THREE LARGEST-KNOWN DIAMETER-THREE GRAPHS MADE EXPLICIT, AND THEIR MAXIMALITY

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ABSTRACT. Isham, Lakhotia, Monroe and Petrini recently proved that the polarity quotient of a Kronecker product of generalized polygons is the Kronecker product of their polarity quotients, and deduced the existence of diameter-three graphs of degree 18, 19 and 20 on 2340, 2470 and 2600 vertices — larger, at those degrees, than anything in the published tables. Their argument is existential: no graph and no polarity is exhibited. We build the three graphs, from a Suzuki–Tits polarity of the symplectic quadrangle $W(8)$ obtained by the Klein correspondence, and certify their order, degree sequence, diameter and connectedness by exhaustive computation. We then record two structural facts about them. The 130 absolute vertices are not merely a set at mutual distance at least three, as Delorme observed, but a perfect 1-code: their closed neighbourhoods partition the vertex set. Consequently the 2-packing number of the 2340-vertex graph is exactly 130, so 2470 and 2600 are the largest orders that Delorme replication can extract from it. Our main results are that all three graphs are *maximal*: no vertex can be added to any of them without raising the maximum degree or the diameter. We have not found this question asked of a largest-known degree/diameter graph before, although the reverse technique — adding vertices to a good graph — is a standard source of table entries. Every statement below is machine-checked in Lean 4, apart from one elementary reduction that is flagged where it occurs.

1. Introduction

The *degree/diameter problem* asks for the largest order $N(\Delta, D)$ of a simple graph with maximum degree Δ and diameter D. Counting the vertices at each distance from a fixed vertex gives the Moore bound

$$N(\Delta, D) \;\le\; \mathrm{MB}(\Delta, D) \;=\; 1 + \Delta \sum_{i=0}^{D-1} (\Delta - 1)^i,$$

which is attained only in a handful of famous cases. Exact values are known for almost no cell, so the subject keeps a *table of largest known graphs*, and progress means a construction that beats a cell of it [7, 8, 5].

This paper is about three cells of the row $D = 3$. Isham, Lakhotia, Monroe and Petrini [1] study the *polarity quotient* $P(\mathbb{G})$ of the incidence graph of a generalized polygon: one vertex per orbit $\{x, \rho x\}$ of a polarity ρ, with a loop at each absolute vertex. Their main theorem is that the polarity quotient of a Kronecker product is the Kronecker product of the polarity quotients, and their Corollary and Lemma for generalized n-gons give, for the loops-deleted quotient $\bar{P}$ of a product of two generalized quadrangles of orders (q, q) and (r, r), the order $(q^3+q^2+q+1)(r^3+r^2+r+1)$, maximum degree $(q+1)(r+1)$ and diameter 3, with $(q^2+1)(r^2+1)$ absolute vertices; replicating the absolute vertices k times, a technique they attribute to Delorme [3], raises the degree to $(q+1)(r+1)+k$ and the order by $k(q^2+1)(r^2+1)$. At $(q, r) = (1, 8)$ this yields diameter-three graphs of degree 18, 19 and 20 on 2340, 2470 and 2600 vertices, which is the content of their Table 1, captioned "New diameter-3 graphs that are larger for their degree than those cited in previous references, as of August 2026".

Their construction is existential. The paper states plainly: "The combinatorial results in this paper hold for any generalized quadrangle that admits polarity, so we do not describe specific polarities here." No graph, no polarity, and no adjacency data appear in it. The three orders are therefore lower bounds for $N(18, 3)$, $N(19, 3)$ and $N(20, 3)$ that rest on the theory of generalized quadrangles rather than on an object one can hold.

What is done here. First, we make the objects explicit. A Suzuki–Tits polarity of the symplectic quadrangle $W(8) = G_4(8, 8)$ is constructed from scratch by the Klein correspondence (Section 3); the resulting graphs are literal adjacency tables, and their order, maximum and minimum degree, diameter and connectedness are established by exhaustive finite checks (Section 4). The existence claims are not ours — they are [1]'s, announced in August 2026, shortly before the present work — but the graphs are, and with them the ability to ask questions of the objects instead of the construction.

Second, we record what the absolute vertices really are (Section 5). Delorme's published observation, quoted in [1], is that the absolute vertices of a polarity quotient lie at mutual distance at least three, which is what licenses replication. In fact they form a *perfect* 1-*code*: every vertex of the graph has exactly one absolute vertex in its closed neighbourhood, so the closed neighbourhoods of the absolute set partition the vertex set. This is a short count on top of Delorme's observation, not a new theorem, but it has not been written down for these graphs, and it is what makes the rest exact: it forces the 2-packing number of the 2340-vertex graph to be exactly 130, so that no replication of that graph can do better than $2340 + 130k$ at degree $18 + k$. The orders 2470 and 2600 are optimal for the method that produced them.

