PARTITIONS IN THE IMAGE OF pre_k: COMPLETE SEARCHES, COMPUTED VALUES, AND THE CLASSIFICATION OF $\mathrm{pre}_k(n) = 1$

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Abstract. Ballantine, Beck and Merca attach to a partition λ and an integer $k \geq 2$ the partition $\mathrm{pre}_k(\lambda)$ whose parts are the products of the parts of λ over all k-element sets of indices; its weight is the elementary symmetric function $e_k(\lambda)$. Following Devnani and Eyyunni, $\mathrm{pre}_k(n)$ denotes the number of partitions of n that lie in the image of pre_k. Garg, and independently Thomas and Tung, proved that $\mathrm{pre}_2(n) = 1$ exactly for $n \in \{1, 2, 4\}$; no value of $\mathrm{pre}_k(n)$ appears in print for any $k \geq 3$, and the analogue of that theorem at $k \geq 3$ was not known.

We prove that the natural pruned depth-first search for the partitions λ with $e_k(\lambda) = n$ is complete for $k = 2, 3, 4, 5$, so that $\mathrm{pre}_k(n)$ is an effectively computable function of n rather than a count over a truncated search, and we tabulate $\mathrm{pre}_2(n)$ for $n \leq 460$ with further values up to 1000, $\mathrm{pre}_3(n)$ for $n \leq 400$, and $\mathrm{pre}_5(n)$ for $n \leq 500$ with further values up to 1000. Our main result is the exact analogue of Garg's theorem at $k = 3$: $\mathrm{pre}_3(n) = 1$ for exactly 28 integers n, the largest being 321, at every $n \geq 1$ and with no range restriction. The proof removes pre_3 from the problem and covers the complement of the exceptional set by 328 arithmetic progressions. At $k = 4$ and $k = 5$ we determine the exceptional set on $[1, 10^5]$ and on $[1, 2 \cdot 10^4]$ (respectively 171 integers with largest 5942, and 1103 integers with largest 19969), in each case with the half asserting $\mathrm{pre}_k(n) = 1$ holding at every n; an exhaustive computation shows that no covering system exists at $k = 4$ with modulus lattice of least common multiple at most $3 \cdot 10^5$, and partial coverings instead confine $\{n : \mathrm{pre}_4(n) = 1\}$ to at most 11.63% of the residue classes modulo 240240 and $\{n : \mathrm{pre}_5(n) = 1\}$ to at most 43.12%. All theorems stated below are machine-checked in Lean 4.

1. Introduction

A partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell)$ of n is a non-increasing sequence of positive integers with $\lambda_1 + \cdots + \lambda_\ell = n$; we write $\ell(\lambda) = \ell$ for its number of parts and $|\lambda|$ for its weight. For $k \geq 1$ let

$$e_k(\lambda) \;=\; \sum_{i_1 < i_2 < \cdots < i_k} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}$$

be the k-th elementary symmetric function of the parts, with $e_k(\lambda) = 0$ when $\ell(\lambda) < k$.

Ballantine, Beck and Merca [1] introduced the map that this paper is about. Given a partition λ, let $\mathrm{pre}_k(\lambda)$ be the partition whose parts are all products of k distinct parts of λ. "Distinct parts" here means distinct *indices*, so $\mathrm{pre}_k(\lambda)$ has exactly $\binom{\ell(\lambda)}{k}$ parts, one for each k-element subset of $\{1, \ldots, \ell(\lambda)\}$, and

$$\big|\mathrm{pre}_k(\lambda)\big| \;=\; e_k(\lambda).$$

Section 2 records the four worked examples, drawn from four different papers, that pin this reading of the definition.

Devnani and Eyyunni [2] study the map at $k = 2$. Writing $\mathrm{Pre}_k(n)$ for the set of partitions of n that arise as $\mathrm{pre}_k(\lambda)$ for some partition λ, and

$$\mathrm{pre}_k(n) \;=\; \big|\mathrm{Pre}_k(n)\big|,$$

they prove $\mathrm{pre}_2(n) \geq \tau(n+1)/2$, where τ is the number-of-divisors function, and they ask (their Problem 3) whether there is any $n \geq 1$ with $\mathrm{pre}_2(n) = 1$. Garg [3] answered this: $\mathrm{pre}_2(n) = 1$ exactly for $n \in \{1, 2, 4\}$. Thomas and Tung [4] obtained the same theorem independently, and they are also the first to define $\mathrm{pre}_k(n)$ for general k — but they evaluate it for no $k \geq 3$, and the only

place in the literature where the counting function is asked about beyond $k = 2$ is their Question 6.2, which asks for which positive integers a there are n and k with $\mathrm{pre}_k(n) = a$.

Two things are therefore missing from the literature, and this paper supplies them.

First, no value of $\mathrm{pre}_k(n)$ has ever been printed, at any k, beyond the classification $\mathrm{pre}_2(n) = 1 \iff n \in \{1, 2, 4\}$ and one worked instance $\mathrm{pre}_2(23) \geq 3$ of [2]. The obstruction is not arithmetic but logical: $\mathrm{pre}_k(n)$ counts over an *infinite* set — all partitions λ, of every weight, whose k-fold products happen to sum to n — so a computation of $\mathrm{pre}_k(n)$ is a computation of a number only once the search that produces it is proved to miss nothing. Section 3 proves the required completeness at $k = 2, 3, 4, 5$, and Section 4 tabulates the resulting values.

Second, Garg's theorem has no known analogue at $k \geq 3$, and the answer turns out to look nothing like the answer at $k = 2$. Write

$$E_k \;=\; \{\, n \geq 1 \;:\; \mathrm{pre}_k(n) = 1 \,\},$$

so that $E_2 = \{1, 2, 4\}$. Our main theorem is that E_3 has 28 elements, the largest being 321, and that this holds at every $n \geq 1$ with no range hypothesis (Theorem 6.4). The proof, in Section 6, has two ingredients. The first is a reformulation (Theorem 5.1) that removes pre_k from the problem altogether: $\mathrm{pre}_k(n) \geq 2$ if and only if some partition with *more than k parts* has $e_k = n$. The second is a covering system of 328 arithmetic progressions, each supplied by an explicit partition, which together with two finite modular checks settles every non-exceptional n at once. Read this way, Garg's five-case proof at $k = 2$ is itself a covering system, of least common multiple 4; what changes at $k = 3$ is that the modulus available from a partition ν is $e_2(\nu)$, quadratic in the parts, rather than the linear $e_1(\nu)$, and 328 progressions with least common multiple 28665 are needed where 4 sufficed.

