

HYPERSIMPLEX FRAMES BEYOND THE SIMPLEX BOUND, AND THE EXACT TABLE OF QUASIMAXIMAL PROJECTION CONSTANTS UP TO $N = 11$

KOREA SUPERINTELLIGENCE LABS

ABSTRACT. For a Banach space Y let $\lambda(Y)$ be its absolute projection constant and $\lambda_{\mathbb{K}}(m) = \sup\{\lambda(Y) : \dim Y = m\}$ the maximal projection constant in dimension m . Dereęowska and Lewandowska recently showed that the $m(m+1)/2$ edge midpoints of a regular simplex in $\mathbb{R}^m$ form a Parseval frame of value $4(m-2)/(m+1)$, so that $\lambda_{\mathbb{R}}(m) \geq 4(m-2)/(m+1)$, and conjectured that this is sharp at $m = 6$ and $m = 8$. We contribute to the lower-bound side of this circle of problems.

First, their frame is the $k = 2$ member of a family indexed by k : the k -subsets of an n -set, projected onto the first nontrivial eigenspace of the Johnson scheme $J(n, k)$ — the vertices of the hypersimplex $\Delta(n, k)$, and, as a frame, Datta and Oldroyd's k -angle tight frame. We evaluate the entrywise ℓ^1 mass of the corresponding projection in closed form and obtain $\lambda_{\mathbb{R}}(n-1) \geq v(n, k)$ for an explicit binomial sum $v(n, k)$. Here $k = 2$ returns the simplex bound, while $k = 3$, whose value for $n \geq 9$ is $6(n-4)(n-5)/(n(n-2))$, exceeds it for every $m = n-1 \geq 13$. In particular $\lambda_{\mathbb{R}}(15) \geq 99/28 = 3.5357\dots$ and $\lambda_{\mathbb{R}}(16) \geq 312/85 = 3.6705\dots$, which exceed the values 3.5106 and 3.6157 that a recent numerical search reports in those dimensions.

Second, we compute the quasimaximal relative projection constants $\mu_{\mathbb{R}}(m, N)$ exhaustively for $1 \leq m \leq N \leq 11$ in exact arithmetic. This reproduces every printed digit of Foucart and Skrzypek's table for $N \leq 10$, supplies a closed algebraic form for each entry, and adds the row $N = 11$, which their method could not reach; two of the new cells are exactly rational, $\mu_{\mathbb{R}}(5, 11) = 2$ and $\mu_{\mathbb{R}}(6, 11) = 23/11$, with exactly rational spectral projections.

Every lower bound here is certified by an explicit integer matrix M and an integer $D > 0$ with $M = M^T$, $M^2 = DM$ and $\text{tr } M = mD$, and every such certificate is machine-checked in Lean 4. We claim nothing about the upper half of the conjecture, and no asymptotic record: $\max_k v(n, k)$ grows like $\sqrt{m}$ with a constant below the one König, Lewis and Lin reached in 1983.

1. INTRODUCTION

1.1. The objects. Let Y be a Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. For $Y \subseteq X$ write $\lambda(Y, X) = \inf\{\|P\| : P: X \rightarrow Y \text{ a projection}\}$, and let $\lambda(Y) = \sup\{\lambda(Y, X) : Y \subseteq X\}$ be the *absolute projection constant* of Y . The quantity this paper is about is the maximum of λ over a dimension,

$$\lambda_{\mathbb{K}}(m) := \sup\{\lambda(Y) : \dim Y = m\},$$

the *maximal absolute projection constant*. Kadec and Snobar [8] give $\lambda_{\mathbb{K}}(m) \leq \sqrt{m}$, and König [11] showed that the order is sharp. Beyond the trivial $m = 1$, the exact values recorded over $\mathbb{R}$ are those at $m = 2, 3, 7, 23$ — namely $4/3$, $(1 + \sqrt{5})/2$, $5/2$, $14/3$ [6, Theorem 1.5] — the four dimensions in which a maximal real equiangular tight frame is known to exist.

Everything below rests on one classical reduction. Writing $\lambda_{\mathbb{K}}(m, N)$ for the supremum of $\lambda(Y, \ell_{\infty}^N)$ over the m -dimensional subspaces Y of $\ell_{\infty}^N(\mathbb{K})$, Chalmers and Lewicki [2] proved that for integers $N \geq m$

$$(1) \quad \lambda_{\mathbb{K}}(m, N) = \max\left\{\sum_{i,j=1}^N t_i t_j |U^* U|_{ij} : t \in \mathbb{R}_+^N, \|t\|_2 = 1, U \in \mathbb{K}^{m \times N}, UU^* = I_m\right\},$$

and $\lambda_{\mathbb{K}}(m) = \sup_{N \geq m} \lambda_{\mathbb{K}}(m, N)$. We take (1) in the form in which Dereęowska and Lewandowska state it, as [6, Theorem 1.1], where it is attributed to [2]; [7, Theorem 1] give a proof valid over $\mathbb{C}$ as well, together with the equivalent expression in terms of Seidel matrices used in (4) below.

Specialising t to the uniform vector $(1/\sqrt{N}, \dots, 1/\sqrt{N})$ in (1) produces the *quasimaximal relative projection constant*

$$(2) \quad \mu_{\mathbb{K}}(m, N) := \max \left\{ \frac{1}{N} \sum_{i,j=1}^N |U^* U|_{ij} : U \in \mathbb{K}^{m \times N}, UU^* = I_m \right\} \leq \lambda_{\mathbb{K}}(m, N) \leq \lambda_{\mathbb{K}}(m).$$

The condition $UU^* = I_m$ says that the columns of U form a Parseval frame, and then $G := U^* U$ is precisely the orthogonal projection of $\mathbb{K}^N$ onto the row space of U ; conversely every rank- m orthogonal projection arises this way. So, writing $\mathcal{P}_m(N)$ for the set of rank- m orthogonal projections of $\mathbb{R}^N$ and $\|G\|_1 = \sum_{i,j} |G_{ij}|$,

$$(3) \quad \mu_{\mathbb{R}}(m, N) = \frac{1}{N} \max_{G \in \mathcal{P}_m(N)} \|G\|_1, \quad \text{so} \quad \lambda_{\mathbb{R}}(m) \geq \frac{\|G\|_1}{N} \text{ for every } G \in \mathcal{P}_m(N).$$

Inequality (3) is the only bridge to functional analysis that this paper uses: past it, everything is a statement about explicit matrices. Basso [1, Theorem 1.2] proved the converse direction $\lambda_{\mathbb{R}}(m) = \sup_N \mu_{\mathbb{R}}(m, N)$, which we quote but never use.

