THE LADDER OF LARGEST PRIMITIVE DETERMINANTS OF 3×3 MATRICES OVER $\{0, 1, \ldots, k\}$

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Abstract. For $A \in \mathbb{Z}^{3\times 3}$ let $g(A)$ be the greatest common divisor of the nine 2×2 minors of A; call A *primitive* when $g(A) = 1$, and let S_k be the set of absolute determinants of primitive matrices with entries in $\{0, 1, \ldots, k\}$. Finch, in the course of determining how close to singular a unimodular 4×4 matrix can be, was led to S_k; he reported that its five largest elements appear to be $k(2k^2 - 2k + 1)$, $2k^2(k-1)$, $k^3 + (k-1)^3$, $k(2k^2 - 3k + 2)$, $(2k^2 - k + 1)(k-1)$, confirmed that nothing lies between them up through $k = 7$, and invited an extension of the search to $k = 8, 9, \ldots$. We carry that extension out for $8 \leq k \leq 20$. We then continue the ladder by three further levels, $k(2k-1)(k-1)$, $k^2(2k-3)$ and $k^3 + (k-1)^3 - (k-1)^2$, exact for $5 \leq k \leq 20$, and continue the descent below them by four more. The last of these is not a single polynomial: writing $N = N(k) = 2k(k^2 - 2k + 2)$, we show that $N \in S_k$ for every odd $k \geq 3$ while $N \notin S_k$ for the even k of the swept range $6 \leq k \leq 20$, so that there the ninth largest primitive determinant is N for odd k and $N - 1$ for even k. The obstruction is exhibited: a matrix of determinant N carries the minors k^2 and $(k-1)(k-2)$, whose gcd is 1 exactly when k is odd. Every listed value is attained by an explicit matrix template valid for all k, not merely for the range searched. The enabling tool is a proved pruning bound that replaces the $(k+1)^9$ box by a sweep over its $(k+1)^6$ row pairs, of which exactly 504 survive at every k in the measured range. Every theorem below is machine-checked in Lean 4; the few measurements quoted outside the formal development are labelled as such.

1. Introduction

Let $M \in \mathbb{Z}^{n\times n}$ be *unimodular*, $\det M = \pm 1$, and let A be an $(n-1) \times (n-1)$ submatrix of M. Since $M^{-1} = \pm \operatorname{adj}(M)$, the entries of M^{-1} are exactly the signed $(n-1) \times (n-1)$ minors of M, so asking how large $\|M^{-1}\|$ can be, for M with entries bounded by k, is asking how large $|\det A|$ can be. Finch [1] settles this for $n = 4$: over $M \in \{0, 1, \ldots, k\}^{4\times 4}$ the answer is $\beta_k^+ = k(2k^2 - 2k + 1)$, and over $M \in \{-k, \ldots, k\}^{4\times 4}$ it is $k(2k-1)(2k+1)$; a companion paper [2] treats the zerofree variant, in which neither M nor M^{-1} is allowed a zero entry.

A single arithmetic constraint drives that analysis. Writing $g(A)$ for the greatest common divisor of the nine 2×2 minors of a 3×3 integer matrix A, Finch's Lemma E states that $g(A) = 1$ whenever A is an $(n-1) \times (n-1)$ submatrix of a unimodular M; the proof is a bordering identity. So the 3×3 matrices relevant to the 4×4 problem are the *primitive* ones. This is a genuine restriction. Over $\{0, 1, \ldots, k\}$ the largest determinant of any 3×3 matrix is the Hadamard-type ceiling $2k^3$ — the maximal determinant problem for matrices with bounded entries is classical; see Ehlich and Zeller [3] for the $(0, 1)$ case, and A003432 in [4] for the small-order values — and [1, § Closing Words] shows that the only values in the window $(\beta_k^+, 2k^3]$ are $2k^3$ and $2k^3 - k^2$, the two multiples of k^2 lying there. Any A

attaining one of them has $k \mid g(A)$ [1, §1.4] and is therefore not primitive, so the primitive maximum drops to β_k^+.

Definition 1.1. For $k \geq 2$ put

$$S_k \;=\; \big\{\, |\det A| \;:\; A \in \{0,1,\ldots,k\}^{3\times 3},\; g(A) = 1 \,\big\} \subset \mathbb{Z}_{\geq 0}.$$

Below the ceiling the spectrum S_k becomes rich, and [1] closes with a report and an invitation. The report: the five largest elements of S_k "appear to be"

(1)

$$k(2k^2-2k+1) \succ 2k^2(k-1) \succ k^3+(k-1)^3 \succ k(2k^2-3k+2) \succ (2k^2-k+1)(k-1)$$

"with successive gaps $k,\ k^2-3k+1,\ k-1,\ 1$", and "the computations confirm there is nothing between them up through $k = 7$". The invitation, verbatim:

> "We would warmly encourage others to extend the search to $k = 8, 9, \ldots,$ to test whether the 'first primitive level = sum of two consecutive cubes' phenomenon persists, and to chase down the analogous signed-alphabet ladder."

The same paragraph records what is open: "a proof that $k^3 + (k-1)^3$ is *exactly* the third value for all $k \geq 3$ — and a full characterization of the descent below it — remains open".

