynomials of* (2) *have closed forms in* h*:*

$$H_x(1,1,h,0,0,0) = H_y(1,1,h,0,0,0) = 12, \qquad N(1,1,h,0,0,0) = 4,$$

$$\Delta(1,1,h,0,0,0) \;=\; 64\,h\,(h+1) \qquad \text{for every real } h.$$

Consequently Δ *is the only one of the four cells that the sign of* h *breaks: the inequality* $\Delta \ge 0$*, which on* $\mathcal{C}$ *is the fourth assertion of Theorem 5.1, is false at that point for every* h *with* $-1 < h < 0$ *and true there for every* $h \ge 0$*, with the value* 0 *at* $h = 0$*, so the window* $h \ge 0$ *is sharp. Since* $0 < q < 1$ *forces* $h_q = 2^q - 2$ *into* $(-1, 0)$*, the relaxation underlying Theorem 1.1 therefore fails at this one point at* every *exponent below* 1*, and not merely at the two that Section 12 leaves as targets, where the values are* $-\frac{128}{9}$ *at* $2^q = \frac{5}{3}$ *and* $-\frac{896}{81}$ *at* $2^q = \frac{16}{9}$*.*

Proof. The three constant identities and the closed form for Δ are polynomial identities in h, proved by expansion in (2). The sign statements are $64h(h+1) < 0$ for $-1 < h < 0$ and $64h(h+1) \ge 0$ for $h \ge 0$; the exponent statement is strict monotonicity of $q \mapsto 2^q$, applied at $q = 0$ and at $q = 1$, which gives $1 < 2^q < 2$ and hence $-1 < h_q < 0$ on $0 < q < 1$; and the two values are $64h(h+1)$ at $h = -\frac{1}{3}$ and $h = -\frac{2}{9}$. □

Two things Theorem 13.1 does *not* say. It says nothing about the metric inequality below $q = 1$: the point it uses is not the image of any metric, exactly by the mechanism just described, so what fails there is the relaxation and not the geometry. And the window it locates is a window in h at one point of the cube, not a claim that the relaxation holds on all of $\mathcal{C}$ for $h \ge 0$ — that is Theorem 5.1, and it is proved separately. Four controls fix the shape of the statement, each an exact rational evaluation: the failure is an interval phenomenon and not a statement about negative h in general, since the value returns to 0 at $h = -1$ and is $+128$ at $h = -2$; the threshold $h = 0$ is attained, so the second half cannot be sharpened to a strict inequality; the witness point is a genuine point of $\mathcal{C}$ for every $h \in [0,1]$, so that half is not vacuous; and at $h = 1$ the same point gives $+128$, the largest value of $64h(h+1)$ on $[0,1]$.

A second change is forced at the same time: below $q = 1$ the inequality $\chi \le 0$, which is what the cell $N \ge 0$ delivers, is false. A scan over 60 000 random three-leaf stars at $q = \log_2 \frac{5}{3}$, with the branch lengths drawn log-uniformly and χ evaluated in floating point, found χ ranging over

$[-0.111108, +0.002559]$ — inside $\pm\kappa_q^{(2)} = \pm\frac{1}{9}$, the constant a three-leaf star carries, but not inside $(-\infty, 0]$. This is a numerical experiment and not a theorem. The certificate must therefore prove the three cells

$$S_{UU} \geq 0, \qquad S_{VV} \geq 0, \qquad \big(\kappa^{(3)}\big)^2 S_{UU} S_{VV} - S_{UV}^2 \geq 0,$$

which give the two-sided conclusion $|\chi| \leq \kappa^{(3)}$ directly, rather than the four cells of Theorem 5.1, whose one-sided reading is no longer available.

13.2. **Measurement: the envelope must pinch, and a pinched one works.**

Remark 13.2 (measurement, outside the formal development). Let a_i, a_j be two branch lengths of a star, $A_i = a_i^q$, $\alpha_i = a_i^{q/2}$, and put

$$\sigma = \frac{2\alpha_i\alpha_j}{A_i + A_j} \in (0, 1], \qquad e = \frac{\eta_q(a_i/a_j)}{h_q} \in [0, 1].$$

Both are functions of $w = (a_i/a_j)^{q/2}$ alone, so $e = \Phi_q(\sigma)$ for an increasing bijection Φ_q of $(0, 1]$; and σ, unlike Φ_q, is a rational function of the chart variables. A certificate therefore needs rational $L \leq \Phi_q \leq U$. Two things were measured, at the two exponents $2^q = \frac{5}{3}$ and $2^q = \frac{16}{9}$.