Over $\mathbb{R}$ the maximum in (2) is finite: writing $A = I_N + B$ with B a *Seidel matrix* (symmetric, zero diagonal, $|B_{ij}| = 1$ off the diagonal) and $\lambda_1^\downarrow \geq \lambda_2^\downarrow \geq \dots$ for the ordered spectrum,

$$(4) \quad \mu_{\mathbb{R}}(m, N) = \max \left\{ \frac{1}{N} \sum_{k=1}^m \lambda_k^\downarrow(A) : A = I_N + B, B \text{ a Seidel matrix of order } N \right\}.$$

Since $B \mapsto DBD$ with D diagonal of signs is a similarity, the maximum runs over the *two-graphs* of order N , that is over switching classes of graphs, of which there are 16, 54, 243, 2038, 33120, 1182004 for $N = 6, \dots, 11$ (sequence A002854 of [13]). This is the route by which Foucart and Skrzypek [7] tabulated $\mu_{\mathbb{R}}(m, N)$ for $1 \leq m \leq N \leq 10$, and the route by which Basso [1] obtained his formula.

1.2. The source, and its questions. Deręgowska and Lewandowska [6] take the $m(m+1)/2$ midpoints of the edges of a regular simplex in $\mathbb{R}^m$, show that this configuration V_m is a biangular Parseval frame with $\lambda_{\mathbb{R}}(V_m) = \mu_{\mathbb{R}}(V_m) = 4(m-2)/(m+1)$, and conclude

$$(5) \quad \lambda_{\mathbb{R}}(m) \geq \frac{4(m-2)}{m+1} \quad (m \geq 3).$$

At $m = 7$ this is the known value $\lambda_{\mathbb{R}}(7) = 5/2$; at $m = 6$ and $m = 8$ it gives $16/7$ and $8/3$, and they conjecture that these two are exact:

$$\lambda_{\mathbb{R}}(6) \stackrel{?}{=} \frac{16}{7}, \quad \lambda_{\mathbb{R}}(8) \stackrel{?}{=} \frac{8}{3}.$$

Two open ends in this picture are the ones we address. The first is that the family (5) is bounded by 4, so it says nothing at all in large dimensions; the second is that the exhaustive data on $\mu_{\mathbb{R}}(m, N)$ stops early. Foucart and Skrzypek write, of their own computation, that the number of Seidel matrices “still limited our computations to $N = 10$ ” [7, §3], and their Appendix B tabulates $\mu_{\mathbb{R}}(m, N)$, $1 \leq m \leq N \leq 10$, to four decimals with no closed forms.

1.3. What is settled here. We work only on the lower half: no upper bound for any $\lambda_{\mathbb{R}}(m)$ is proved or improved here, and the conjecture of [6] is untouched.

Theorem A. *Let $n \geq 3$ and $1 \leq k \leq n-1$; index $\mathbb{R}^N$, $N = \binom{n}{k}$, by the k -subsets of $[n]$ and put $M_{P,Q} = n|P \cap Q| - k^2$. Then $M^2 = n \binom{n-2}{k-1} M$ and $\text{tr } M = \binom{n}{k} k(n-k)$, so $G = M / (n \binom{n-2}{k-1})$ is an orthogonal projection of $\mathbb{R}^N$ of rank $n-1$, and*

$$\lambda_{\mathbb{R}}(n-1) \geq v(n, k) = \frac{1}{n \binom{n-2}{k-1}} \sum_{t=0}^k \binom{k}{t} \binom{n-k}{k-t} |nt - k^2|.$$

Here $v(n, 2) = 4(n-3)/n$ is exactly the bound (5), while $v(n, 3) = 6(n-4)(n-5)/(n(n-2))$ for $n \geq 9$; and for $n \geq 9$ one has $v(n, 3) > v(n, 2)$ precisely when $n^2 - 17n + 48 > 0$, that is for every $n = n-1 \geq 13$.

The frame behind Theorem A is not new: it is Datta and Oldroyd's k -angle tight frame [4], obtained by normalising the sums of k vectors of the regular simplex equiangular tight frame, and the identity $M^2 = n \binom{n-2}{k-1} M$ is their tightness statement in matrix form. Their paper computes the usual ℓ^2 frame potential and never mentions projection constants; what is new here is the evaluation of $\sum_{i,j} |G_{ij}|$ and its use in (3). The two literatures appear not to have met: none of [6, 7, 1, 9, 14] mentions k -angle frames, hypersimplices or the Johnson scheme, and [4] does not contain the phrase “projection constant”.

Theorem B. $\lambda_{\mathbb{R}}(15) \geq 99/28 = 3.53571\dots$ and $\lambda_{\mathbb{R}}(16) \geq 312/85 = 3.67058\dots$, certified by explicit rational orthogonal projections of $\mathbb{R}^{560}$ and $\mathbb{R}^{680}$ of ranks 15 and 16. These exceed 3.5106 and 3.6157, the values reported for those dimensions in the last two rows of [14, Table 1], obtained there by a numerical local search.

Theorem C. The exhaustive maxima $\mu_{\mathbb{R}}(m, N)$, $1 \leq m \leq N \leq 11$, were computed in exact arithmetic (Computation 7 and Table 2). For $N \leq 10$ the values agree with [7, Appendix B, Table 1] in every printed digit; the closed algebraic forms and the row $N = 11$ are new. Two cells of that row are exactly rational,

$$\mu_{\mathbb{R}}(5, 11) = 2, \quad \mu_{\mathbb{R}}(6, 11) = \frac{23}{11},$$

with exactly rational spectral projections; for $\mu_{\mathbb{R}}(6, 11)$ the projection has common denominator 210 and satisfies the sign condition of [6] exactly.

Theorem C is stated as a computation, not as a formally verified theorem. For three cells — $(m, N) = (4, 10)$, $(5, 10)$ and $(6, 11)$ — the inequality half, an explicit projection together with its value, is machine-checked; the rest of the table, and in particular maximality over the 1182004 two-graphs of order 11, rests on an exhaustive floating-point sweep followed by an exact rational post-pass. Section 7 says exactly which is which.

1.4. Certificates. All the lower bounds above are produced by one device, which is what makes them checkable. A *certificate* on a finite index set ι with $|\iota| = N$ is a triple (M, D, m) with $M \in \mathbb{Z}^{\iota \times \iota}$, $D \in \mathbb{Z}_{>0}$, $m \in \mathbb{Z}$ and

$$(6) \quad M = M^T, \quad M^2 = D M, \quad \text{tr } M = m D.$$

Then $G := M/D$ is a symmetric idempotent real matrix of trace m , hence an orthogonal projection of rank m , and by (3)

$$(7) \quad \lambda_{\mathbb{R}}(m) \geq \mu_{\mathbb{R}}(m, N) \geq \text{value}(M, D) := \frac{\sum_{i,j} |M_{ij}|}{D N} \in \mathbb{Q}.$$

Every hypothesis in (6) is a finite integer identity, so a certificate is checkable by a proof kernel with no floating-point arithmetic anywhere; the whole content of (6)–(7) is verified in Lean 4 once and for all, and each concrete matrix below instantiates it. See Section 7.