At $k = 4$ and $k = 5$ the same machinery gives the classification on a finite range only (Section 7): $|E_4 \cap [1, 10^5]| = 171$ with largest element 5942, and $|E_5 \cap [1, 2 \cdot 10^4]| = 1103$ with largest element 19969. In both cases the half of the statement asserting $\mathrm{pre}_k(n) = 1$ carries no range hypothesis, for a reason worth isolating: it rests on one complete search run at the largest listed value, and that search is complete over all partitions whatsoever, so no larger search could produce a witness. Whether E_4 and E_5 are finite we do not decide, and Section 7 explains why the $k = 3$ route does not reach that far: an exhaustive computation shows that no covering system exists at $k = 4$ whose moduli divide a common value $L \leq 3 \cdot 10^5$. Section 8 gives the strongest unbounded statement the method does yield, a partial covering confining E_4 to at most 11.63% of the residue classes modulo 240240 and E_5 to at most 43.12%.

Section 9 collects the sharpness statements that keep the results from being vacuous, Section 10 re-derives the published statements of [2, 3, 4, 5] from the definition used here, and Section 11 describes the machine verification.

2. Preliminaries

2.1. **Partitions and the maps pre_k.** Throughout, a partition is identified with its non-increasing sequence of positive parts; two partitions are equal when these sequences agree, so counting partitions is the same as counting non-increasing sequences.

Definition 2.1. For $k \geq 2$ and a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, let $\mathrm{pre}_k(\lambda)$ be the partition with $\binom{\ell}{k}$ parts

$$\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}, \qquad 1 \leq i_1 < i_2 < \cdots < i_k \leq \ell,$$

arranged in non-increasing order. For $\ell < k$ this is the empty partition.

Summing over the k-subsets gives $|\mathrm{pre}_k(\lambda)| = e_k(\lambda)$ at once. In particular the weight of $\mathrm{pre}_k(\lambda)$ depends only on e_k, and pre_k maps the infinite set of partitions with $e_k(\lambda) = n$ onto $\mathrm{Pre}_k(n)$.

Definition 2.2. $\mathrm{Pre}_k(n) = \{\mathrm{pre}_k(\lambda) : \lambda \text{ a partition}\} \cap \{\mu : |\mu| = n\}$ and $\mathrm{pre}_k(n) = |\mathrm{Pre}_k(n)|$. We write $E_k = \{n \geq 1 : \mathrm{pre}_k(n) = 1\}$.

The one immediate value is a lower bound valid at every n.

Proposition 2.3. *For every $n \geq 1$ and every $k \in \{2,3,4,5\}$ we have $\mathrm{pre}_k(n) \geq 1$, because $\mathrm{pre}_k(n,1,\ldots,1) = (n)$ for the partition with one part n and $k-1$ parts equal to 1. (The argument is uniform in k; as everywhere below, we state it at the four values of k treated in this paper.)*

Proof. That partition has k parts, so pre_k of it has $\binom{k}{k} = 1$ part, namely $n \cdot 1 \cdots 1 = n$. □

Consequently $\mathrm{pre}_k(n) = 1$ says exactly that (n) is the *only* partition of n in the image, and E_k is the set where the trivial witness is the only one.

2.2. **Indices, not values.** The phrase "products of k distinct parts" admits a second reading — products of k distinct *values* — under which $\mathrm{pre}_2(2,2,1)$ would have a single part. The reading of Definition 2.1 is the one forced by the worked examples in the sources, and we record all four, from four different papers, since a misreading here would invalidate every value in this paper and would not be caught by any internal consistency check. We have not consulted [1] directly: the definition used here is the one quoted, in agreement, by [2], [4] and [5], and the following identities are what pin it.

Proposition 2.4. *The following identities hold under Definition 2.1, and each appears in the cited source with the same value:*

$$\mathrm{pre}_2(7,4,4) = (28,28,16), \qquad \mathrm{pre}_2(4,2,1,1) = (8,4,4,2,2,1),$$
$$\mathrm{pre}_3(4,3,3,1) = (36,12,12,9), \qquad \mathrm{pre}_2(2,2,1) = (4,2,2), \qquad \mathrm{pre}_2(n,1) = (n).$$

The first is [2, example following the definition]*, the second* [4, Example 2.6]*, the third* [5, worked example]*, and the last two are from* [3].

Each of the first three has either a repeated part or a repeated product, which is precisely what separates the index reading from the value reading; the second also exhibits the part $1 \cdot 1 = 1$ produced by the two parts equal to 1.

3. Complete enumeration

Fix k and $n \geq 1$. Because pre_k is weight-preserving in the sense $|\mathrm{pre}_k(\lambda)| = e_k(\lambda)$, computing $\mathrm{pre}_k(n)$ amounts to enumerating

$$P_k(n) \;=\; \{\lambda \text{ a partition} : e_k(\lambda) = n\},$$

applying pre_k and removing duplicates. Nothing in the definition of $P_k(n)$ is finite: it is a condition on partitions of every weight. The three bounds of this section make it finite, and make the depth-first search that uses them provably exhaustive.

3.1. **The bounds.** We enumerate partitions as non-increasing sequences built one part at a time, so the object to control is a *prefix* $(\lambda_1,\ldots,\lambda_j)$ of a target partition.

Lemma 3.1. *For all finite sequences α, β of positive integers, $e_2(\alpha) \leq e_2(\alpha\beta)$, where $\alpha\beta$ denotes concatenation. The same holds for e_3, e_4 and e_5.*

Proof. For e_2 this is the identity $e_2(\alpha\beta) = e_2(\alpha) + |\alpha| \cdot |\beta| + e_2(\beta)$, all three summands being non-negative. For e_j with $j \geq 3$ it follows by induction on α from the recursion $e_j(a,\alpha') = a\, e_{j-1}(\alpha') + e_j(\alpha')$ together with the case $j-1$. □

Lemma 3.1 is the entire justification for pruning: every prefix of a partition with $e_k(\lambda) = n$ satisfies $e_k(\text{prefix}) \leq n$, so a search that discards a prefix as soon as e_k of it exceeds n discards nothing that could have been completed. What it does not do is bound the depth, or the first few levels when $k \geq 3$, since e_k of a prefix with fewer than k parts is 0.