A uniform band cannot work. Substituting $|e - \Phi_{\mathrm{fit}}(\sigma)| \leq \delta$ for any fixed $\delta > 0$ leaves a negative cell: at the balanced star the target inequality is an *equality*, so a band of positive width there is fatal to first order. Measured worst cells at $x = 1$ were $-1.00 \cdot 10^{-6}$ at $\delta = 10^{-4}$ and $-1.006 \cdot 10^{-4}$ at $\delta = 10^{-2}$, decreasing linearly in δ. What is required is an envelope that *pinches*: $L(1) = U(1) = 1$.

A pinched envelope does work. The pair

$$\sigma \;\leq\; \Phi_q(\sigma) \;\leq\; \frac{m\,\sigma}{1 + (m-1)\sigma}, \qquad m = 2 \text{ at } 2^q = \tfrac{5}{3}, \quad m = 3 \text{ at } 2^q = \tfrac{16}{9},$$

was valid at every point of a 200 001-point grid in w. Substituting $e = L + (U - L)\hat{e}$ with $\hat{e} \in [0, 1]$ — a sound relaxation, since the true e lies between $L(\sigma)$ and $U(\sigma)$ — the minimum of the three cells above over 60 restarts of 20 000 annealing steps each was

	three leaves	four leaves
$2^q = \frac{5}{3}$	$+0.0 \cdot 10^{0}$	$-3.33 \cdot 10^{-16}$
$2^q = \frac{16}{9}$	$-5.20 \cdot 10^{-18}$	$-3.33 \cdot 10^{-16}$

and a systematic 13-point grid at three leaves gave $+2.70 \cdot 10^{-8}$ and $+7.49 \cdot 10^{-8}$ at the two exponents. As a negative control the same search with the defect ratios free returned the violations $-1.389 \cdot 10^{-2}$ and $-1.080 \cdot 10^{-2}$ (three leaves) and $-2.083 \cdot 10^{-2}$ and $-1.543 \cdot 10^{-2}$ (four leaves), as it must. In the closed form of the substitution the pinching appears as an algebraic factor $(\alpha_i - \alpha_j)^2$ multiplying $\hat{e}$, so the extremal configuration is pinned by the algebra rather than by a side condition.

This is not a proof. The validity of the envelope is a grid check and needs a one-variable analytic lemma at fixed rational 2^q; the nonnegativity of the cells is a search and needs a certificate.

13.3. **Measurement: the endpoint principle does not generalise.**

Remark 13.3 (measurement, outside the formal development). The proof in [1] reduces a varying distance to an endpoint by an argument about the pencil $K(t) = K_0 - e^t uu^{\mathsf{T}} - e^{-t} vv^{\mathsf{T}}$ with u in the all-ones direction: a negative interior critical point of the resulting partial correlation is a strict local maximum, so the negative minimum sits at an endpoint. Taking one further Schur complement, as the passage from two anchors to p anchors requires, destroys the conclusion. Over 1500 random instances per anchor count, restricted to the instances in which the correlation is defined and $|\chi| \leq 1$, with each candidate refined by ternary search to 10^{-11} and confirmed by a central second difference

at three step sizes:

anchors	usable	interior negative local minima
$p = 2$ (the proved case, control)	1226	0
$p = 3$	1113	1
$p = 4$	957	11
$p = 5$	799	16

and the conclusion actually used — that the minimum over the feasible interval is attained at an endpoint — failed 0, 0, 9 and 13 times respectively. The worst witness at $p = 4$ has $\chi(t^*) = -0.4998834$ at $t^* = +0.075649$ with second derivative $+0.850$ and $\chi(t^* \pm 0.05) = -0.4988$, and was recomputed from its printed data by an independent fine scan. The control is what makes this a measurement rather than a modelling artefact: a first version of the experiment used a generic rank-one direction in place of the all-ones one and refuted the proved case in 11 of 231 instances; with the all-ones hypothesis restored the proved case has no violations at all. So the route through the second derivative is closed for $p \geq 4$; at $p = 3$ only the differential clause failed, and a salvage, if there is one, must prove the endpoint conclusion directly.

Every number in Remarks 13.2 and 13.3 was reproduced, on 2026-08-29, from four single-file fixed-seed standard-library Python scripts archived with the Lean development under the same repository reference as the formal proofs (Section 16): `envelope.py` and `envelope_confirm.py` for Remark 13.2, `pencil_psd.py` for the counts of Remark 13.3, and `verify_witness.py`, which recomputes each worst witness from its printed (K_0, d) by an independent fine scan. The one figure not reproducible from them is the generic rank-one control at the end of Remark 13.3, which belongs to a superseded version of that experiment and is quoted from its recorded output.