2. PRELIMINARIES AND NOTATION

Throughout, $[n] = \{1, \dots, n\}$, $\binom{n}{k}$ is a binomial coefficient, J_n is the all-ones $n \times n$ matrix, I_n the identity, and

$$L_n := nI_n - J_n$$

the Laplacian of the complete graph K_n . We use repeatedly

$$(8) \quad J_n^2 = nJ_n, \quad L_n J_n = 0, \quad L_n^2 = nL_n.$$

For a set system $\mathcal{S} = \{P_1, \dots, P_N\}$ of subsets of $[n]$ we write $B \in \mathbb{Z}^{n \times N}$ for its incidence matrix, $B_{aP} = 1$ if $a \in P$ and 0 otherwise.

For $G \in \mathcal{P}_m(N)$ we abbreviate $\|G\|_1 = \sum_{i,j} |G_{ij}|$; the quantity bounded in (3) is $\|G\|_1/N$, and we call it the *value* of the projection, or of the Parseval frame whose Gram matrix it is. The notation

$\gamma(N, r)$, $\gamma(r)$ of [14] is the same object: $\gamma(N, r) = \mu_{\mathbb{R}}(r, N)$ and $\gamma(r) = \lambda_{\mathbb{R}}(r)$, so the numbers of [14, Table 1] are directly comparable with ours.

Finally, $\delta_{m,N} = \frac{m}{N} \left(1 + \sqrt{(N-1)(N-m)/m}\right)$ denotes the König–Lewis–Lin bound [10], which satisfies $\mu_{\mathbb{R}}(m, N) \leq \delta_{m,N}$ with equality if and only if a real equiangular tight frame of N vectors in $\mathbb{R}^m$ exists [7, Theorem 5]. It is used below only as a consistency check on computed data.

3. THE EDGE-MIDPOINT FRAME IN RATIONAL FORM

The frame V_m of [6] is given there by explicit vectors involving $\sqrt{m+1}$: with $n = m+1$,

$$v_{i,k} = \frac{e_i + e_k - \frac{2}{m}(1+n^{-1/2})\mathbf{1}_m}{\sqrt{m-1}} \quad (i < k \leq m), \quad v_{i,i} = \frac{e_i + \frac{1}{m}((m-1)n^{-1/2}-1)\mathbf{1}_m}{\sqrt{m-1}}.$$

Evaluating the inner products exactly in $\mathbb{Q}(\sqrt{n})$, every surd cancels.

Proposition 1. *Let $4 \leq m \leq 10$ and $n = m+1$. Re-index V_m by the 2-subsets of $[n]$, sending $v_{i,k}$ to $\{i, k\}$ and $v_{i,i}$ to $\{i, n\}$. Then the Gram matrix of V_m is rational and equals*

$$G_{P,Q} = \frac{n|P \cap Q| - 4}{n(n-2)}, \quad P, Q \in \binom{[n]}{2}.$$

Consequently $M := n(n-2)G$ has integer entries $M_{P,Q} = n|P \cap Q| - 4$ and satisfies $M^2 = n(n-2)M$, $\text{tr } M = n(n-1)(n-2)$ and $\sum_{P,Q} |M_{P,Q}| = 2n(n-1)(n-2)(n-3)$; so $(M, n(n-2), m)$ is a certificate of value $4(n-3)/n = 4(m-2)/(m+1)$. At $m = 7$ its value is $5/2 = \lambda_{\mathbb{R}}(7)$.

Proof sketch. The identification of the Gram matrix is a finite exact computation, carried out independently for each of $m = 4, \dots, 10$: the vectors above are represented as pairs (a, b) standing for $a + b\sqrt{n}$ with $a, b \in \mathbb{Q}$, all $\binom{n}{2}^2$ inner products are formed in $\mathbb{Q}(\sqrt{n})$, and in each case the coefficient of $\sqrt{n}$ vanishes identically and the resulting rational matrix agrees entrywise with the displayed formula. The matrix identities are then the case $k = 2$ of Theorem 3 below, whose proof is uniform in n ; the value is $4(n-2)(n-3)/(n(n-2))$. Each of the seven certificates $(M, n(n-2), m)$, $n = 5, \dots, 11$, is checked by kernel evaluation, as is each of the seven values. $\square$

The closed form of Proposition 1 is not in [6]; it is what the vectors of [6] compute to, and it is what makes the following bound a few hundred integer operations rather than an argument about irrational vectors. The bound itself is [6]’s.

Theorem 2. $\lambda_{\mathbb{R}}(6) \geq 16/7$ and $\lambda_{\mathbb{R}}(8) \geq 8/3$. The first is certified by the 21×21 integer matrix $M_{P,Q} = 7|P \cap Q| - 4$ on the 2-subsets of [7], with $M^2 = 35M$, $\text{tr } M = 210$ and $\sum |M_{P,Q}| = 1680$; the second by the 36×36 matrix $M_{P,Q} = 9|P \cap Q| - 4$ on the 2-subsets of [9], with $M^2 = 63M$, $\text{tr } M = 504$ and $\sum |M_{P,Q}| = 6048$.

Proof. Proposition 1 at $m = 6$ and $m = 8$; the values are $1680/(35 \cdot 21) = 16/7$ and $6048/(63 \cdot 36) = 8/3$. $\square$

These two inequalities are the content of (5) at $m = 6, 8$ and are due to [6]; the two values also appear, to four decimals, as the rows $r = 6$ and $r = 8$ of [14, Table 1]. What Theorem 2 adds is that the two numbers the conjecture of [6] is about are certified by a pair of explicit integer matrices, so that the lower half of that conjecture is checkable by hand or by machine and no longer depends on the geometry of the simplex.

4. HYPERSIMPLEX FRAMES

Nothing in Section 3 forces the “2” in “2-subsets”.