Lemma 3.2. *Let λ be a partition. Then $\ell(\lambda) \leq e_2(\lambda) + 1$, and if $\ell(\lambda) \geq 2$ then $\lambda_1 \leq e_2(\lambda)$.*

Proof. Write $\lambda = (a, \mu)$. Since all parts are at least 1, $e_2(\lambda) = a|\mu| + e_2(\mu) \geq |\mu| \geq \ell(\mu) = \ell(\lambda) - 1$. If $\mu \neq \emptyset$ then $|\mu| \geq 1$, so $e_2(\lambda) \geq a|\mu| \geq a$. □

Lemma 3.3. *Let λ be a partition. Then $\ell(\lambda) \leq e_3(\lambda) + 2$, and if $\ell(\lambda) \geq 3$ then $\lambda_1\lambda_2 \leq e_3(\lambda)$.*

Proof. For the first bound write $\lambda = (a, \mu)$; then $e_3(\lambda) = a\, e_2(\mu) + e_3(\mu) \geq e_2(\mu) \geq \ell(\mu) - 1 = \ell(\lambda) - 2$ by Lemma 3.2. For the second write $\lambda = (a, b, \mu)$ with $\mu \neq \emptyset$; then $e_3(\lambda) \geq a\, e_2(b, \mu) \geq a\, b|\mu| \geq ab$. □

Lemma 3.4. *Let λ be a partition. Then $\ell(\lambda) \leq e_4(\lambda) + 3$ and $\ell(\lambda) \leq e_5(\lambda) + 4$; if $\ell(\lambda) \geq 4$ then $\lambda_1\lambda_2\lambda_3 \leq e_4(\lambda)$, and if $\ell(\lambda) \geq 5$ then $\lambda_1\lambda_2\lambda_3\lambda_4 \leq e_5(\lambda)$.*

Proof. Both are the induction of Lemma 3.3 carried one and two levels further, using $e_j(a, \mu) = a\, e_{j-1}(\mu) + e_j(\mu)$ at each step. □

3.2. **The search, and its completeness.** For $k = 2$ the search enumerates non-increasing sequences of positive integers, extending a prefix by a part no larger than the last one used, and discarding a prefix when $e_2(\text{prefix}) > n$; the depth is bounded by $n + 2$ and the first part by n, both by Lemma 3.2. For $k \geq 3$ the same scheme is used from level k onwards, while the first $k - 1$ parts are enumerated directly under the bound $\lambda_1 \cdots \lambda_{k-1} \leq n$ of Lemmas 3.3 and 3.4.

Theorem 3.5. *Let $n \geq 1$ and let $I_2(n)$ be the duplicate-free list obtained by applying* pre_2 *to every partition returned by the pruned search above. Then for every partition μ,*

$$\mu \in I_2(n) \quad \Longleftrightarrow \quad \big(\mu = \mathrm{pre}_2(\lambda) \text{ for some partition } \lambda\big) \ \text{ and } \ |\mu| = n.$$

In particular $\mathrm{pre}_2(n) = |I_2(n)|$, *and* $\mathrm{pre}_2(n)$ *is an effectively computable function of n.*

Proof. Soundness — everything returned is a partition with $e_2 \leq n$ — is an induction on the recursion depth: the prefix is non-increasing by construction, its parts are positive, and the prune is exactly the assertion $e_2 \leq n$. Completeness is the converse induction. Let λ be a partition with $e_2(\lambda) = n$. By Lemma 3.2 it has at most $n + 1$ parts and largest part at most n, so it lies inside the depth and branching bounds; and by Lemma 3.1 every prefix of it satisfies $e_2 \leq n$, so no prune removes it at any level. Hence λ is visited. Applying pre_2 and deduplicating turns the enumeration of $P_2(n)$ into an enumeration of $\mathrm{Pre}_2(n)$, and the displayed equivalence is the statement that the two agree. The quantifier over λ on the right is over all partitions whatsoever, which is the point: no bound on $|\lambda|$ is assumed. □

Theorem 3.6. *The corresponding statement holds at $k = 3$, at $k = 4$ and at $k = 5$: for each of these k and each $n \geq 1$ the pruned search returns exactly the partitions λ with $e_k(\lambda) = n$, and after applying* pre_k *and deduplicating one obtains exactly* $\mathrm{Pre}_k(n)$. *Hence* $\mathrm{pre}_k(n)$ *is effectively computable for $k = 2, 3, 4, 5$.*

Proof. Identical to Theorem 3.5, with Lemma 3.3 or Lemma 3.4 in place of Lemma 3.2. The only structural change is at the top of the search: for $k \geq 3$ the bound $e_k(\text{prefix}) \leq n$ is vacuous while the prefix has fewer than k parts, and the product bound $\lambda_1 \cdots \lambda_{k-1} \leq e_k(\lambda) = n$ supplies the missing finiteness at those levels. □

Remark 3.7. The gain over an unpruned enumeration is not a constant factor. At $n = 50$ the set of partitions of weight at most $n + 1$, which is the crudest finite superset of $P_2(n)$, has 1 535 914 elements against 381 nodes visited by the pruned search; at $n = 1000$ the crude bound is of order 10^{31} while the pruned search visits 1 118 869 nodes. Proving the bound is what makes the computation possible, not merely what makes it trustworthy.

4. Values

Theorem 4.1. *The values* $\mathrm{pre}_2(n)$ *for* $1 \le n \le 460$ *are determined; those for* $n \le 60$ *are*

n	1	2	3	4	5	6	7	8	9	10
0+	1	1	2	1	2	2	2	2	3	2
10+	3	3	3	3	5	2	4	4	4	4
20+	6	3	5	6	5	5	7	4	7	6
30+	5	7	9	5	8	8	6	7	11	6
40+	9	9	8	9	12	7	10	11	10	10
50+	12	8	13	13	10	12	15	7	14	15

Moreover

$$\mathrm{pre}_2(100) = 17, \quad \mathrm{pre}_2(200) = 82, \quad \mathrm{pre}_2(300) = 189, \quad \mathrm{pre}_2(400) = 302,$$
$$\mathrm{pre}_2(500) = 603, \quad \mathrm{pre}_2(600) = 983, \quad \mathrm{pre}_2(700) = 1376,$$
$$\mathrm{pre}_2(800) = 2232, \quad \mathrm{pre}_2(900) = 3200, \quad \mathrm{pre}_2(1000) = 4219.$$

Proof. Each value is the cardinality of the finite set $I_2(n)$ of Theorem 3.5, evaluated by exhaustive search; the completeness half of that theorem is what makes the result a value of $\mathrm{pre}_2(n)$ rather than of a truncated enumeration. The searches are described in Section 11. □

Theorem 4.2. *The values* $\mathrm{pre}_3(n)$ *for* $1 \le n \le 400$ *are determined; those for* $n \le 60$ *are*

n	1	2	3	4	5	6	7	8	9	10
0+	1	1	1	2	1	1	2	1	1	3
10+	1	2	2	1	1	3	2	1	2	3
20+	1	4	1	2	3	1	2	4	1	2
30+	3	3	1	4	2	2	3	3	2	5
40+	1	2	3	4	2	4	2	1	3	4
50+	2	7	1	2	2	4	2	6	2	5

and $\mathrm{pre}_3(400) = 23$.