Third — the new results — all three graphs are maximal (Section 6). The only vertices with spare degree are the 130 absolute ones and their replicas; each of them is within distance two of at most one absolute vertex; and there are 130 absolute vertices against at most $\Delta \leq 20$ available neighbours for a new vertex. Hence no vertex can be attached to any of the three graphs without pushing the maximum degree above Δ or the diameter above three.

We have not found the question "is a largest-known degree/diameter graph maximal?" asked or answered anywhere, for any cell of the table. The technique is thoroughly documented in the opposite direction. Comellas' table paper [8] has a section headed "Addition of vertices" recording real table entries won that way, for instance $(13, 3) = 856$, obtained in 2021 by Pelekhaty from an 851-vertex graph of degree 13 and diameter 3 "by adding five new vertices and reconnecting several vertices". Nor is the technique historical, or remote from the graphs studied here: the per-cell descriptions accompanying the same table record that the degree-15 diameter-three entry was raised from 1215 to 1224 in August 2026, because "[n]ine new vertices are added to the 1215-vertex $(\otimes Q_{2,4})'$ Delorme–Farhi graph which remains unchanged", and they describe that 1215-vertex graph as a "[c]omponent with polarity of the cartesian product of the quocient of a quadrangle by a polarity by itself" — as they do the $(16, 3)$ entry of order 1600, which they credit to Delorme and Farhi [4] (read 29 August 2026). Vertices do get added to graphs built out of polarity quotients of quadrangles, and recently. The question of when they *cannot* be seems simply not to have been put.

Section 7 collects what we could not settle, including a measured but inconclusive ceiling on the one obvious way to beat 2470 at degree 19 without replicating, and Section 8 describes the verification.

The published baseline. The comparison values quoted in [1] were checked independently. The only *table of largest known graphs* we found that reaches degrees 17 to 20 is the Combinatorics Wiki [9], whose page carries the header "the table of the largest known graphs (as of September 2009) … for graphs of degree at most $3 \leq d \leq 20$ and diameter $2 \leq k \leq 10$" and the footer "This page was last modified on 18 February 2022". Its $D = 3$ column reads 1610, 1620, 1638, 1958 at degrees $17, 18, 19, 20$ — re-read on 29 August 2026 — matching the first comparison column of [1, Table 1] exactly. Comellas' table [8], last changed 28 August 2026, and the dynamic survey of Miller and Širáň [7], whose Table 2 is announced for "degree $\Delta \leq 16$ and diameter $D \leq 10$", both stop at degree 16 and so record nothing at these degrees. The only other 2026 paper we found improving a degree/diameter cell is [13], with $N(12, 5) \geq 34\,992$ and $N(16, 5) \geq 147\,456$: diameter five, and disjoint from the cells considered here.

Remark 1.1 (on the second comparison column of [1, Table 1]). Table 1 of [1] carries a second comparison column, attributed to the PolarStar network family [2], with the values 1830, 2128, 2394 at degrees 18, 19, 20. The attribution is exact, but it holds only against the first arXiv version — which is not the version [1] cites. Table 1 of arXiv:2302.07217v1 (14 February 2023) is precisely this comparison, with rows 18 | 1620 | 1830, 19 | 1638 | 2128 and 20 | 1958 | 2394 (interleaved with two Moore-bound-efficiency columns) and its "Best known Order" column taken from the Combinatorics Wiki: *both* comparison columns of [1, Table 1] reproduce that one table. The bibliography of [1], however, points at the SPAA '24 conference version, and by then the table was gone. In arXiv:2302.07217v2 (6 August 2024) and in the published paper the three integers do not occur at all, and nothing replaces the leaderboard: the orders appear only as a scatter plot of feasible radix/order pairs and as the formula $|V(G^*)| = (q^2+q+1)(2d'+2)$ subject to the supernode constraints tabulated there. A reader who follows the citation will not find the numbers. They are, independently, exactly right: maximising $(q^2+q+1)(2d'+2)$ over prime powers q with $d' = d_* - q - 1$ and $d' \equiv 0$ or $3 \pmod 4$ gives 1830, 2128 and 2394 at $d_* = 18, 19, 20$, attained at $(q, d') = (13, 4)$, $(11, 7)$ and $(11, 8)$; and no other supernode family tabulated in [2] beats those three values at those three radices. Nothing below depends on any of this.

2. Preliminaries

All graphs are finite and simple unless they are explicitly called *loop-graphs*, in which case a vertex may carry a loop. For a graph G we write $N(v)$ for the neighbourhood of v, $N[v] = \{v\} \cup N(v)$ for the closed neighbourhood, $\mathrm{d}(u, v)$ for the graph distance, $\Delta(G)$ and $\delta(G)$ for the maximum and minimum degree, and $\mathrm{diam}(G)$ for the diameter. A set $S \subseteq V(G)$ is a 2-*packing* if $\mathrm{d}(u, v) \geq 3$ for all distinct $u, v \in S$; the largest size of a 2-packing is the 2-*packing number*. A set $A \subseteq V(G)$ is a *perfect* 1-*code*, or an efficient dominating set, if every vertex of G has exactly one member of A in its closed neighbourhood; equivalently, the closed neighbourhoods $N[a]$, $a \in A$, partition $V(G)$.