This paper answers the first request and makes a start on the second. Everything here concerns the nonnegative alphabet $\{0, 1, \ldots, k\}$; the signed alphabet $\{-k, \ldots, k\}$ is untouched.

Results. Write $v_1(k), \ldots, v_5(k)$ for the five polynomials of (1) and set

(2)

$$v_6(k) = k(2k-1)(k-1), \qquad v_7(k) = k^2(2k-3), \qquad v_8(k) = k^3+(k-1)^3-(k-1)^2.$$

The closed form for v_8 expands to $2k^3 - 4k^2 + 5k - 2$; the shape displayed says that the eighth level is Finch's "sum of two consecutive cubes" level reduced by $(k-1)^2$. Finally put

$$N(k) \;=\; 2k^3 - 4k^2 + 4k \;=\; 2k(k^2 - 2k + 2) \;=\; v_8(k) - (k-1). \tag{3}$$

(1) *The requested extension.* For every k with $8 \leq k \leq 20$, the elements of S_k that are at least $v_5(k)$ are exactly $v_1(k), \ldots, v_5(k)$ (Theorem 4.2). The sum-of-two-consecutive-cubes phenomenon persists on that whole range. The same statement for $3 \leq k \leq 7$, inside the range [1] reports on, is reproduced here from the definitions and independently of the source's own computation (Proposition 4.1).

(2) *Three further levels.* For every k with $5 \leq k \leq 20$, the elements of S_k that are at least $v_8(k)$ are exactly $v_1(k), \ldots, v_8(k)$ (Theorem 5.1). The seven successive gaps are then

$$k, \quad k^2-3k+1, \quad k-1, \quad 1, \quad k-1, \quad k, \quad k^2-5k+2,$$

the first four being those of [1]: the run $k-1, 1, k-1$ is symmetric about the fourth and fifth levels, and the two gaps flanking it are both equal to k (Proposition 5.3).

(3) *Attainment, uniformly in k.* Each of v_6, v_7, v_8 is the absolute determinant of an explicit primitive matrix over $\{0, 1, \ldots, k\}$ for *every* $k \geq 2$ (Theorem 5.2), not merely for the range searched; likewise for Finch's five (Proposition 4.3).

The templates use only the entries $0, 1, 2, k-2, k-1, k$, and their primitivity has a one-line proof for all k at once.

(4) *The descent below v_8, and where it stops being polynomial.* For every k with $5 \le k \le 20$, the elements of S_k that are at least $N(k) - k$ are exactly $v_1(k), \dots, v_8(k)$ together with $N-1$, $N-2$, $N-k$ — and, when k is odd, N itself (Theorem 6.1). The values $N-1$, $N-2$, $N-k$ are again attained for every $k \ge 2$ by all-k templates.

(5) *A parity split.* $N(k) \in S_k$ for every odd $k \ge 3$, and $N(k) \notin S_k$ for every even k with $6 \le k \le 20$ (and, by an independent computation, for every even k with $6 \le k \le 24$; the lower limit is real, since $N(2) = v_2(2)$ and $N(4) = v_7(4)$, and every v_i lies in S_k for every $k \ge 2$). On the range covered the ninth largest element of S_k is therefore N for odd k and $N-1$ for even k (Theorem 6.3). The obstruction is visible in the witness: a matrix of determinant N given below carries the minors k^2 and $(k-1)(k-2)$, and $\gcd\bigl(k^2, (k-1)(k-2)\bigr) = 1$ if and only if k is odd.

Relation to existing tables. None of the ladders above appears to have been tabulated before. Six of the eight level polynomials do occur individually in the On-Line Encyclopedia of Integer Sequences [4]: $v_1(k)$, $v_2(k)$, $v_4(k)$ and $v_7(k)$ are A059722, A035006, A212133 and A015238 at the same index, while $v_3(k)$ and $v_6(k)$ are A005898$(k-1)$ and A055112$(k-1)$. The readings those entries carry — rook moves on a chessboard, centered cubes, quadruples with median equal to mean, areas of Pythagorean triangles — have nothing to do with the present problem, and not one of the six mentions determinants, minors or primitivity. The other two levels are absent altogether: neither $v_5(k) = (2k^2 - k + 1)(k-1)$ nor the sequence of eighth levels $31, 82, 173, 316, 523, 806, 1177, 1648, \dots$ is in the OEIS, nor is $N(2), N(3), \dots = 2, 8, 30, 80, 170, 312, 518, 800, \dots$, nor is any row of Table 1 read across. We have not found the primitivity-restricted determinant spectrum S_k treated anywhere outside [1]. (The OEIS lookups reported here, and the arXiv metadata, full-text and local-corpus searches behind them, were run on 28 August 2026.)

The method behind the exhaustive half of (1), (2) and (4) is a pruning bound (Lemma 3.1) that is itself proved rather than trusted. Naively, deciding the membership question at a single k means inspecting all $(k+1)^9$ matrices over $\{0, 1, \dots, k\}$ — 7.9×10^{11} at $k = 20$. The bound lets the search enumerate the $(k+1)^6$ row *pairs* instead and discard almost all of them without ever forming a third row. What makes it decisive is not the exponent alone: at every k from 5 to 24 exactly 504 row pairs survive the prune at the threshold v_8, and exactly 612 at the threshold $N-k$. We record this in Remark 3.3; it is the reason we regard a uniform-in-k proof of the top of the ladder as within reach, and it is discussed again in §9.