Theorem 4.3. *The values* $\mathrm{pre}_5(n)$ *for* $1 \le n \le 500$ *are determined; those for* $n \le 60$ *are*

n	1	2	3	4	5	6	7	8	9	10
0+	1	1	1	1	1	2	1	1	1	1
10+	2	1	1	1	1	2	1	1	1	2
20+	3	1	1	1	1	2	1	1	2	1
30+	2	1	1	1	1	4	1	2	1	1
40+	2	2	1	1	1	2	2	1	1	1
50+	3	2	1	1	2	4	1	1	1	1

Moreover $\mathrm{pre}_5(200) = 3$, $\mathrm{pre}_5(300) = 2$, $\mathrm{pre}_5(400) = 3$, $\mathrm{pre}_5(500) = 4$, $\mathrm{pre}_5(600) = 2$, $\mathrm{pre}_5(700) = 3$, $\mathrm{pre}_5(800) = 4$, $\mathrm{pre}_5(900) = 8$ *and* $\mathrm{pre}_5(1000) = 4$.

Remark 4.4. The counting function thins out quickly in k: against $\mathrm{pre}_2(1000) = 4219$ we have $\mathrm{pre}_5(1000) = 4$, the largest value of pre_5 below 1000 is $\mathrm{pre}_5(756) = 11$, and 294 of the first 1000 values of pre_5 equal 1. This is why the question "for which n is $\mathrm{pre}_k(n) = 1$?" becomes harder, not easier, as k grows.

Remark 4.5. The *preimage* counts of the same search — the number of partitions λ with $e_2(\lambda) = n$ — form the sequence [7, A026903], and they agree with its 1000-term table for every n from 1 to 1000 without exception. Since the agreement is with the enumeration of $P_2(n)$ rather than with the counting step, this is a check on precisely the part of the computation that a bound error would corrupt. No analogous external table exists at $k \ge 3$, nor for the sequences $\mathrm{pre}_k(n)$ themselves.

5. Removing pre_k from the problem

The classification results all rest on the following reformulation, which replaces a question about the image of pre_k by a question about the value set of e_k. The proof is uniform in k; we state it at the four values of k treated here.

Theorem 5.1 (Reformulation)**.** *Let $k \in \{2,3,4,5\}$ and $n \geq 1$. Then $\mathrm{pre}_k(n) \geq 2$ if and only if there is a partition λ with $\ell(\lambda) \geq k+1$ and $e_k(\lambda) = n$.*

Proof. ($\Leftarrow$) By Proposition 2.3, $(n) \in \mathrm{Pre}_k(n)$. Given λ with $\ell(\lambda) \geq k+1$ and $e_k(\lambda) = n$, the partition $\mathrm{pre}_k(\lambda)$ lies in $\mathrm{Pre}_k(n)$ and has $\binom{\ell(\lambda)}{k} \geq k+1 \geq 2$ parts, so it differs from (n).

($\Rightarrow$) Let $\mu \in \mathrm{Pre}_k(n)$ with $\mu \neq (n)$, say $\mu = \mathrm{pre}_k(\lambda)$. Then μ has $\binom{\ell(\lambda)}{k}$ parts, and $\mu \neq \emptyset$ because $n \geq 1$, so $\ell(\lambda) \geq k$. If $\ell(\lambda) = k$ then μ has one part and $\mu = (n)$, a contradiction. Hence $\ell(\lambda) \geq k+1$. □

So E_k is the set of positive integers that are *not* values of e_k on partitions with more than k parts. The next observation is what turns that into arithmetic.

Lemma 5.2 (Progressions)**.** *Let $k \in \{3,4,5\}$ and let ν be a partition with $\ell(\nu) \geq k$. For every $a \geq 1$, inserting a part a into ν gives a partition with $\ell(\nu)+1 \geq k+1$ parts and*

$$e_k(\nu \cup \{a\}) \;=\; a\, e_{k-1}(\nu) + e_k(\nu).$$

Consequently the whole arithmetic progression

$$\mathrm{AP}(\nu) \;=\; \left\{\, e_k(\nu) + a\, e_{k-1}(\nu) \;:\; a \geq 1 \,\right\}$$

— of modulus $e_{k-1}(\nu)$, residue $e_k(\nu)$, and least element $s(\nu) = e_{k-1}(\nu) + e_k(\nu)$ — consists of integers n with $\mathrm{pre}_k(n) \geq 2$.

Proof. The k-subsets of the enlarged index set are those avoiding the new index, contributing $e_k(\nu)$, and those containing it, contributing $a\, e_{k-1}(\nu)$. Since $e_{k-1}(\nu) > 0$ when $\ell(\nu) \geq k-1$ and all parts are positive, the progression is genuinely infinite. Theorem 5.1 converts each of its members into $\mathrm{pre}_k(n) \geq 2$. □

Remark 5.3. The identity of Lemma 5.2 holds verbatim at $k = 2$, where the modulus becomes $e_1(\nu) = |\nu|$, and read through it Garg's proof is exactly a covering system. His five cases use the witnesses $((n-1)/2, 1, 1)$, $(n/3 - 1, 1, 1, 1)$, $((n-4)/4, 2, 2)$, $((n-6)/4, 1, 1, 1, 1)$ and $(2, 2, 1)$; in the present notation the first four are $\mathrm{AP}(\nu)$ for $\nu = (1,1)$, $(1,1,1)$, $(2,2)$, $(1,1,1,1)$, with moduli $e_1(\nu) = 2, 3, 4, 4$ and residues $1, 0, 0, 2$, and $\{1 \bmod 2\} \cup \{0 \bmod 4\} \cup \{2 \bmod 4\} = \mathbb{Z}$. The fifth case is an artefact of inserting the new part at the front rather than in sorted position, and disappears once Lemma 5.2 is used as stated. At $k = 2$ the modulus $e_1(\nu) = |\nu|$ is linear in the parts, so every modulus is cheaply available; at $k = 3$ it is the quadratic $e_2(\nu)$, which is why the covering system of the next section is so much larger.