13.4. **Measurement: what a certificate would cost.**

Remark 13.4 (measurement, outside the formal development). A chart for (12) is a four-leaf star, one leaf more than the three-leaf star behind (2), the reduction to it being the analogue of the one [1] carries out for four points; we do not carry that reduction out here, but its cost can be measured. Under the pinched envelope that chart has nine box variables: three branch coordinates and six relaxation variables $\hat{e}_{ij}$. Expanding the analogues of H_x, H_y, N and Δ there exactly, at either target exponent, gives a Δ of multidegree $(66, 64, 64, 6, 4, 4, 4, 4, 4)$, that is about $6.19 \cdot 10^9$ Bernstein coefficients, with the three constituent cells already at 3.5 to $3.7 \cdot 10^7$ each; a chart using chain ratios instead of the four branch lengths directly is three times worse ($1.96 \cdot 10^{10}$). Against the 21 875 coefficients of Theorem 5.1 this is five orders of magnitude, and it is out of reach.

One rung lower the *size* is not the obstacle: the three-leaf chart — the cube of Theorem 5.1 itself, below $q = 1$, under the pinched envelope — has five box variables and a Δ of 27 012 monomials at multidegree $(38, 38, 4, 4, 4)$, a dense array of $1.90 \cdot 10^5$, which is 8.7 times the present cell and well inside what the method of Section 4 handles. Size is not certifiability, though, and expanding that cell answers the two questions differently. Its first two cells, $H_U \geq 0$ and $H_V \geq 0$, have *no* negative Bernstein coefficient at the root, at either exponent, so they need no subdivision at all. Its third cell — the one carrying the constant — has 247 negative coefficients at $2^q = \frac{5}{3}$ and 212 at $2^q = \frac{16}{9}$, all of them at the corner $x_1 = x_2 = 1$, which is the balanced star; and there the inequality is an *equality*, precisely because the envelope was pinched to make it one. Splitting the two branch axes alternately leaves exactly one bad box at every level and reduces its worst coefficient by a factor of about 2 per split. Since a Bernstein array bounds the minimum of the cell from below, and that minimum is 0, the bound approaches 0 from below and never reaches it: no finite subdivision tree certifies the third cell at the sharp constant. That is the same obstruction as in Remark 13.5 below, seen a second time, and it turns the three-leaf extension from a compute question into a design question.

Remark 13.5 (verified in the same development, belonging to the certificate infrastructure). Theorem 7.1 sandwiches $\sup_{\mathcal{C}} \Delta$ between 3125/16 and 11807/54, a relative width of about 12 percent,

and the upper end is the root Bernstein bound, which cannot be improved: in integers the root expansion certifies $9\Delta \le 1968$ and nothing smaller. Binary subdivision of the box is a different instrument, and where the subdivisions are spent is the whole story: eight leaves spent on x, y, h gain nothing at all, while sixty-four leaves spent on the three defect axes e, f, d certify $9\Delta \le 1761$, so that $\sup_{\mathcal{C}} \Delta \le 1761/9 = 195.66\ldots$ against the attained $3125/16 = 195.3125$ — a sandwich of relative width about 0.18 percent. In integers, $1757 < 9 \sup_{\mathcal{C}} \Delta \le 1761$, the lower end being the same checker's exact-integer refutation mode at $(1, 1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. That is the numerical signature of the maximiser sitting at a corner in (x, y, h) and at an interior midpoint in (e, f, d): a Bernstein array sees a corner exactly, and only subdivision resolves an interior extremum. No finite tree can reach $3125/16$ itself: a Bernstein array bounds the minimum of $C - 9\Delta$ from below, and at the exact $C = 9 \cdot 3125/16$ that minimum is 0, so the bound approaches it and never attains it — the same structural fact as in Remark 13.4. Three controls make the reading of that allocation a measurement rather than an anecdote: the root expansion rejects $9\Delta \le 1761$, so subdivision is necessary; an eight-leaf tree on e, f, d rejects it too, while certifying its own threshold 1769, so depth is necessary; and an eight-leaf tree on x, y, h still rejects the root's own 1967, so the choice of axes is.

14. What remains

Three things are worth naming separately, and they are of different kinds.