What is not claimed. Every exactness statement below is proved at each of finitely many values of k, by exhaustive search; none of them is proved for all k. In particular the open problem of [1] — that $k^3 + (k-1)^3$ is exactly the third element of S_k for *all* $k \ge 3$ — is not settled here. What is uniform in k is the attainment half throughout, and the odd half of the parity split. The absence of N from S_k for even k is computation over a finite range, not a theorem for all even k; §6 says so again where it matters.

2. Preliminaries

We write a 3×3 integer matrix as an ordered triple of rows $A = (a; b; c)$ with $a, b, c \in \mathbb{Z}^3$, and use the scalar triple product

$$\det A \;=\; a \cdot (b \times c).$$

Deleting row a from A leaves three 2×2 minors, which up to sign are the three components of $b \times c$; deleting b leaves those of $a \times c$, and deleting c those of $a \times b$. Hence

$$g(A) \;=\; \gcd\bigl(b \times c,\; a \times c,\; a \times b\bigr), \tag{4}$$

the gcd being taken over all nine components. This grouping is the one the computations use; it costs one cross product per row pair rather than nine 2×2 determinants per matrix.

Lemma 2.1 (faithfulness)**.** *For all $a, b, c \in \mathbb{Z}^3$, the scalar triple product $a \cdot (b \times c)$ equals the determinant of the matrix with rows a, b, c; the nine numbers appearing in* (4) *are, up to sign, the nine 2×2 minors of that matrix, minor (i, j) being obtained by deleting row i and column j; and the right-hand side of* (4) *equals the gcd of those nine minors. Consequently g as computed here is Finch's g.*

Since $g(A) = g(A^\top)$ is not used anywhere, no symmetry of the alphabet is assumed: the searches are over ordered triples of rows from $\{0, 1, \ldots, k\}^3$, and the resulting statements are about the set S_k of Definition 1.1, which is invariant under all of the row and column operations that preserve $\{0, 1, \ldots, k\}^{3\times3}$.

The statements proved below all have one shape.

Definition 2.2. For $k \geq 2$, a threshold $T \geq 0$ and a finite list L of nonnegative integers, say that L *is exact at* (k, T) when

$$S_k \cap [T, \infty) \;=\; L$$

as sets: every entry of L is the absolute determinant of some primitive matrix over $\{0, 1, \ldots, k\}$, and no primitive matrix over $\{0, 1, \ldots, k\}$ has absolute determinant $\geq T$ outside L.

Both halves of Definition 2.2 matter and they are proved differently: the membership half by exhibiting a matrix, uniformly in k; the exhaustion half by a finite search, one k at a time. Throughout, L is listed in decreasing order and T is its last entry, so that "L is exact at (k, T)" reads as "the top $|L|$ elements of S_k are exactly the entries of L, in that order".

3. A pruning bound, and the cost of a sweep

Fix k and a pair of rows $b, c \in \{0, 1, \ldots, k\}^3$, and put $v = b \times c$. Every matrix with those two rows in place has determinant $a \cdot v$ for a third row $a \in \{0, 1, \ldots, k\}^3$. Writing $p(v) = \sum_i \max(v_i, 0)$ and $n(v) = \sum_i \max(-v_i, 0)$ for the sums of the positive and of the negative parts of v, the constraint $0 \leq a_i \leq k$ gives $a \cdot v \leq k\, p(v)$ and $-(a \cdot v) \leq k\, n(v)$ at once.

Lemma 3.1. *For every $a \in \{0, 1, \ldots, k\}^3$ and every $v \in \mathbb{Z}^3$,*

$$|a \cdot v| \;\leq\; \operatorname{bnd}(k, v) \;:=\; k \cdot \max\bigl(p(v),\, n(v)\bigr).$$

k	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	N
5	205	200	189	185	184	180	175	173	170
6	366	360	341	336	335	330	324	316	312
7	595	588	559	553	552	546	539	523	518
8	904	896	855	848	847	840	832	806	800
9	1305	1296	1241	1233	1232	1224	1215	1177	1170
10	1810	1800	1729	1720	1719	1710	1700	1648	1640
11	2431	2420	2331	2321	2320	2310	2299	2231	2222
12	3180	3168	3059	3048	3047	3036	3024	2938	2928
13	4069	4056	3925	3913	3912	3900	3887	3781	3770
14	5110	5096	4941	4928	4927	4914	4900	4772	4760
15	6315	6300	6119	6105	6104	6090	6075	5923	5910
16	7696	7680	7471	7456	7455	7440	7424	7246	7232
17	9265	9248	9009	8993	8992	8976	8959	8753	8738
18	11034	11016	10745	10728	10727	10710	10692	10456	10440
19	13015	12996	12691	12673	12672	12654	12635	12367	12350
20	15220	15200	14859	14840	14839	14820	14800	14498	14480

TABLE 1. The eight ladder levels (1)–(2) and the value N of (3). For $5 \le k \le 20$ the first eight columns are exactly the eight largest elements of S_k (Theorem 5.1); N belongs to S_k in the odd rows and not in the even ones (Theorem 6.3).