Theorem 3. *Let $n \geq 3$ and $1 \leq k \leq n-1$, put $N = \binom{n}{k}$, index $\mathbb{R}^N$ by the k -subsets of $[n]$, and let*

$$M_{P,Q} = n|P \cap Q| - k^2, \quad P, Q \in \binom{[n]}{k}, \quad D = n \binom{n-2}{k-1}.$$

Then:

- (1) $M^2 = D M$ and $\text{tr } M = \binom{n}{k} k(n-k) = (n-1)D$;
- (2) $G := M/D$ is an orthogonal projection of $\mathbb{R}^N$ of rank $n-1$;
- (3) $\frac{\|G\|_1}{N} = v(n, k) := \frac{1}{D} \sum_{t=0}^k \binom{k}{t} \binom{n-k}{k-t} |nt - k^2|$;
- (4) $\lambda_{\mathbb{R}}(n-1) \geq \mu_{\mathbb{R}}(n-1, \binom{n}{k}) \geq v(n, k)$.

Proof. Let $B \in \mathbb{Z}^{n \times N}$ be the incidence matrix of the k -subsets of $[n]$. Counting the k -sets through a point and through a pair of points,

$$(9) \quad BB^T = \binom{n-2}{k-1} I_n + \binom{n-2}{k-2} J_n,$$

and every column of B has weight k . By the second of these, $(B^T J_n B)_{P,Q} = k^2$, so $B^T L_n B = n B^T B - B^T J_n B$ has entries $n|P \cap Q| - k^2$, i.e. $M = B^T L_n B$. Now (8) kills the J_n term of (9):

$$M^2 = B^T L_n (BB^T) L_n B = \binom{n-2}{k-1} B^T L_n^2 B = n \binom{n-2}{k-1} M.$$

For the trace, $M_{P,P} = nk - k^2 = k(n-k)$, so $\text{tr } M = \binom{n}{k} k(n-k)$; and $\binom{n}{k} k = n \binom{n-1}{k-1}$ together with $\binom{n-1}{k-1} (n-k) = \binom{n-2}{k-1} (n-1)$ gives $\text{tr } M = (n-1)D$. Thus $(M, D, n-1)$ satisfies (6), which is (1) and (2). For (3), the number of k -subsets Q with $|P \cap Q| = t$ is $\binom{k}{t} \binom{n-k}{k-t}$, independently of P , so every row of M has the same ℓ^1 mass $\sum_t \binom{k}{t} \binom{n-k}{k-t} |nt - k^2|$; divide by D and by N . Part (4) is (7). $\square$

Geometrically the columns of G are the vertices of the hypersimplex $\Delta(n, k)$ recentred at the centroid and rescaled — equivalently, G is the projection onto the first nontrivial eigenspace V_1 of the Johnson scheme $J(n, k)$. As a frame this is exactly Datta and Oldroyd's construction [4, Theorem 4.7]: they normalise the sums $\sum_{j \in P} f_j$ of k vectors of the regular simplex equiangular tight frame $\{f_j\} \subseteq \mathbb{R}^{n-1}$ and compute $\langle g_P, g_Q \rangle = (|P \cap Q|n - k^2)/(k(n-k))$, which is the unit-normalised form of G ; their $KK^T = \binom{n-2}{k-1} I + \binom{n-2}{k-2} J$ [4, Lemma 4.9] is (9). Parts (1) and (2) of Theorem 3 are therefore theirs, in matrix language. Parts (3) and (4) — the entrywise ℓ^1 mass, and the resulting bound for $\lambda_{\mathbb{R}}$ — are what this paper adds.

Proposition 4. $v(n, 2) = \frac{4(n-3)}{n}$ for $n \geq 4$, and $v(n, 3) = \frac{6(n-4)(n-5)}{n(n-2)}$ for $n \geq 9$. Hence, for $n \geq 9$,

$$v(n, 3) > v(n, 2) \iff n^2 - 17n + 48 > 0 \iff n \geq 14,$$

i.e. $k=3$ improves on the bound (5) exactly in the dimensions $m = n-1 \geq 13$.

Proof. For $k=2$ and $n \geq 4$ the sum in Theorem 3(3) is $4\binom{n-2}{2} + 2(n-2)(n-4) + 2(n-2) = 4(n-2)(n-3)$ and $D = n(n-2)$. For $k=3$ and $n \geq 9$ (so that $|n-9| = n-9$) it is

$$9\binom{n-3}{3} + 3\binom{n-3}{2}(n-9) + 3(n-3)(2n-9) + 3(n-3) = 3(n-3)(n-4)(n-5),$$

while $D = n\binom{n-2}{2} = n(n-2)(n-3)/2$; divide. Finally $6(n-4)(n-5)/(n(n-2)) > 4(n-3)/n$ is $6(n-4)(n-5) > 4(n-3)(n-2)$, i.e. $2n^2 - 34n + 96 > 0$, whose roots are $(17 \pm \sqrt{97})/2 = 3.575\dots, 13.42\dots$. $\square$

The threshold is real, not an artefact of the estimate: at $m=8$, where (5) gives $8/3$, the $k=3$ frame on $\binom{9}{3} = 84$ vectors has value $v(9, 3) = 40/21 = 1.904\dots$, which is smaller. Table 1 lists the crossover range.

TABLE 1. The $k = 3$ hypersimplex frames against the bound (5) of [6]. Each value is certified by an explicit integer matrix of the size shown.

$m = n - 1$	$N = \binom{n}{3}$	$D = n \binom{n-2}{2}$	$v(n, 3)$	$4(m-2)/(m+1)$
13	364	924	$45/14 = 3.2142\dots$	$22/7 = 3.1428\dots$
14	455	1170	$44/13 = 3.3846\dots$	$16/5 = 3.2$
15	560	1456	$99/28 = 3.5357\dots$	$13/4 = 3.25$
16	680	1785	$312/85 = 3.6705\dots$	$56/17 = 3.2941\dots$
17	816	2160	$91/24 = 3.7916\dots$	$10/3 = 3.3333\dots$
18	969	2584	$1260/323 = 3.9009\dots$	$64/19 = 3.3684\dots$
19	1140	3060	4	$17/5 = 3.4$

Theorem 5. $\lambda_{\mathbb{R}}(15) \geq 99/28 = 3.535714\dots$ and $\lambda_{\mathbb{R}}(16) \geq 312/85 = 3.670588\dots$. Both are exact rationals, certified by the integer matrices $M_{P,Q} = 16|P \cap Q| - 9$ on the 560 three-subsets of [16], with $D = 1456$, and $M_{P,Q} = 17|P \cap Q| - 9$ on the 680 three-subsets of [17], with $D = 1785$. They exceed 3.5106 and 3.6157, the entries reported for $\gamma(15)$ and $\gamma(16)$ in the last two rows of [14, Table 1].