6. The classification at $k = 3$

Set

$$E_3^* = \{1, 2, 3, 5, 6, 8, 9, 11, 14, 15, 18, 21, 23, 26, 29,$$
$$33, 41, 48, 53, 75, 89, 111, 141, 173, 179, 251, 273, 321\},$$

a set of 28 integers with maximum 321.

6.1. The bounded statement.

Proposition 6.1. *For every n with $1 \leq n \leq 1000$ we have $\mathrm{pre}_3(n) = 1$ if and only if $n \in E_3^*$. In particular $\mathrm{pre}_3(n) \geq 2$ for every n with $322 \leq n \leq 1000$, and $\mathrm{pre}_3(321) = 1$.*

Proof. Direct evaluation of $\mathrm{pre}_3(n)$ for each n in the range, using Theorem 3.6. □

Proposition 6.1 already shows that the naive transfer of Garg's theorem to $k = 3$ fails badly, and fails past any threshold one would guess from $k = 2$: there, $\mathrm{pre}_2(n) \geq 2$ for all $n \geq 5$, whereas here $\mathrm{pre}_3(321) = 1$. It says nothing beyond $n = 1000$. The rest of this section removes that restriction.

6.2. **The covering system.** By Theorem 5.1 and Lemma 5.2 it suffices to exhibit a set G of partitions, each with at least three parts, such that every $n \geq 1$ outside E_3^* lies in $\mathrm{AP}(\nu)$ for some $\nu \in G$. Write, for $\nu \in G$,

$$m(\nu) = e_2(\nu), \qquad r(\nu) = e_3(\nu), \qquad s(\nu) = m(\nu) + r(\nu),$$

and call n *good* if $s(\nu) \leq n$ and $n \equiv r(\nu) \pmod{m(\nu)}$ for some $\nu \in G$. Being good implies $n \in \mathrm{AP}(\nu)$, hence $\mathrm{pre}_3(n) \geq 2$.

The set G used here has 328 partitions, 2443 parts in total, largest part 68, greatest number of parts 23; its moduli $m(\nu)$ take 49 distinct values, the largest being 819, and $\max_\nu s(\nu) = 7173$. The 26 moduli that divide $28665 = 3^2 \cdot 5 \cdot 7^2 \cdot 13$ do the covering work; the remaining ones serve only the range below 7173. Nothing in what follows depends on this particular choice: any G passing the two checks below would do.

Proposition 6.2 (The two checks)**.** (A) *For every n with $1 \leq n < 7173$: n is good if and only if $n \notin E_3^*$. (7172×328 modular tests.)*

(B) *For every residue x with $0 \leq x < 28665$ there is $\nu \in G$ with $m(\nu) \mid 28665$, $s(\nu) \leq 7173$ and $x \equiv r(\nu) \pmod{m(\nu)}$. ($28665 \times 328$ modular tests.)*

Both hold.

Proof. Each is a finite computation over the tabulated pairs $(m(\nu), r(\nu))$, performed once and reported in Section 11. Note that neither check evaluates pre_3, or even e_3, at any n: the pairs are precomputed from G and the checks are purely modular, which is what makes them cheap. □

Proposition 6.3. *Every $n \geq 1$ with $n \notin E_3^*$ is good.*

Proof. For $n < 7173$ this is check (A). For $n \geq 7173$, put $x = n \bmod 28665$ and take ν as in check (B). Since $m(\nu) \mid 28665$ we get $n \equiv x \equiv r(\nu) \pmod{m(\nu)}$, and $s(\nu) \leq 7173 \leq n$. □

Theorem 6.4. *For every $n \geq 1$,*

$$\mathrm{pre}_3(n) = 1 \quad \Longleftrightarrow \quad n \in E_3^*.$$

Consequently E_3 is finite, $|E_3| = 28$, $\max E_3 = 321$, and $\mathrm{pre}_3(n) \geq 2$ for every $n \geq 322$.

Proof. ($\Leftarrow$) Every element of E_3^* is at most 321, so this is Proposition 6.1.

($\Rightarrow$) Let $n \geq 1$ with $n \notin E_3^*$. By Proposition 6.3 there is $\nu \in G$ with $s(\nu) \leq n$ and $n \equiv e_3(\nu) \pmod{e_2(\nu)}$. Put $a = (n - e_3(\nu))/e_2(\nu)$, an integer, and $a \geq 1$ because $n \geq s(\nu) = e_2(\nu) + e_3(\nu)$. Lemma 5.2 then produces a partition with at least four parts and $e_3 = n$, and Theorem 5.1 gives $\mathrm{pre}_3(n) \geq 2$; with Proposition 2.3 this is $\mathrm{pre}_3(n) \neq 1$. □

Remark 6.5. Both bounds in check (B) are sharp for this G: restricting to $m(\nu) \leq 700$ leaves the residue 746 uncovered, and restricting to $s(\nu) \leq 7000$ leaves the residue 7076 uncovered. The smallest least common multiple admitting a covering system at all is $1311 = 3 \cdot 19 \cdot 23$, for which the corresponding start bound is 21398; an exhaustive sweep shows no $L < 1311$ admits one. The value 28665 was chosen because it makes the start bound 7173, small enough that the range below it is settled by the same tabulated data.

Remark 6.6. It is natural to look instead for two-parameter witnesses, splitting $\lambda = \{a, b\} \cup \mu$ and writing $p_i = e_i(\mu)$: then $n = abp_1 + (a + b)p_2 + p_3$ and

$$(ap_1 + p_2)(bp_1 + p_2) \;=\; p_1 n + p_2^2 - p_1 p_3,$$

so such a witness exists exactly when a fixed linear function of n factors suitably. For $\mu = (1, 1)$ the condition is that $2n + 1$ be composite — which already covers a set of density 1, and indeed $2n + 1$ is

prime for all 28 elements of E_3^*. But if the relevant linear form is prime it supplies no witness, and under Dickson's conjecture any admissible finite collection of linear forms is simultaneously prime infinitely often. So no finite collection of two-parameter witnesses can settle all large n, and the one-parameter progressions of Lemma 5.2, whose images are full arithmetic progressions, are the right tool. This paragraph is a heuristic obstruction, not a theorem.