The consequence is the algorithm. Enumerate the $(k+1)^6$ ordered pairs (b, c); for each, form $v = b \times c$ once; if $\mathrm{bnd}(k, v) < T$, no completion of (b, c) can have $|\det| \ge T$, so the $(k+1)^3$ candidate third rows are skipped entirely. Only for the surviving pairs is the inner loop run, and there each a is tested for $|a \cdot v| \ge T$, for primitivity, and for membership of $|a \cdot v|$ in L. Call the Boolean result of this procedure $\mathrm{sweep}(k, T, L)$.

Proposition 3.2 (soundness of the sweep). *If* $\mathrm{sweep}(k, T, L)$ *returns true, then every primitive* $A \in \{0, 1, \ldots, k\}^{3\times 3}$ *with* $|\det A| \ge T$ *has* $|\det A| \in L$.

Proof sketch. Let $A = (a; b; c)$ be primitive with $|\det A| \ge T$. The pair (b, c) was enumerated. If it was pruned, then $\mathrm{bnd}(k, b \times c) < T \le |a \cdot (b \times c)|$, contradicting Lemma 3.1. Otherwise the inner loop reached a, and the only branch that can have returned true at (a, b, c), given $|\det A| \ge T$ and $g(A) = 1$, is membership in L. $\square$

So the prune costs nothing in trust: the certificate $\mathrm{sweep}(k, T, L) = \text{true}$ decides a question about all $(k+1)^9$ matrices while evaluating $(k+1)^6$ cross products and a negligible remainder.

Remark 3.3 (measured, outside the formal development). The remainder is far smaller than the exponent suggests. An independent implementation in C, run at each k from 5 to 24, reports that the number of row pairs surviving the prune is *exactly* 504 *at every one of those* k when $T = v_8(k)$, and exactly 612 when $T = N(k) - k$. (At $k = 3$ and $k = 4$ the counts are 228 and 300; these are the same small-k degeneracies that make the eight-term list fail at $k = 4$, see §7.) Enumerating the surviving pairs themselves at $k = 8, 9, 11$ shows that their entries are drawn from exactly the eight values $\{0, 1, 2, 3\} \cup \{k-3, k-2, k-1, k\}$ at each of those k. This suggests that the surviving pairs form finitely many polynomial families in k, in which case the exactness statements below would become uniform in k by a finite case analysis together with Lemma 3.1; this is the natural route to the open problem of [1]. The counts and the entry sets are computations outside the formal development and are quoted here as data, not as theorems.

4. Finch's five levels

We first reproduce the range [1] reports on, from the definitions and independently of the source's own program.

Proposition 4.1. *For $3 \le k \le 7$ the list $(v_1(k), \dots, v_5(k))$ is exact at $(k, v_5(k))$: the five largest elements of S_k are*

$$\begin{array}{ll} 39, 36, 35, 33, 32 & (k = 3), \\ 100, 96, 91, 88, 87 & (k = 4), \\ 205, 200, 189, 185, 184 & (k = 5), \\ 366, 360, 341, 336, 335 & (k = 6), \\ 595, 588, 559, 553, 552 & (k = 7). \end{array}$$

This is the claim of [1] — "the computations confirm there is nothing between them up through $k = 7$" — with the numerical values, which the source does not print, made explicit. The lower endpoint $k = 3$ is forced: at $k = 2$ the five polynomials evaluate to $10, 8, 9, 8, 7$, so the second and third are already out of order and the list is not a ladder there. The next statement is what the source asks for.

Theorem 4.2. *For every k with $8 \le k \le 20$, the list $\big(v_1(k), \dots, v_5(k)\big)$ is exact at $\big(k, v_5(k)\big)$. That is, the five largest elements of S_k are*

$$k(2k^2-2k+1) > 2k^2(k-1) > k^3+(k-1)^3 > k(2k^2-3k+2) > (2k^2-k+1)(k-1),$$

with no element of S_k strictly between two consecutive ones and none above the first.

In particular the "first primitive level = sum of two consecutive cubes" phenomenon of [1] — that the third level is $k^3 + (k-1)^3$, the first level of the ladder to escape the sublattice $k\mathbb{Z}$, since $k^3 + (k-1)^3 \equiv -1 \pmod{k}$ — persists on the whole range $3 \le k \le 20$.

Proof sketch. The membership half is Proposition 4.3, which is uniform in k. The exhaustion half is a separate finite search for each of the thirteen values $k = 8, \dots, 20$: the certificate

$$\mathrm{sweep}\big(k,\ v_5(k),\ (v_1, \dots, v_5)\big)$$

of §3 is evaluated and returns true, and Proposition 3.2 converts it into the statement about all of $\{0, 1, \dots, k\}^{3\times 3}$. In practice the certificate actually evaluated is the sharper one at threshold $v_8(k)$ used for Theorem 5.1; the five-term statement follows because the two lists agree above $v_5(k)$. Nothing in this half is uniform in k: thirteen searches were run, one per k. $\square$

The lower-bound half is uniform, and this is where the source's "each value is exhibited by an explicit matrix" becomes a family of identities in k.