Proof. Theorem 3 and Proposition 4 at $(n, k) = (16, 3)$ and $(17, 3)$: $v(16, 3) = 6 \cdot 12 \cdot 11 / (16 \cdot 14) = 99/28$ and $v(17, 3) = 6 \cdot 13 \cdot 12 / (17 \cdot 15) = 312/85$. The comparisons are rational arithmetic: $99/28 - 3.5106 > 0.025$ and $312/85 - 3.6157 > 0.054$. $\square$

The values of [14, Table 1] are themselves lower bounds, obtained by a deterministic local search over weighted two-graph cores whose ancillary data file records that they are “best found by the recorded deterministic local search; not certified globally optimal”; Theorem 5 does not contradict them, it improves on them, and it does so with exact rationals rather than floating-point output. Their table stops at $r = 16$, so the entries of Table 1 at $m = 17, 18, 19$ have nothing to be compared with.

Remark 6 (no asymptotic record). The family of Theorem 3 is unbounded, in contrast with (5): at $m = n - 1$ the best value $\max_k v(n, k)$ is of order $\sqrt{m}$. This is *not* a record and we claim none. Numerically, and outside the formal development, $\max_k v(n, k) / \sqrt{m}$ equals 0.8864, 0.8783, 0.8671, 0.8608 at $m = 99, 199, 999, 9999$, with the optimal k near $\sqrt{n/2}$; approximating the hypergeometric sum in Theorem 3(3) by a Poisson sum suggests the limit $\sup_{\lambda > 0} \mathbb{E}|\text{Pois}(\lambda) - \lambda| / \sqrt{\lambda} = \sqrt{2/e} = 0.85776\dots$, attained at $\lambda = 1/2$. We do not prove this, and nothing above depends on it. König, Lewis and Lin [10, p. 342] already exhibit spaces with $\lambda(X_k) / \sqrt{k} \rightarrow \sqrt{4/5} \approx 0.894$, “which is the largest known value so far” as they put it, and König [11] reaches $\rightarrow 1$ — over $\mathbb{R}$ “for a subsequence of integers k ”. Those constructions live on sparse subsequences of the dimension, whereas $v(n, k)$ is an elementary explicit rational frame in *every* dimension; that coverage, not the constant, is what the family offers.

5. EXHAUSTIVE MAXIMA OVER TWO-GRAPHS

Computation 7. For every $1 \leq m \leq N \leq 11$ the value $\mu_{\mathbb{R}}(m, N)$ was determined exactly.

This is a computation carried out outside the proof assistant, and we report it as such; the certificates of Theorem 8 and Proposition 9 below are the parts of it that are machine-checked. The method is the one of [7, §3] and [1], in two passes.

Enumeration. By (4) the maximum runs over the two-graphs of order N , and every switching class has a representative in which a chosen vertex is isolated; so enumerating the isomorphism classes of graphs on $N - 1$ vertices covers every two-graph. Using **nauty-geng** [12] this consumed 1, 2, 4, 11, 34, 156, 1044, 12346, 274668, 12 005 168 graphs for $N = 2, \dots, 11$, matching sequence A000088 of [13] term by term — the completeness check on the enumeration. For each graph

a cyclic Jacobi sweep in double precision produced the spectrum of $A = I_N + B$, and for each m a running maximum of $\sum_{k \leq m} \lambda_k^\downarrow(A)$ and one argmax were kept.

Exact post-pass. For each argmax the integer characteristic polynomial of A was computed by Faddeev–Le Verrier over $\mathbb{Q}$, all N roots were isolated by Sturm sequences to width 10^{-28} with rational endpoints, and the exact top- m eigenvalue sum was identified; when the set of the m largest roots is Galois-stable, that is when the corresponding monic factor has integer coefficients, the exact rational spectral projection was recovered from a Bézout identity $uf + vg = 1$ as $G = (vg)(A)$. This is what produces the closed forms of Table 2. The whole computation took about a tenth of a CPU-hour for the enumeration and about ten CPU-minutes for the exact pass.

TABLE 2. The exhaustive maxima $\mu_{\mathbb{R}}(m, N)$ for $1 \leq m \leq N \leq 11$, in exact form (Computation 7). The rows $m = 1$ and $m = N$, where $\mu_{\mathbb{R}} = 1$, are omitted. A star marks a value that is algebraic of degree > 2 over $\mathbb{Q}$; such a value is pinned by the characteristic polynomial of the maximising Seidel matrix together with “the sum of the m largest roots”, and only its numerical value is printed here. The block $N \leq 10$ agrees with [7, Appendix B, Table 1] in every printed digit of every cell; the exact forms, and the whole row $N = 11$, are new. The two boldface cells are Theorem 8.

N	m	$\mu_{\mathbb{R}}(m, N)$	numeric	N	m	$\mu_{\mathbb{R}}(m, N)$	numeric
3	2	$4/3$	1.333333	9	4	*	1.802045
4	2	$(3 + \sqrt{5})/4$	1.309017	9	5	*	1.913156
4	3	$3/2$	1.500000	9	6	*	1.922276
5	2	$(9 + \sqrt{17})/10$	1.312311	9	7	$17/9$	1.888889
5	3	$(11 + \sqrt{17})/10$	1.512311	9	8	$16/9$	1.777778
5	4	$8/5$	1.600000	10	2	$(4 + \sqrt{7})/5$	1.329150
6	2	$4/3$	1.333333	10	3	*	1.593624
6	3	$(1 + \sqrt{5})/2$	1.618034	10	4	$(5 + 3\sqrt{2})/5$	1.848528
6	4	$5/3$	1.666667	10	5	2	2.000000
6	5	$5/3$	1.666667	10	6	$(6 + 3\sqrt{2})/5$	2.048528
7	2	$(11 + \sqrt{57})/14$	1.324988	10	7	*	1.993624
7	3	*	1.596243	10	8	$(7 + \sqrt{7})/5$	1.929150
7	4	*	1.739100	10	9	$9/5$	1.800000
7	5	$(17 + \sqrt{57})/14$	1.753560	11	2	$(19 + \sqrt{105})/22$	1.329407
7	6	$12/7$	1.714286	11	3	*	1.604167
8	2	$(7 + \sqrt{13})/8$	1.325694	11	4	*	1.838037
8	3	*	1.587521	11	5	2	2.000000
8	4	*	1.789219	11	6	23/11	2.090909
8	5	*	1.837521	11	7	*	2.110764
8	6	$(11 + \sqrt{13})/8$	1.825694	11	8	*	2.058712
8	7	$7/4$	1.750000	11	9	$(33 + \sqrt{105})/22$	1.965770
9	2	$4/3$	1.333333	11	10	$20/11$	1.818182
9	3	*	1.588943				

5.1. What the table reproduces, and what is new. For $N \leq 10$ every cell of Table 2 agrees with [7, Appendix B, Table 1] in all four printed digits — the comparison was made against the authors’ posted full version of [7], cited there. That block is a replication — an independent implementation agreeing with a computation published in 2017 — and we claim it as nothing else. What is new in Computation 7 is (i) the exact algebraic value of every cell, where [7] print four decimals and give no closed form anywhere, and (ii) the entire row $N = 11$, beyond the limit [7, §3] states for their own computation.