The first is analytic and unchanged from the first version of this paper: the sharp two-summand estimate $0 \le \eta_q(t) \le h_q$ for $1 \le q \le 2$, which is what puts the three defect ratios of Section 5 in $[0, 1]$. It needs no computation at all: the lower half is a standard power-sum inequality, and the upper half is two derivative computations and a concavity argument on $F_x(q) = \log\big(\sinh(qx)/\sinh x\big) - (q-1)\log(2\cosh x)$, which vanishes at $q = 1$ and $q = 2$. Closing it is a matter of formalising real analysis, not of certifying a polynomial.

The second is the estimate (12) itself, and the state of the question is now sharper than it was. Its constant is determined (Theorems 11.1 and 11.3); its exponent range is determined from above (Corollary 12.2); and the two conjectural values $P(7)$ and $P(8)$ that [1] reduces to it lie inside that range. What is not determined is whether (12) is true on $0 < q \le 1$. Two routes to it were examined here. The route through the endpoint principle of [1] is closed at four anchors and above (Remark 13.3). The route through a certificate on a box is open in principle but not at the size that matters: the relaxation has to be repaired by a pinched envelope, which numerically it is (Remark 13.2), and the resulting four-leaf expansion is five orders of magnitude beyond the present one (Remark 13.4). The honest summary is that the conjecture at $n \ge 8$ remains open, and that what has been added is the constant, the window, and the price.

The third is one rung lower, and it is now measured rather than guessed. Extending Theorem 5.1 itself below $q = 1$ — which Theorem 13.1 shows cannot be done naively, since the relaxation is false there at a single explicit point for every exponent in $(0, 1)$ — becomes, once the pinched envelope of Remark 13.2 replaces the free defect ratios, a certificate problem of the size this paper already handles: five box variables and about $1.9 \cdot 10^5$ coefficients. Two of its three cells are then immediate, having no negative Bernstein coefficient even at the root. The third is not, and not for want of size: by Remark 13.4 its negative coefficients all sit at the balanced star, where the pinched envelope makes the inequality an equality, and no finite subdivision tree removes them. So what stands between the measurement and the theorem is not two lemmas but two lemmas and a design decision: validity of the envelope, which is a one-variable inequality at a fixed rational 2^q in the shape of the $F_x(q)$ argument above; nonnegativity of the first two cells, which is exactly the instrument of Section 4; and, for the third, something other than subdivision — a constant strictly inside the sharp one, at the cost of sharpness; a change of variables moving the equality point off the interior of a face; or an analytic treatment of a neighbourhood of the balanced star with a certificate on the complement.

15. Cost

The following two paragraphs are engineering measurements, not theorems, and they are not part of the formal development.

Remark 15.1 (Replication of the source's numbers; measured, not formalised). Before any of the above was written, the four polynomials of (2) were expanded independently from the printed equations with exact rational arithmetic and converted to the tensor-product Bernstein basis by the conversion formula of [1]. All 24,305 resulting coefficients agree, term for term, with the stored list of the ancillary package [2], and every entry of the source's appendix table reproduces: multidegrees $(2,2,2,2,2,2)$, $(2,2,2,2,2,2)$, $(2,2,3,2,2,2)$ and $(4,4,6,4,4,4)$; coefficient counts 729, 729, 972, 21,875; zero counts 0, 0, 190, 415; least positive coefficients 3, 3, 1/12, 4/9; largest coefficients 27, 27, 27, 11807/54. The source's own verifier was also run and reports success. Timings on one core: the source's verifier 5.9 s; the independent expansion, conversion and full coefficient-by-coefficient comparison 39.5 s in 17 MB; a prototype of the homogenisation of Section 4 5.6 s. This replication is a check on the source, carried out outside the formal development and in a different language; it is not among the machine-checked statements of this paper, which prove the same inequalities from scratch.

Remark 15.2 (The price of storing nothing; measured). The design of Section 4 costs four soundness facts about the sparse polynomial arithmetic — that addition, multiplication and the homogenising factors behave as expected under evaluation, plus Lemma 4.2 — and buys the removal of two layers: the stored coefficient array and the basis conversion. The certificate half of the development is 1288 lines and one clean rebuild costs 107 s of processor time (1.8 minutes) at a peak resident size of 7.10 GB, almost all of it the library import; no shared library is built and no dynamically loaded object is used. The algebraic half — Sections 10–12 — adds 832 lines in four modules and 40 s of processor time at the same peak resident size, with no compiled evaluation anywhere, and Theorem 13.1 with its controls one further module of 140 lines, measured at 73 s and 6.87 GB, again with none. For contrast, a companion development in the same repository replays the stored Bernstein *subdivision* certificate of a different recent paper (Soberón [9], on Grünbaum's four-hyperplane problem): 1872 lines, 1.75 hours of processor time, a compiled shared library, and — because its certified object is a stored array of integers — a transcription step between the paper's polynomials and the checked data that no proof inside the development can close. The two designs are complementary rather than competing: a certificate *with* subdivision genuinely needs a basis-conversion layer, and a certificate *without* subdivision genuinely does not, because then the homogenisation of the polynomial is already the Bernstein array up to binomials.