Proposition 4.3. *For every $k \geq 2$, each of $v_1(k), \dots, v_5(k)$ lies in S_k. Explicit primitive matrices are*

$$\begin{aligned} &v_1: \quad (0, k-1, k;\ k, 0, k-1;\ k, k, 0), && v_2: \quad (0, k-1, k;\ k-1, 0, k;\ k, k, 0), \\ &v_3: \quad (0, k-1, k;\ k, 0, k-1;\ k-1, k, 0), && v_4: \quad (0, k-1, k;\ k, 0, k-1;\ k, k, 1), \\ &v_5: \quad (0, k-1, k;\ k-1, 0, k;\ k, k, 1). \end{aligned}$$

Two of these five matrices already appear in [1]. The third is, up to a cyclic permutation of rows, the circulant $(k, 0, k-1;\ k-1, k, 0;\ 0, k-1, k)$ printed in the Closing Words passage — the only 3×3 matrix displayed there. The first is, up to one row swap and one column swap, the matrix $(0, k, k-1;\ k, 0, k;\ k, k-1, 0)$, which [1, §1.4] displays together with $(0, k, k-1;\ k, 1, k;\ k, k, 0)$ as two representatives, modulo permutations of rows and of columns, of matrices attaining β_k^+; the second of those two is not a row-and-column permutation of any template listed here. The templates for v_2, v_4 and v_5 we have not seen written down, though the values they realise are Finch's. Their primitivity is uniform in k for the reason isolated next.

Lemma 4.4. *For every $K \in \mathbb{Z}$, $\gcd\big(K^2, (K-1)^2\big) = 1$. Consequently, a 3×3 integer matrix two of whose 2×2 minors are $\pm K^2$ and $\pm (K-1)^2$ is primitive.*

Proof. $K - (K-1) = 1$, so $\gcd(K, K-1) = 1$ and hence $\gcd(K^2, (K-1)^2) = 1$. The gcd of the nine minors divides both of the two named minors, hence divides 1. □

Proof of Proposition 4.3. Each listed matrix has entries in $\{0, 1, \dots, k\}$ for $k \geq 2$. Its determinant is the stated polynomial identically in k: for instance $\det(0, k-1, k;\ k, 0, k-1;\ k, k, 0) = k(2k^2 - 2k + 1)$ as polynomials. Two of its nine minors are $\pm k^2$ and $\pm (k-1)^2$. For the three templates whose second row is $(k, 0, k-1)$ — those for v_1, v_3, v_4 — these are the minors obtained by deleting the first row and third column, and by deleting the third row and first column. For the two whose second row is $(k-1, 0, k)$ — those for v_2 and v_5 — they are the minors obtained by deleting the first row and first column, and by deleting the third row and third column. Lemma 4.4 applies in either case. □

5. Three further levels

The searches of §4 are run at a threshold lower than v_5, and reveal three more levels which [1] does not describe.

Theorem 5.1. *For every k with $5 \leq k \leq 20$, the list $\big(v_1(k), \dots, v_8(k)\big)$ of (1) and (2) is exact at $\big(k, v_8(k)\big)$: the eight largest elements of S_k are*

$$v_1 > v_2 > v_3 > v_4 > v_5 > k(2k-1)(k-1) > k^2(2k-3) > k^3 + (k-1)^3 - (k-1)^2,$$

with nothing of S_k between consecutive ones. Numerically these are the first eight columns of Table 1.

Proof sketch. As before, in two halves. Membership is Proposition 4.3 for $v_1, \dots, v_5$ and Theorem 5.2 for v_6, v_7, v_8; both are uniform in k. Exhaustion is sixteen separate finite searches, one for each k from 5 to 20: the certificate sweep(k, T, L) with $T = v_8(k)$ and $L = (v_1, \dots, v_8)$ evaluates to true, and Proposition 3.2 does the rest. At $k = 20$ the search visits $21^6 = 85{,}766{,}121$ row pairs, of which 504 survive to the inner loop; the box it decides has $21^9 \approx 7.9 \times 10^{11}$ elements. □

Theorem 5.2. *For every $k \geq 2$, each of $v_6(k) = k(2k-1)(k-1)$, $v_7(k) = k^2(2k-3)$ and $v_8(k) = k^3 + (k-1)^3 - (k-1)^2$ lies in S_k. Explicit primitive matrices are*

$$\begin{aligned} v_6 &: \quad (0,\, k-1,\, k-1;\;\; k,\, 0,\, k-1;\;\; k,\, k,\, 0), \\ v_7 &: \quad (0,\, k-1,\, k;\;\; k-1,\, 1,\, k;\;\; k,\, k,\, 0), \\ v_8 &: \quad (0,\, k-1,\, k;\;\; k,\, 0,\, k-1;\;\; k-2,\, k,\, 0). \end{aligned}$$

Proof. Each matrix has entries in $\{0, 1, \ldots, k\}$ for $k \geq 2$, and its determinant equals the stated polynomial identically in k; this is a polynomial identity in one variable, verified by expansion. Each has $\pm k^2$ and $\pm(k-1)^2$ among its nine 2×2 minors, so Lemma 4.4 gives primitivity for every k. □

The point of Theorem 5.2 is that it is not a search result. The exhaustion half of Theorem 5.1 holds for $5 \leq k \leq 20$ only; the attainment half holds for all $k \geq 2$, so $v_6, v_7, v_8 \in S_k$ is known well beyond the searched range. The templates were found by searching the restricted alphabet $\{0, 1, 2, k-2, k-1, k\}$ at $k = 8, 9, 11, 13$ and intersecting; that search is a heuristic, and its output is verified by the identities above.

Two features of the extended ladder are worth recording separately.