5.2. Consistency checks. The table is over-determined, and passes every independent check we know of.

- $\mu_{\mathbb{R}}(2, N) = 4/3 = \lambda_{\mathbb{R}}(2)$ exactly for $N = 3, 6, 9$ and strictly less for $N = 4, 5, 7, 8, 10, 11$; $\mu_{\mathbb{R}}(3, 6) = (1 + \sqrt{5})/2 = \lambda_{\mathbb{R}}(3)$ and $\mu_{\mathbb{R}}(3, N) < (1 + \sqrt{5})/2$ for $7 \leq N \leq 11$. Kobos [9] proves that if a maximal equiangular tight frame exists in $\mathbb{K}^n$ then $\mu_{\mathbb{K}}(n, N) = \lambda_{\mathbb{K}}(n)$ if and only if $d_{\mathbb{K}}(n) = n(n+1)/2$ divides N ; for $n = 2$ and $n = 3$ these are $3 \mid N$ and $6 \mid N$, which is exactly the pattern above.
- $\mu_{\mathbb{R}}(3, 5) = (11 + \sqrt{17})/10 = 1.5123\dots$ sits strictly below the exact $\lambda_{\mathbb{R}}(3, 5) = (5 + 4\sqrt{2})/7 = 1.5224\dots$ of [2], quoted in [7, Remark 3]; this same pair, with the same numerical value $\mu_{\mathbb{R}}(3, 5) \approx 1.5123$, is the illustration [7, §3] give that $\mu < \lambda$ in general.
- $\mu_{\mathbb{R}}(4, 6) = 5/3$, matching Basso's $\Pi(4, 6) = 5/3$ [1], which is a maximum over *weighted* Seidel matrices and hence an upper bound for $\mu_{\mathbb{R}}(4, 6)$.
- $\mu_{\mathbb{R}}(5, 10) = 2 = \delta_{5,10}$, attained at the regular two-graph on ten vertices, i.e. at the real equiangular tight frame of 10 vectors in $\mathbb{R}^5$; there A has spectrum $4^{(5)}, (-2)^{(5)}$.
- The complementation duality of Remark 10 holds in every row.

5.3. The two rational cells of the new row.

Theorem 8. $\mu_{\mathbb{R}}(5, 11) = 2$ and $\mu_{\mathbb{R}}(6, 11) = 23/11$ (Computation 7), and in both cases the exact post-pass returns a rational spectral projection for the maximising Seidel matrix. Of the two, only the cell $(6, 11)$ carries a certificate: there $G = M/210$ with

$$M = \begin{pmatrix} 111 & 29 & 41 & 29 & 29 & 29 & 41 & 29 & 29 & 41 & 30 \\ 29 & 111 & 29 & 41 & -29 & -29 & 29 & 41 & -29 & -41 & -30 \\ 41 & 29 & 111 & 29 & -41 & 29 & -29 & -41 & 29 & -29 & 30 \\ 29 & 41 & 29 & 111 & 41 & -29 & -41 & -29 & -29 & 29 & -30 \\ 29 & -29 & -41 & 41 & 111 & 41 & 29 & -29 & -29 & 29 & -30 \\ 29 & -29 & 29 & -29 & 41 & 111 & 29 & -29 & 41 & -41 & -30 \\ 41 & 29 & -29 & -41 & 29 & 29 & 111 & 29 & -41 & -29 & 30 \\ 29 & 41 & -41 & -29 & -29 & -29 & 29 & 111 & 41 & 29 & -30 \\ 29 & -29 & 29 & -29 & -29 & 41 & -41 & 41 & 111 & 29 & -30 \\ 41 & -41 & -29 & 29 & 29 & -41 & -29 & 29 & 29 & 111 & 30 \\ 30 & -30 & 30 & -30 & -30 & -30 & 30 & -30 & -30 & 30 & 150 \end{pmatrix},$$

a certificate in the sense of (6) with $D = 210$, $M^2 = 210M$, $\text{tr } M = 1260$ and $\sum_{i,j} |M_{ij}| = 4830$, so that $\lambda_{\mathbb{R}}(6) \geq 4830/(210 \cdot 11) = 23/11$. Moreover the sign condition of [6] holds exactly: $A := \text{sgn}(G)$ is the Seidel matrix $I + B$ of the maximising two-graph, and $\text{tr}(AM) = \sum_{i,j} |M_{ij}|$.

At $m = 5$ the certified projection is one order lower: the regular two-graph on ten vertices gives a rational rank-5 projection of $\mathbb{R}^{10}$, with common denominator 6, of value 2 and with the same sign condition, so that $\lambda_{\mathbb{R}}(5) \geq 2$.

Proof sketch. The two equalities are the corresponding cells of Computation 7: the maximum of $\frac{1}{11} \sum_{k \leq m} \lambda_k^\downarrow(I_{11} + B)$ over the 1 182 004 two-graphs of order 11 was found by the exhaustive sweep described above, and its exact value was then recovered by the Sturm and Bézout pass. For $m = 6$ the top six eigenvalues of the maximiser are 4, 4, 4, 4, 4, 3, a Galois-stable set, so the projection is rational, and clearing denominators gives the displayed M . That M is a certificate of value $23/11$, and both halves of the sign condition, are finite integer identities and are machine-checked; hence the inequality $\lambda_{\mathbb{R}}(6) \geq 23/11$ depends on no computation outside the proof kernel, while the equality $\mu_{\mathbb{R}}(6, 11) = 23/11$ depends on the enumeration. The order-10 rank-5 certificate is likewise machine-checked; the equality $\mu_{\mathbb{R}}(5, 11) = 2$ is not, and rests on the enumeration alone. $\square$

Both cells matter because $N = 11$ is where the enumeration stops: $\mu_{\mathbb{R}}(6, 11) = 23/11 = 2.0909\dots$ is still below the simplex bound $16/7 = 2.2857\dots$ of Theorem 2, so the exhaustive search over all

two-graphs of order at most 11 does not reach the value conjectured for $\lambda_{\mathbb{R}}(6)$ — as it cannot, since the configuration of [6] that attains $16/7$ consists of 21 vectors.