Remark 6.7. Theorem 6.4 should not be confused with the much harder question of which integers are *not* of the form $abc+abd+acd+bcd$, that is, e_3 of a partition with *exactly* four parts [7, A027563]. That set is known only conjecturally (126 terms, largest 52061), and its $k = 2$ analogue [7, A025052] is complete only under the generalised Riemann hypothesis [6]. All 28 elements of E_3^* lie in A027563, but A027563 has 23 further terms below 321, the smallest being 30, and $30 = e_3(2, 1, 1, 1, 1, 1)$ shows why: relaxing "exactly four parts" to "at least four parts" is exactly what makes a covering system possible.

7. The classifications at $k = 4$ and $k = 5$

At $k = 4$ and $k = 5$ the machinery of Section 5 still applies, but the two halves of the classification now have different scopes, and it is worth saying precisely why.

7.1. **The half that is unbounded.**

Proposition 7.1. *Let $k \in \{4, 5\}$ and let S be a finite set of positive integers with $\max S = B$. Suppose that among the partitions λ with $\ell(\lambda) \geq k$ and $e_k(\lambda) \leq B$ — a finite set, by Lemmas 3.3 and 3.4 — none with $\ell(\lambda) \geq k+1$ has $e_k(\lambda) \in S$. Then $\mathrm{pre}_k(n) = 1$ for every $n \in S$, with no further hypothesis on n.*

Proof. A witness for $\mathrm{pre}_k(n) \geq 2$ at some $n \in S$ would, by Theorem 5.1, be a partition λ with $\ell(\lambda) \geq k + 1$ and $e_k(\lambda) = n \leq B$; such a λ is inside the enumerated set by Theorem 3.6, and is excluded by hypothesis. Now apply Proposition 2.3. □

The point is that the search is complete over *all* partitions, not merely over those a bounded search would reach, so a single run at B rules out every witness for every listed n at once.

Theorem 7.2. *There is an explicit set E_4^* of* 171 *integers, the smallest being* 1 *and the largest* 5942*,*

$$E_4^* = \{1, 2, 3, 4, 6, 7, 8, 10, 11, 12, 14, 18, 19, 20,$$
$$22, 24, 26, 27, 31, 32, \ldots, 4439, 4754, 5246, 5702, 5822, 5942\},$$

with the following two properties.

1. $\mathrm{pre}_4(n) = 1$ *for every $n \in E_4^*$, with no range hypothesis.*
2. *For every n with $1 \leq n \leq 100000$: $\mathrm{pre}_4(n) = 1$ if and only if $n \in E_4^*$.*

In particular $E_4 \cap [1, 100000]$ has exactly 171 *elements and greatest element* 5942*, and $\mathrm{pre}_4(n) \geq 2$ for $5943 \leq n \leq 100000$.*

Proof. Part (1) is Proposition 7.1 with $B = 5942$: the enumeration of partitions with at least four parts and $e_4 \leq 5942$ has 112072 members, and none of those with at least five parts has $e_4 \in E_4^*$. Part (2) needs, in addition, a witness at every non-exceptional $n \leq 100000$; these are supplied by 668 partitions ν with $\ell(\nu) \geq 4$ through Lemma 5.2, and the check that their progressions cover exactly the non-exceptional $n \leq 100000$ is a finite modular computation. Being an equivalence, that check also certifies that no progression passes through a member of E_4^*, which would contradict part (1). □

Theorem 7.3. *There is an explicit set E_5^* of* 1103 *integers, the smallest being* 1 *and the largest* 19969*,*

$$E_5^* = \{1, 2, 3, 4, 5, 7, 8, 9, 10, 12, 13, 14, 15, 17, 18, 19, 22, 23, 24, 25, \ldots, 19969\},$$

such that $\mathrm{pre}_5(n) = 1$ *for every* $n \in E_5^*$ *with no range hypothesis, and such that for every* n *with* $1 \le n \le 20000$ *we have* $\mathrm{pre}_5(n) = 1$ *if and only if* $n \in E_5^*$. *In particular* $E_5 \cap [1, 20000]$ *has exactly* 1103 *elements and greatest element* 19969.

Proof. As for Theorem 7.2, with $B = 19969$: the enumeration of partitions with at least five parts and $e_5 \le 19969$ has 439399 members and none with at least six parts has $e_5 \in E_5^*$; and 651 partitions ν with $\ell(\nu) \ge 5$ supply, through Lemma 5.2, progressions covering exactly the non-exceptional $n \le 20000$. □

The ranges 100000 and 20000 are properties of the covering data rather than arbitrary cuts: at $k = 4$ the first non-exceptional integer missed by the 668 progressions is 100094, and at $k = 5$ the first missed by the 651 progressions is 20019 — for which a witness $20019 = e_5(62, 13, 3, 3, 1, 1)$ can still be written down by hand.

7.2. **Why the $k = 3$ argument stops here.** Theorem 6.4 needed a genuine covering system: progressions whose moduli divide a single L and which together cover $\mathbb{Z}/L\mathbb{Z}$. At $k = 4$ the modulus available from ν is $e_3(\nu)$, cubic in the parts rather than quadratic, and there are far fewer residues per modulus. The following is a computation, performed outside the formal development; we state it as such.

Remark 7.4. Using all progressions of modulus at most 3000, every $n \le 2 \cdot 10^6$ outside E_4^* is covered, so progressions are not in short supply. What fails is the common modulus. Writing $R(m)$ for the set of residues realised at modulus m and $\mathrm{dens}(L) = \sum_{m \mid L,\ m \le 3000} |R(m)|/m$, a union of such progressions covers at most $L \cdot \mathrm{dens}(L)$ residues, so $\mathrm{dens}(L) \ge 1$ is necessary; this makes an exhaustive sweep possible, and it shows that *no* $L \le 300000$ admits a covering system at $k = 4$. The best is $L = 240240$, which leaves 11.63% of the residues uncovered. The mean of $|R(m)|/m$ over $m \le 3000$ is 0.665 at $k = 3$, 0.0203 at $k = 4$ and 0.00553 at $k = 5$, a drop of a factor 33 and then 3.7; and the best achievable density grows more slowly in $\log L$ than the requirement does, so the deficit widens rather than closes. This last sentence is a measurement together with a heuristic, not a proof.

Conjecture 7.5. E_4 is finite with maximum 5942, and E_5 is finite with maximum 270007.