Proposition 5.3. *As polynomials in k,*

$$\begin{aligned} &v_1 - v_2 = k, && v_2 - v_3 = k^2 - 3k + 1, && v_3 - v_4 = k - 1, \\ &v_4 - v_5 = 1, && v_5 - v_6 = k - 1, && v_6 - v_7 = k, \\ &v_7 - v_8 = k^2 - 5k + 2, && v_8 - \big(N - 1\big) = k - 1, && v_8 = v_3 - (k-1)^2. \end{aligned}$$

The first four identities are those printed in [1]; the rest follow from (2) and (3). Displayed as a sequence, the seven gaps of the eight-term ladder are

$$k, \quad k^2 - 3k + 1, \quad k - 1, \quad 1, \quad k - 1, \quad k, \quad k^2 - 5k + 2,$$

in which the run $k-1,\, 1,\, k-1$ is symmetric about the fourth and fifth levels and is flanked on both sides by a gap equal to k. That symmetry is the reason we regard the three new levels as continuing Finch's ladder rather than merely following it.

Proposition 5.4. *For every integer $k \geq 5$, $v_1(k) > v_2(k) > \cdots > v_8(k)$.*

Without Proposition 5.4 the phrase "the eight largest elements, in order" would be a statement about each searched k separately; with it, the ordering of the eight polynomials is settled at every integer $k \geq 5$, searched or not. The hypothesis $k \geq 5$ is sharp: at $k = 4$ one has $v_7 = 80 < 82 = v_8$, and the two swap; see §7.

6. Below the eighth level: the descent and a parity split

Lowering the threshold one step further, to $N(k) - k$ with N as in (3), reaches four more elements of S_k — and the first place at which the answer stops being a single polynomial in k.

Theorem 6.1. *Let $5 \leq k \leq 20$ and $N = N(k) = 2k(k^2 - 2k + 2)$. Then*

$$S_k \cap [N - k, \infty) \;=\; \begin{cases} \{v_1, \ldots, v_8\} \cup \{N,\, N-1,\, N-2,\, N-k\} & \text{if } k \text{ is odd}, \\ \{v_1, \ldots, v_8\} \cup \{N-1,\, N-2,\, N-k\} & \text{if } k \text{ is even}. \end{cases}$$

In particular the ninth largest element of S_k is N for odd k and $N-1$ for even k, and the gaps below v_8 are $k-1, 1, 1, k-2$ for odd k and $k-1, 1, k-2$ for even k.

Proof sketch. Membership of $N-1$, $N-2$, $N-k$ is Proposition 6.2 and membership of N for odd k is Theorem 6.3, both uniform in k. Exhaustion is sixteen further finite searches, one per k from 5 to 20, at the lower threshold $N-k$; at that threshold 612 row pairs survive the prune at each k. The two cases differ only in the list handed to the search: twelve entries for odd k, eleven for even k. $\square$

Proposition 6.2. *For every $k \geq 2$, each of $N-1$, $N-2$ and $N-k$ lies in S_k. Explicit primitive matrices are*

$$\begin{aligned} N-1: &\quad (0,\, k-1,\, k;\;\; k,\, 0,\, k-1;\;\; k-1,\, k,\, 1),\\ N-2: &\quad (0,\, k-1,\, k;\;\; k-1,\, 0,\, k;\;\; k,\, k,\, 2),\\ N-k: &\quad (0,\, k-1,\, k;\;\; k,\, 0,\, k-1;\;\; k,\, k,\, 2). \end{aligned}$$

Proof. As in Theorem 5.2: the determinants are $2k^3-4k^2+4k-1$, $2k^3-4k^2+4k-2$ and $2k^3-4k^2+3k$ identically in k, and each matrix carries $\pm k^2$ and $\pm(k-1)^2$ among its minors, so Lemma 4.4 applies. $\square$

The remaining value, N itself, behaves differently.

Theorem 6.3. *Let $N(k) = 2k(k^2-2k+2)$.*

(i) *For every odd $k \geq 3$, $N(k) \in S_k$, witnessed by*

$$W_k \;=\; (0,\, k-1,\, k;\;\; k,\, 1,\, k;\;\; k-2,\, k,\, 0), \qquad \det W_k = N(k) \textit{ identically in } k.$$

(ii) *For every even k with $6 \leq k \leq 20$, $N(k) \notin S_k$: no primitive matrix over $\{0,1,\dots,k\}$ has absolute determinant $N(k)$.*

Consequently, on the range $5 \leq k \leq 20$, $N(k) \in S_k$ if and only if k is odd.

Proof sketch. For (i), $\det W_k = 2k^3-4k^2+4k$ is a polynomial identity, and W_k has entries in $\{0,1,\dots,k\}$ for $k \geq 2$. Among its nine minors are $\pm k^2$ (delete the first row and the first column) and $\pm(k-1)(k-2)$ (delete the second row and the third column). Now $(k-1)(k-2) = k^2-3k+2 \equiv 2 \pmod{k}$, so any prime dividing both k^2 and $(k-1)(k-2)$ divides k and hence divides 2; therefore

$$\gcd\bigl(k^2,\,(k-1)(k-2)\bigr) = 1 \iff k \text{ is odd,} \tag{5}$$

since for even k both numbers are even. For odd k this makes W_k primitive, and for odd $k \geq 3$ it gives $N(k) \in S_k$ with no upper bound on k. (For even k the nine minors of W_k are k^2, $k(k-2)$, k^2-k+2, $(k-1)(k-2)$ and $k(k-1)$ up to sign, all even, so $g(W_k) = 2$ and this witness fails; part (ii) says much more than that.)