Proposition 9. $\mu_{\mathbb{R}}(4, 10) = (5 + 3\sqrt{2})/5 = 1.8485281374\dots$, the maximum over the 33 120 two-graphs of order 10. An explicit rational rank-4 certificate of order 10 has value $1.848528130419\dots$, whence $\lambda_{\mathbb{R}}(4) \geq 1.8485281304$.

Proof sketch. The cell $(m, N) = (4, 10)$ of Computation 7. The maximiser’s top-4 eigenvalue sum is irrational, so no rational spectral projection exists; the certificate is instead $G = X(X^{\top}X)^{-1}X^{\top}$ for an integer rounding X of the top-4 eigenvectors at scale 10^4 . Because $G \mapsto \text{tr}(AG)$ is maximised over the Grassmannian at the true spectral projection, a rounding of size ε costs $O(\varepsilon^2)$; here it costs $7 \cdot 10^{-9}$. The certificate is machine-checked, as are the comparisons $1.848528 < \text{value} < 1.8485282$ and the exactness of its sign pattern $\text{sgn}(G) = A$. $\square$

The numeric 1.8485 is [7, Table 1], cell $(N, m) = (10, 4)$; what is added here is the closed form and the certificate. We stress what this is *not*: it is not a record for $\lambda_{\mathbb{R}}(4)$. [7, Table 2] reports $\lambda_{\mathbb{R}}(4, 10) \geq 1.8496$ and [14, Table 1] reports $\gamma(4) \geq 1.8500$, both from numerical optimisation over the weights t of (1); the best closed-form proved lower bound in the literature is $\lambda_{\mathbb{R}}(4) \geq \Pi(4, 6) = 5/3$, the exact value computed in [1], whose lower half is attributed there to Sokołowski. Proposition 9 contradicts none of these, and the development contains a control refuting the two larger claims for this certificate (Proposition 13).

Remark 10 (complementation duality). Foucart and Skrzypek [7, Proposition 2, item 2) β)] prove $\tilde{\mu}_{\mathbb{K}}(N - m, N) = \tilde{\mu}_{\mathbb{K}}(m, N)$ for $\tilde{\mu}_{\mathbb{K}}(m, N) = \mu_{\mathbb{K}}(m, N) - m/N$, i.e.

$$\mu_{\mathbb{R}}(N - m, N) = \mu_{\mathbb{R}}(m, N) + \frac{N - 2m}{N},$$

by the argument $B \mapsto -B$. Every row of Table 2 satisfies it; this is a check on the enumeration, not a claim, and it halves the work of extending the table.

Remark 11 (the next row, priced). Extending Table 2 to $N = 12$ by the same route needs the 1 018 997 864 isomorphism classes of graphs on 11 vertices, 84.9 times the input at $N = 11$; at the measured rate that is about 8 CPU-hours for the eigenvalue sweep and 2–4 for the generation. Two routes would bring it down: restricting to K_{m+2} -free two-graphs, which [1] shows suffices for the supremum over N and which prunes the eigenvalue work; or generating switching classes directly, of which there are 87 723 296 of order 12 against $1.02 \cdot 10^9$ graphs, an 11.6-fold saving. This is an engineering measurement, not a mathematical statement.

6. CONTROLS

A development in which every theorem is an identity about an explicit integer matrix can be satisfied by a misstated hypothesis while proving nothing, so the controls are part of the result.

Proposition 12. Run on the exhaustive maximisers at $(m, N) = (2, 3)$ and $(3, 6)$, the certificate machinery returns $\lambda_{\mathbb{R}}(2) \geq 4/3$ exactly, and at $(3, 6)$ a rational certificate whose value is machine-checked to lie in $(1.618033, 1.618034)$ and falls short of $(1 + \sqrt{5})/2$ by $2.27 \cdot 10^{-7}$. These are the known exact values $\lambda_{\mathbb{R}}(2) = 4/3$ (Grünbaum’s conjecture, proved in [3]) and $\lambda_{\mathbb{R}}(3) = (1 + \sqrt{5})/2$, from the six equiangular lines in $\mathbb{R}^3$.

Proposition 13. The following are machine-checked, each refuting a way the certificates above could have been vacuous or misstated.

- (1) The $m = 6$ simplex certificate of Theorem 2 has value neither $7/3 > 16/7$ nor $9/4 < 16/7$; the certificate of Proposition 9 has value neither ≥ 1.85 nor $5/3$.
- (2) The denominators are not free: the 21×21 simplex matrix does not satisfy $M^2 = 34M$, and the order-10 rank-5 matrix satisfies $M^2 = DM$ for $D = 6$ and for no other $D < 200$.

- (3) *The shape of the entry formula does work: replacing the constant 4 by 3 in $M_{P,Q} = 7|P \cap Q| - 4$ leaves a matrix that is a multiple of an idempotent for no $D < 200$.*
- (4) *The idempotence hypothesis of (6) is not vacuous for matrices of this shape: the Seidel matrix $A = I + B$ of the order-10 regular two-graph satisfies $A^2 = DA$ for no $D < 400$.*
- (5) *The crossover of Proposition 4 is a genuine threshold: at $m = 8$ the $k = 3$ frame has value $40/21 < 8/3$, and at $m = 13$ the $k = 3$ value $45/14$ is $> 22/7$ but $< 10/3$.*

7. VERIFICATION

Every matrix identity stated in this paper has been formally verified in the Lean 4 proof assistant [15] (toolchain v4.32.0) against *Mathlib* [16]; the development contains no `sorry` and no hand-written axiom. The repository is private; repository reference to be added at submission. What is verified, and what is not, is worth stating precisely, because the two halves of this paper have different status.

Kernel-clean, and general. The bridge (6) $\Rightarrow$ (7) — that an integer (M, D, m) with M symmetric, $M^2 = DM$ and $\text{tr } M = mD$ yields a real symmetric idempotent M/D of trace m whose entrywise ℓ^1 mass divided by N is the rational value $\text{value}(M, D)$ — and the whole algebra behind Theorem 3, namely (8), the entry formula for $B^\top L_n B$, and the implication

$$BB^\top = dI_n + cJ_n \implies (B^\top L_n B)^2 = (nd) (B^\top L_n B) \quad \text{for all } n, N, d, c,$$

depend on nothing beyond Lean’s three standard axioms: `propext`, `Classical.choice` and `Quot.sound`. So the mechanism behind Theorem 3 is verified once, uniformly in the parameters n , N and k , and not instance by instance.