The evidence is computational and is not asserted anywhere above: the 171 values of E_4^* are unchanged when the search is pushed to $n = 2 \cdot 10^6$, and at $k = 5$ a search to $n = 2 \cdot 10^7$ returns 1503 values with maximum 270007, unchanged from $n = 10^6$ onwards — of which the 1103 below 20000 are the ones covered by Theorem 7.3 and the remaining 400 are asserted nowhere in this paper. For comparison, at the moment E_3 was settled the ratio of search reach to largest exception was about 310; the corresponding ratios here are about 340 and about 74.

7.3. **Which moduli are available.** One consequence of Theorem 6.4 is worth isolating, because it is the one place where the levels interact.

Proposition 7.6. *For* $m \ge 1$, *the integers* m *that occur as* $e_3(\nu)$ *for some partition* ν *with* $\ell(\nu) \ge 4$ *are exactly the* $m \notin E_3^*$ *; and the integers* m *that occur as* $e_4(\nu)$ *for some* ν *with* $\ell(\nu) \ge 5$ *are exactly those with* $\mathrm{pre}_4(m) \ge 2$.

Proof. Both are Theorem 5.1 read backwards, combined at $k = 3$ with Theorem 6.4. □

So each level's exceptional set is exactly the set of moduli the next level may not use: 4 is the least modulus available at $k = 4$ and 8 is unavailable there, while 5 is the least available at $k = 5$ and 8 is again unavailable.

8. Unbounded density statements at $k = 4$ and $k = 5$

A covering system is out of reach at $k \ge 4$ by Remark 7.4, but a *partial* covering still yields statements holding at every n.

Theorem 8.1. *Let* $L = 240240$.

(1) *There are* 444 *partitions* ν *with* $\ell(\nu) \geq 4$, *all with* $e_3(\nu) \mid L$ *and* $s(\nu) \leq 17380$, *whose progressions meet exactly* 212302 *of the* L *residue classes modulo* L. *Consequently at least* 212302 *residue classes modulo* L *consist entirely, above* 17380, *of integers* n *with* $\mathrm{pre}_4(n) \geq 2$; *equivalently, at most* 27938 *classes — that is,* 11.63% *— contain any* $n \geq 17380$ *with* $\mathrm{pre}_4(n) = 1$.

(2) *There are* 148 *partitions* ν *with* $\ell(\nu) \geq 5$, *all with* $e_4(\nu) \mid L$ *and* $s(\nu) \leq 9262$, *whose progressions meet exactly* 136652 *of the* L *classes. Consequently at least* 136652 *residue classes modulo* L *consist entirely, above* 9262, *of integers* n *with* $\mathrm{pre}_5(n) \geq 2$; *so at most* 103588 *classes — that is,* 43.12% *— contain any* $n \geq 9262$ *with* $\mathrm{pre}_5(n) = 1$.

Proof. If $e_{k-1}(\nu) \mid L$ then membership of n in the residue class of $\mathrm{AP}(\nu)$ is decided by $n \bmod L$; if in addition $n \geq s(\nu)$ then Lemma 5.2 applies and gives $\mathrm{pre}_k(n) \geq 2$. Counting the classes met is a finite computation over $\{0, \ldots, L-1\}$, and the complementary count follows. □

These are the first statements about $\{n : \mathrm{pre}_4(n) = 1\}$ and $\{n : \mathrm{pre}_5(n) = 1\}$ that hold at every n. They do not bound those sets: a partial covering says nothing about the uncovered classes, and uncovered classes exist and are exhibited — the class of 26 modulo 240240 at $k = 4$ and the class of 2 at $k = 5$. Whether the coverage tends to 1 as $L \to \infty$ is precisely Conjecture 7.5, and no finite certificate decides it. The certificates above are not optimal: $L = 720720$ reaches 90.56% at $k = 4$ and $L = 2162160$ reaches 65.17% at $k = 5$.

9. Sharpness

The statements of this section are what keep the results above from being satisfiable by a misplaced quantifier or a mis-tuned search bound.

Proposition 9.1. (1) $\mathrm{pre}_2(n) \geq 3$ *for all* $n \geq 5$ *is false:* $\mathrm{pre}_2(5) = 2$. *So the constant* 2 *in Garg's theorem cannot be raised.*

(2) *Devnani–Eyyunni's bound cannot lose its halving:* $\mathrm{pre}_2(n) \geq \tau(n+1)$ *fails at* $n = 4$, *where* $\tau(5) = 2$ *and* $\mathrm{pre}_2(4) = 1$.

(3) $E_2 = \{1, 2, 4\}$ *cannot be enlarged by* 8, *the one value Garg's proof handles by a separate case:* $\mathrm{pre}_2(8) \neq 1$.

(4) pre_2 *is not monotone:* $\mathrm{pre}_2(16) = 2 < 5 = \mathrm{pre}_2(15)$. *So no proof of the classification can proceed by monotonicity from the small values.*

(5) pre_3 *is not* pre_2*:* $\mathrm{pre}_2(5) = 2$ *while* $\mathrm{pre}_3(5) = 1$.

Proposition 9.2. *The literal transfer of Garg's theorem to* $k = 3$ *fails, and fails past any threshold suggested by* $k = 2$*: the assertion* $\mathrm{pre}_3(n) \geq 2$ *for all* $n \geq 5$ *fails at* $n = 5$, *and the assertion* $\mathrm{pre}_3(n) \geq 2$ *for all* $n \geq 200$ *fails at* $n = 321$.

Proposition 9.3. *Every bound used by the searches of Section 3 is at its minimum useful size. Truncating the* $k = 2$ *search to depth* 3 *undercounts at* $n = 6$, *where the partition* $(1, 1, 1, 1)$ *and its image* (1^6) *are lost, while depth* 4 *suffices there; capping the largest part at* 2 *undercounts at* $n = 5$, *where* $(5, 1)$ *and its image* (5) *are lost. At* $k = 3$, *allowing only three parts undercounts at* $n = 4$, *and capping* $\lambda_1\lambda_2$ *at* 2 *instead of* n *loses* $(4, 1, 1)$ *and* $(2, 2, 1)$; *in each case one further unit of budget restores the correct value. Finally the hypothesis* $n \geq 1$ *in Theorems 3.5 and 3.6 is load-bearing: at* $n = 0$ *the right-hand side holds for the empty partition while the enumeration is empty.*

Proposition 9.4. *Reading "products of* k *distinct parts" as products of* k *distinct* values *is refuted by the sources' own arithmetic: it would make* $\mathrm{pre}_2(5, 1, 1)$ *a single part of weight* 5, *where Garg's Case 1 records weight* $2a + 1 = 11$, *and it would give* $\mathrm{pre}_2(2, 2, 1)$ *one part where the source records three. Reading the index set as ordered pairs is refuted the same way: it would give six parts and weight* 16 *for* $\mathrm{pre}_2(2, 2, 1)$ *against the recorded three parts and weight* 8.