Part (ii) is the exhaustion half of Theorem 6.1 at even k: for each even k in $\{6,8,\dots,20\}$ the search at threshold $N(k)-k$ returns a list not containing $N(k)$, and $N(k) > N(k)-k$, so the absence is genuine and not an artefact of the threshold. This is a separate finite computation at each of eight values of k. $\square$

Remark 6.4 (the shape of the obstruction)*.* Comparing (5) with Lemma 4.4 explains why N is the first level to split. All eleven templates of Proposition 4.3, Theorem 5.2 and Proposition 6.2 carry the coprime pair $k^2, (k-1)^2$, and $\gcd(k^2,(k-1)^2) = 1$ unconditionally. The template for N carries $k^2, (k-1)(k-2)$ instead, and there the gcd is 1 only for odd k. For even k the computation says more than that this particular witness fails: it says that no matrix over $\{0,1,\dots,k\}$ at all realises N primitively.

Remark 6.5 (what part (ii) does and does not assert). Part (ii) is a statement about eight specific even values of k. An independent implementation in C extends the same absence to every even k with $6 \le k \le 24$. It does not extend below: $N(2) = 8 = v_2(2)$ and $N(4) = 80 = v_7(4)$, and those values lie in S_k by Proposition 4.3 and Theorem 5.2, so 6 is where the even-k absence begins. We do not know whether $N(k) \notin S_k$ for all even $k \ge 6$. A rigidity argument in the style of the one [1] uses for $2k^3$ and $2k^3 - k^2$ would settle it, and would be the first result about the descent that is uniform in k on both halves.

Remark 6.6 (the descent continues, measured only). The same pruned search, run in C at the still lower threshold $N-3k$, returns for every k from 6 to 21 the offsets below $N-k$ as exactly $k+1,\ k+2,\ 2k,\ 2k+1,\ 3k-1,\ 3k$. Combining with Theorem 6.1, the whole of $S_k \cap [N-3k, \infty)$ is, for $6 \le k \le 20$, the eight polynomial levels together with the offsets $\{0 \text{ (odd } k \text{ only)}, 1, 2, k, k+1, k+2, 2k, 2k+1, 3k-1, 3k\}$ from N — and the offset 0 is the only one whose presence depends on the parity of k. These six further offsets are computations outside the formal development; they are recorded here as data and as the natural next target.

7. Negative controls

Exactness statements are easy to state vacuously or with a misplaced quantifier, so the boundaries of the claims above were made explicit.

Proposition 7.1. (i) (the eight-term list starts at $k = 5$) *The list*

$$100,\ 96,\ 91,\ 88,\ 87,\ 84,\ 80$$

is not *exact at* $(4, 80)$*: the value* $82 = v_8(4)$ *lies in* S_4 *and exceeds* $v_7(4) = 80$*, so at* $k = 4$ *the seventh and eighth polynomials have swapped.*

(ii) (the ladder does not continue with $v_8 - (k-1)$) *The nine-term list*

$$1305,\ 1296,\ 1241,\ 1233,\ 1232,\ 1224,\ 1215,\ 1177,\ 1169$$

is not *exact at* $(9, 1169)$*: the value* $1170 = N(9)$ *lies in* S_9 *strictly between* 1169 *and* $v_8(9) = 1177$*.*

(iii) (too large is refuted) *No primitive matrix over* $\{0, \dots, 8\}$ *has absolute determinant* $905 = v_1(8) + 1$*.*

(iv) (“nothing between”, spelled out) *No primitive matrix over* $\{0, \dots, 8\}$ *has absolute determinant strictly between* 848 *and* 855*.*

(v) (too small is refuted) $904 \in S_8$*; no bound below* 904 *can hold.*

(vi) (primitivity is load-bearing) *The matrix* $(0, 8, 8;\ 8, 0, 8;\ 8, 8, 0)$*, with entries in* $\{0, \dots, 8\}$*, has determinant* $2k^3 = 1024$ *and* $g = 64$*, so dropping the primitivity hypothesis raises the maximum from* 904 *to* 1024*.*

Item (i) is why Theorem 5.1 and Proposition 5.4 begin at $k = 5$. Item (ii) is worth a comment: by Proposition 5.3 one has $v_8 - (k-1) = N - 1$ identically, so the naive guess “the ladder continues with $v_8 - (k-1)$” is the guess “the ninth level is $N - 1$”. Theorem 6.3 says exactly when that is right — for even k, where N is absent — and exactly when it is wrong, namely for odd k, where N intervenes. The guess is in fact correct at $k = 8$, which is why the refutation had to be run at $k = 9$.

8. Questions

Question 8.1 (Finch)**.** Is $k^3 + (k-1)^3$ exactly the third largest element of S_k for every $k \geq 3$? More generally, is the eight-term list of Theorem 5.1 exact for every $k \geq 5$?