16. Verification

Every inequality, identity and numerical value asserted by the theorems, propositions and corollaries of Sections 5–9 and Sections 10–12, together with Lemmas 3.1 and 4.2 and with Theorem 13.1 and the four controls stated after it, has been formally verified in Lean 4 (toolchain `v4.32.0`) against *Mathlib* [10, 11]; the development contains no `sorry` and declares no axiom by hand. The short arithmetic bridges written out in the proofs above are not themselves Lean statements: this concerns the positivity of $\mathcal{H}_0\mathcal{H}_1$ in Theorem 8.2, which the Lean statement of that bound does not carry, and the step from the vanishing of Δ on the face $x = y = e = f = d = 1$ to the sharpness of the constant $(1+2h)/3$. Proposition 4.1 is classical and is used only to say where the certificates come from: its parts (i) and (iii) are verified in the form the proofs need, namely as evaluation identities for the built objects, while part (ii), the identification of the coefficients with the Bernstein basis, is deliberately not part of the formal development and is used by no proof. The reasoning layer is free of compiled evaluation: the soundness facts behind Lemma 4.2, the identities between each built certificate and its closed form, Lemma 3.1, the three identities of Theorem 8.1, the two exponent facts of Proposition 5.3 ($h_q \in [0,1]$ for $1 \le q \le \log_2 3$, and $h_{\log_2(9/4)} = \frac{1}{4}$) and every point evaluation in Theorems 6.2, 7.1 and 9.1 depend on no axioms beyond `propext`, `Classical.choice` and

`Quot.sound`. What is delegated to compiled evaluation, through Lean's `native_decide`, is exactly the nonnegativity tests on explicit integer term lists: four Booleans for Theorem 5.1 (lists of 699, 699, 782 and 21,460 entries), three for Theorem 6.1 (213, 213, 226), four for the upper bounds of Theorem 7.1, and five returning *false* for Theorem 9.1(i). Those sixteen Booleans are the only source of extra axioms in any statement of this paper: each carries the single generated axiom that records its own compiled evaluation, and every statement above depends on exactly the ones it uses. (The development also contains a shared, arity-generic certificate checker with a subdivision layer, the instrument behind Remark 13.5; it has compiled Booleans of its own, and no theorem of this paper depends on them.) Each of those Booleans is an evaluation of a compiled program on data the same program computed from the equations (2) and (3); no coefficient list is supplied as input anywhere in the development, and no dynamically loaded shared library is involved. Every numerical value quoted was first produced by an independent implementation in Python and the formal proof was then written against it (Remark 15.1); the interior point of Theorem 7.1 was located by a floating-point pattern search and its value then established exactly.

The statements of Sections 10–12 use no compiled evaluation at all: every one of them depends on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`. Two points about their scope should be stated plainly. They are theorems about explicit matrices and about the rational functions (9)–(11), not about metric spaces: what is verified in Theorems 11.3 and 12.1 is that the displayed blocks admit the displayed Schur witness and that the resulting partial correlation has the displayed value, while the identification of those blocks with the Schoenberg matrix of $CS(p, 2)$ or $CS(2, 3)$, and the fact that these spaces are Ptolemaic, are read off the definitions and quoted from [1] respectively. And the two halves of base independence, Proposition 10.5, are verified separately; their composite over a concrete point set is not.

Section 13 mixes three kinds of statement and labels each. Theorem 13.1 and the four controls after it are results of this paper, verified like the rest and with no compiled evaluation at all: they are polynomial identities in one variable and exact rational evaluations, depending on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`. The three remarks labelled as measurements are floating-point experiments and are not part of the formal development. Remark 13.5 is of the third kind: a Lean theorem of the same standing as the rest, proved through the shared certificate checker rather than through the private pipeline of Section 4, and therefore resting on one further compiled Boolean, the 64-leaf subdivision tree; it belongs to the certificate infrastructure and is quoted here as such rather than claimed among this paper's results.

The source of the development is currently held in a private repository: repository reference to be added at submission.

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