Remark 9.5. The classification data are similarly pinned. At $k = 3$ no progression of G passes through an element of E_3^*, as none may; at $k = 4$ the exceptional set does not stop at its 170th element 5822, and 321 — the largest $k = 3$ exception — is not exceptional at $k = 4$, so the two sets are genuinely different objects; at $k = 5$ the largest $k = 4$ exception 5942 satisfies $\mathrm{pre}_5(5942) \geq 2$, witnessed by $5942 = e_5(104, 14, 1, 1, 1, 1)$, while 5 is exceptional at $k = 5$ and not at $k = 4$.

10. Published statements re-derived

Every statement in the sources that can be tested against Definition 2.1 has been, and all of them reproduce.

Theorem 10.1 (Garg; independently Thomas–Tung)**.** *For every $n \geq 1$, $\mathrm{pre}_2(n) = 1$ if and only if $n \in \{1, 2, 4\}$.*

Proof. The values $\mathrm{pre}_2(1) = 1$, $\mathrm{pre}_2(2) = 1$, $\mathrm{pre}_2(3) = 2$ and $\mathrm{pre}_2(4) = 1$ are computed directly. For $n \geq 5$ the source's five constructions give, in each case, a partition λ with $e_2(\lambda) = n$ and at least three parts, hence with $\binom{3}{2} = 3$ or more pairwise products; by Theorem 5.1 (or directly, since $\mathrm{pre}_2(\lambda) \neq (n)$) this gives $\mathrm{pre}_2(n) \geq 2$. The five cases are: $\lambda = ((n-1)/2, 1, 1)$ for n odd; $\lambda = (n/3 - 1, 1, 1, 1)$ for n even with $3 \mid n$; $\lambda = ((n-4)/4, 2, 2)$ for $4 \mid n$, $3 \nmid n$, $n \geq 12$; $\lambda = ((n-6)/4, 1, 1, 1, 1)$ for $n \equiv 2 \pmod 4$, $3 \nmid n$; and $\lambda = (2, 2, 1)$ for $n = 8$. No search and no bound on n enters. □

We claim no novelty for Theorem 10.1: it is the theorem of [3], proved independently in [4, §3.2]. It is restated here because it is the $\Leftarrow$ half of nothing below and the $k = 2$ case of everything below, and because the proof above is the covering system of Remark 5.3 in disguise.

Proposition 10.2. *Devnani–Eyyunni's lower bound holds for every $n \leq 60$: writing τ for the number-of-divisors function, $\mathrm{pre}_2(n) \geq \lfloor \tau(n+1)/2 \rfloor$, with $\lfloor (\tau(n+1)+1)/2 \rfloor$ in place of it when $n + 1$ is a perfect square. The bound is frequently far from sharp; for instance at $n = 39$ it gives 4 where $\mathrm{pre}_2(39) = 11$.*

Proposition 10.3. *Devnani–Eyyunni's worked instance at $n = 23$ reproduces: $\mathrm{pre}_2(11, 1, 1) = (11, 11, 1)$, $\mathrm{pre}_2(7, 2, 1) = (14, 7, 2)$ and $\mathrm{pre}_2(5, 3, 1) = (15, 5, 3)$ are three distinct partitions of 23 in the image, so $\mathrm{pre}_2(23) \geq 3$; the exact value is $\mathrm{pre}_2(23) = 5$.*

Remark 10.4. One incidental observation, recorded rather than claimed. Thomas and Tung prove that pre_3 is injective on the three-part partitions of n exactly for $3 \leq n \leq 12$ and for $n = 15, 18$, verifying non-injectivity by hand for $n = 14$, $n = 16$ and $n > 18$; the value $n = 13$ is absent from that list. Their conclusion is nevertheless correct, and the missing case is supplied by a pair already printed in [2]: $(6, 6, 1)$ and $(9, 2, 2)$ are distinct partitions of 13 with $\mathrm{pre}_3(6, 6, 1) = \mathrm{pre}_3(9, 2, 2) = (36)$.

11. Verification

Every theorem, proposition and lemma stated above has been formalised and machine-checked in Lean 4 against Mathlib. The structural part of the development — the monotonicity and depth bounds of Section 3, the soundness and completeness of the searches at $k = 2, 3, 4, 5$, the reformulation of Theorem 5.1 and the progression lemma Lemma 5.2 at each of those k, the deductions of Theorems 6.4, 7.2, 7.3 and 8.1 from their finite checks, and Theorem 10.1 together with its five constructions — uses no axioms beyond `propext`, `Classical.choice` and `Quot.sound`, and several of the worked examples of Proposition 2.4 depend on no axioms at all. What is delegated to compiled evaluation, through Lean's `native_decide`, is exactly the finite searches and modular checks: the values of $\mathrm{pre}_2(n)$ for $61 \leq n \leq 460$ and the further values to 1000, of $\mathrm{pre}_3(n)$ for $61 \leq n \leq 400$ and of $\mathrm{pre}_5(n)$ for $61 \leq n \leq 1000$ (the values up to $n = 60$ are checked by the kernel itself); the bounded classification of Proposition 6.1; checks (A) and (B) of Proposition 6.2; the two checks behind each of Theorems 7.2 and 7.3; and the residue counts of Theorem 8.1. Each such evaluation contributes its own axiom, so the axiom list of a statement names exactly which of its ingredients were evaluated

rather than proved. Every numerical value quoted was first produced by independent implementations outside Lean and the formal statement was then written against them: the searches were implemented twice in C and once in Python, the Python implementation recomputing e_k from the combinatorial definition rather than from the recursion the other implementations share, and the exceptional set at each of $k = 3, 4, 5$ was reproduced by several mutually independent programs — four of them at $k = 4$, where the enumerating search, a sieve over the progressions, a combinatorial re-derivation and a separately written depth-first search all agree. The external agreement of Remark 4.5 with the tabulated [7, A026903] for every $n \leq 1000$ is the strongest single check, because it tests the enumeration rather than the counting step. The formal development is available for inspection; repository reference to be added at submission.

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