The searches settle Question 8.1 for $5 \leq k \leq 20$ and, in the five-term form, for $3 \leq k \leq 20$. Remark 3.3 suggests the shape a uniform answer might take: if the 504 row pairs that survive the prune are, at every $k \geq 5$, the values at k of finitely many fixed polynomial row pairs — which the measured entry set $\{0, 1, 2, 3\} \cup \{k-3, k-2, k-1, k\}$ makes plausible — then Lemma 3.1 reduces the whole exactness statement to a case analysis over those 504 shapes, uniformly in k. We have not carried this out.

Question 8.2. Is $N(k) = 2k(k^2 - 2k + 2)$ absent from S_k for *every* even $k \geq 6$?

Theorem 6.3(ii) gives this for the eight even k in $[6, 20]$, and the C computation extends it to every even k with $6 \leq k \leq 24$; at $k = 2$ and $k = 4$ the answer is no, which is why the question starts at 6. A proof would presumably run like the rigidity argument of [1] for $2k^3$ and $2k^3 - k^2$: show that any matrix over $\{0, 1, \ldots, k\}$ of determinant N is forced into a shape all of whose minors are even. Note that the parity of k has to enter, since by Theorem 6.3(i) the same value *is* attained primitively for every odd k.

Question 8.3. How far below N does the descent remain a finite union of polynomial families in k?

Remark 6.6 records that down to $N - 3k$ it does, with the offsets $0, 1, 2, k, k+1, k+2, 2k, 2k+1, 3k-1, 3k$ from N, only the first of which depends on the parity of k. Whether a second parity split — or a split at some other modulus — occurs lower down we do not know; the measured range stops there.

Question 8.4 (Finch)**.** What is the corresponding ladder over the signed alphabet $\{-k, \ldots, k\}$?

Nothing here addresses this. We note only that the method transfers verbatim: Lemma 3.1 is stated for arbitrary v and uses the bound $0 \leq a_i \leq k$ only through $|a_i| \leq k$, so the analogue over $\{-k, \ldots, k\}$ holds with bnd unchanged, and the pair sweep becomes one over $(2k+1)^6$ pairs.

9. Verification

Every theorem, proposition and lemma of this paper has been formally verified in Lean 4 against *Mathlib*; the development is free of `sorry`. (The remarks explicitly marked as measurements are the exception, and are discussed at the end of this section.) Repository reference to be added at submission. Lemma 2.1 is what ties the formal development to the definitions used here: it identifies the scalar triple product with `Matrix.det` and the nine numbers of (4) with the nine 2×2 minors as `Matrix.dets`, so that "primitive" means Finch's $g(A) = 1$ and not a lookalike. Lemma 3.1, Proposition 3.2, Lemma 4.4, the odd half of (5) — its even half being the observation that both numbers are even — Propositions 4.3, 5.3, 5.4 and 6.2, Theorem 5.2 and part (i) of Theorem 6.3 depend on no axioms beyond `propext`, `Classical.choice` and `Quot.sound` — in particular the attainment half of every theorem, and the odd half of the parity split, are kernel-checked and uniform in k.

What is delegated to compiled evaluation, through Lean's `native_decide`, is exactly the finite searches, one Boolean per value of k: five for Proposition 4.1 ($k = 3, \ldots, 7$), sixteen for Theorem 5.1 ($k = 5, \ldots, 20$), from which Theorem 4.2 is deduced by narrowing the threshold, sixteen more for Theorem 6.1 and hence for part (ii) of Theorem 6.3, and one each for parts (i) and (ii) of Proposition 7.1, both of which return *false*. Each such Boolean is the certificate $\text{sweep}(k, T, L)$ of §3, and Proposition 3.2 is the proved bridge from the certificate to the quantified statement; the pruning branch inside it is discharged by Lemma 3.1, so no matrix of the box escapes the search. Each theorem proved this way carries, on top of the three standard axioms, exactly one axiom of compiled evaluation per sweep it invokes; those are the only additional axioms in the development. The total cost of the searches is about half a CPU-hour, dominated by the four largest k.

Every numerical value quoted was first produced by an independent implementation in C and the formal statement was written against it. Two C programs were used: a naive exhaustive $(k+1)^9$ enumeration, run for $3 \leq k \leq 15$ (114 seconds at $k = 15$), and the pruned $(k+1)^6$ algorithm of §3, run for $3 \leq k \leq 24$ (about one second at $k = 24$). Three independent computations of g — two in C, differing in whether the nine minors are written out or grouped as three cross products, and one in Lean — agree at every k in their common range, and the Lean certificates agree with the C output at every k from 3 to 20. The measurements quoted in Remarks 3.3, 6.5 and 6.6 come from the C programs alone and are labelled as such.

References

[1] S. Finch, *Small Matrices with Large Inverses: Unimodular* 4×4 *Cases*, arXiv:2607.07688v4 (17 August 2026), 21 pp. All quotations and section numbers above refer to v4, the latest version as of 28 August 2026; the paper carries no journal reference.

[2] S. Finch, *Small Matrices with Small Inverses: Unimodular Zerofree Cases*, arXiv:2605.03691 (2026).

[3] H. Ehlich and K. Zeller, *Binäre Matrizen*, Z. Angew. Math. Mech. **42** (1962), no. S1, T20–T21; `doi:10.1002/zamm.19620421310`.

[4] OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, `https://oeis.org`; entries A003432, A005898, A015238, A035006, A055112, A059722 and A212133, consulted 28 August 2026.