Checked by compiled evaluation. Everything numerical — each certificate’s symmetry, idempotence and trace; each value; each comparison of Theorems 2, 5, 8 and Propositions 1, 9, 12, 13; the sign conditions; and the two finite inputs (9) and “every column of B has weight k ” for each concrete (n, k) — is decided inside the proof assistant by compiled evaluation (Lean’s `native_decide`), so each such statement is a theorem carrying, in addition to the three standard axioms, the compiled-evaluation axiom generated for that declaration. No external program’s output is trusted. The instances certified are $(n, k) = (7, 2)$ and $(n, 3)$ for $n = 9, 14, 15, 16, 17, 18, 19, 20$ on the hypersimplex side, the seven simplex certificates $n = 5, \dots, 11$, and the five Seidel certificates at $(m, N) = (2, 3), (3, 6), (4, 10), (5, 10), (6, 11)$. The largest index set has $N = 1140$ elements; no quantity of size N^3 is ever evaluated, which is what makes that size affordable. The whole development compiles in about 0.85 CPU-hours.

Cited, not formalised. *Mathlib* has no projection constants, so (3) — the Chalmers–Lewicki formula (1) together with the identification of rank- m orthogonal projections with Gram matrices of Parseval frames — is quoted from [2, 6] and is the one step of every lower bound above that is not machine-checked. Each formal statement is therefore an assertion about an explicit finite matrix, and the passage to $\lambda_{\mathbb{R}}(m)$ is by (3).

Outside the development. Computation 7 and Table 2 are not formalised: the exhaustive sweep over two-graphs is a C program using double-precision eigenvalues, and the exact post-pass is a rational-arithmetic program. What the kernel checks of that computation is the inequality half of the cells (4, 10), (5, 10) and (6, 11), through the certificates of Theorem 8 and Proposition 9. The identification of the Gram matrix in Proposition 1 is likewise an external computation, though an exact one, in $\mathbb{Q}(\sqrt{n})$ rather than in floating point. The numerics of Remark 6 and the cost figures of Remark 11 are measurements.

ACKNOWLEDGEMENT OF PRIOR WORK

The frame of Theorem 3 is Datta and Oldroyd’s [4] — see also [5] by the same authors — and the bound (5) is Deręowska and Lewandowska’s [6]; the block $N \leq 10$ of Table 2 replicates Foucart and Skrzypek [7], and the reduction to two-graphs is theirs and Basso’s [1].

REFERENCES

- [1] G. Basso, *Computation of maximal projection constants*, J. Funct. Anal. **277** (2019), no. 10, 3560–3585; doi:10.1016/j.jfa.2019.05.011; arXiv:1901.07866. Theorem numbers quoted here are those of arXiv:1901.07866v1, where the two-graph formula is Theorem 1.2 and the existence of a polyhedral maximiser is Theorem 1.4. See also the author’s correction, *Corrigendum to “Computation of maximal projection constants”*, J. Funct. Anal. **287** (2024), no. 5, Paper No. 110491; doi:10.1016/j.jfa.2024.110491 (posted as arXiv:2402.06672 under the title *Erratum to ...*). It withdraws Lemma 3.1(2) and with it the proof of Theorem 1.4, and states that “[t]he other results of the article, in particular the alternative proof of Grünbaum’s conjecture, do not depend on Lemma 3.1 and are thus not affected by this error”; the two-graph formula quoted here is among those.
- [2] B. L. Chalmers and G. Lewicki, *Three-dimensional subspace of $\ell_\infty^{(5)}$ with maximal projection constant*, J. Funct. Anal. **257** (2009), no. 2, 553–592; doi:10.1016/j.jfa.2009.01.005. Formula (1) is quoted here in the form given by [6, Theorem 1.1] and [7, Theorem 1], both of which attribute it to this paper; we quote no statement number from the paper itself.
- [3] B. L. Chalmers and G. Lewicki, *A proof of the Grünbaum conjecture*, Studia Math. **200** (2010), no. 2, 103–129; doi:10.4064/sm200-2-1.
- [4] S. Datta and J. Oldroyd, *Construction of k -angle tight frames*, Numer. Funct. Anal. Optim. **37** (2016), no. 8, 975–989; doi:10.1080/01630563.2016.1176580; arXiv:1605.09429. Statement numbers quoted here are those of arXiv:1605.09429v2.
- [5] S. Datta and J. Oldroyd, *Properties of certain multi-angle tight frames in $\mathbb{R}^d$* , Linear Algebra Appl. **650** (2022), 75–91; doi:10.1016/j.laa.2022.05.022.
- [6] B. Deręgowska and B. Lewandowska, *Maximal projection constants and extremal vector configurations: some conjectures and examples*, arXiv:2608.08695v1 (9 August 2026). All statement numbers quoted here are those of v1, which was still the only version on 3 September 2026.
- [7] S. Foucart and L. Skrzypek, *On maximal relative projection constants*, J. Math. Anal. Appl. **447** (2017), no. 1, 309–328; doi:10.1016/j.jmaa.2016.09.066. The paper is not on arXiv; the text quoted here, and the table compared with in Section 5, are those of the authors’ posted full version, [foucart.github.io/publi/MaxProjCst_rev.pdf](https://github.com/foucart/publi/MaxProjCst_rev.pdf).
- [8] M. I. Kadec and M. G. Snobar, *Certain functionals on the Minkowski compactum* (Russian), Mat. Zametki **10** (1971), 453–457; English translation: Math. Notes **10** (1971), 694–696.
- [9] T. Kobos, *Spaces with the maximal projection constant revisited*, arXiv:2505.24526.
- [10] H. König, D. R. Lewis and P.-K. Lin, *Finite dimensional projection constants*, Studia Math. **75** (1983), no. 3, 341–358; doi:10.4064/sm-75-3-341-358.
- [11] H. König, *Spaces with large projection constants*, Israel J. Math. **50** (1985), no. 3, 181–188; doi:10.1007/BF02761398.
- [12] B. D. McKay and A. Piperno, *Practical graph isomorphism, II*, J. Symbolic Comput. **60** (2014), 94–112; doi:10.1016/j.jsc.2013.09.003.
- [13] OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, <https://oeis.org>. The sequences used here are A000088 (simple graphs on n unlabelled nodes) and A002854 (two-graphs on n nodes).
- [14] V. Sivashankar, Q. Tang and T. Wakhare, *Graph eigenvalues and projection constants*, arXiv:2608.02429v1 (3 August 2026), with the ancillary files `anc/weighted_cores.json` and `anc/verify_weighted_cores.py`; v1 was still the only version on 3 September 2026.
- [15] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, in: Automated Deduction — CADE 28, Lecture Notes in Computer Science 12699, Springer, 2021, pp. 625–635; doi:10.1007/978-3-030-79876-5_37.
- [16] The mathlib Community, *The Lean mathematical library*, in: CPP 2020, ACM, 2020, pp. 367–381; doi:10.1145/3372885.3373824.