2.1. Generalized quadrangles and polarity quotients.

A generalized n-gon of order (s, t) is a point-line geometry whose incidence graph is bipartite of diameter n and girth $2n$, with $s + 1$ points on every line and $t + 1$ lines through every point. A *polarity* ρ is an involutive automorphism of the incidence graph interchanging the two sides; a vertex x with x incident to ρx is *absolute*. The *polarity quotient* $P(\mathbb{G})$ is the loop-graph on the orbits $\{x, \rho x\}$, two orbits being adjacent when some representatives are incident; an absolute vertex acquires a loop. We write $\bar{P}(\mathbb{G})$ for $P(\mathbb{G})$ with its loops deleted.

The generalized quadrangles $G_4(q, q)$ admit a polarity exactly when $q = 1$ or $q = 2^{2m+1}$; for q even the symplectic quadrangle $W(q)$ is self-dual, and for $q = 2^{2m+1}$ the Suzuki–Tits polarity has $q^2 + 1$ absolute points, forming an ovoid [10, 11]. The loop-graph $P(G_4(q, q))$ is $(q + 1)$-regular on $q^3 + q^2 + q + 1 = (q + 1)(q^2 + 1)$ vertices, so $\bar{P}(G_4(q, q))$ has maximum degree $q + 1$, minimum degree q, and exactly $q^2 + 1$ vertices of degree q. Bachratý, Šiagiová and Širáň [6], in the only other paper we found that uses the phrase *polarity quotient*, state both the degree split and diam $= 3$ for this graph verbatim: their $A(q)$, defined for $q = 2^{2n+1}$ as the quotient of the incidence graph of $W(q)$ by a polarity with the edges between u and $\pi(u)$ suppressed, is our Γ_q. It is Delorme's family [3]; at $q = 8$ it is the degree-9, diameter-3 entry of order 585 in the published tables, listed there as "[q]uotient of the incidence graph of a regular generalized quadrangle by a polarity" under Delorme's name, with the degree-10 entry of order $650 = 585 + 65$ listed as the same quotient "with duplication of some vertices" [8].

We write $\widehat{\Gamma}_q = P(G_4(q, q))$ for the loop-graph and $\Gamma_q = \bar{P}(G_4(q, q))$ for the simple graph. For $q = 1$ the quadrangle $G_4(1, 1)$ is an ordinary quadrangle, whose incidence graph is an octagon; a reflection polarity has two absolute vertices and $\widehat{\Gamma}_1$ is the path on $0, 1, 2, 3$ with a loop at each end.

2.2. The Kronecker product and its quotient.

The Kronecker (tensor, categorical) product $\widehat{\Gamma}_1 \otimes \widehat{\Gamma}_2$ of two loop-graphs has vertex set $V_1 \times V_2$, with (a, b) adjacent to (a', b') when a is adjacent to a' and b is adjacent to b', loops counting as adjacencies. The main theorem of [1] identifies $P(\mathbb{H}\odot\mathbb{K})$ with $P(\mathbb{H})\otimes P(\mathbb{K})$ for bipartite $\mathbb{H},\mathbb{K}$ admitting polarity, where $\odot$ is the reduced Kronecker product of incidence graphs. We only use the right-hand side. Write

$$\Gamma(q,r) \;:=\; \big(\widehat{\Gamma}_q\otimes\widehat{\Gamma}_r\big) \text{ with its loops deleted,}$$

a graph on $(q^3+q^2+q+1)(r^3+r^2+r+1)$ vertices with maximum degree $(q+1)(r+1)$ and $(q^2+1)(r^2+1)$ absolute vertices — the pairs of absolute vertices, which are exactly the vertices carrying a loop before deletion, and hence exactly the vertices whose degree drops to $(q+1)(r+1)-1$. We index the vertices of $\Gamma(q,r)$ by $v = a\,n_r + b$ where $n_r = r^3+r^2+r+1$, a is a vertex of $\widehat{\Gamma}_q$ and b one of $\widehat{\Gamma}_r$.

2.3. **Replication.** Let G be a graph and $S\subseteq V(G)$ a 2-packing. The *k-fold Delorme replication* of S adds, for each $s\in S$, k new vertices $s^{(1)},\dots,s^{(k)}$ with $N(s^{(j)}) = N_G(s)$. Since S is a 2-packing, no vertex of G has two neighbours in S, so every degree grows by at most k; the new vertices inherit the degree of their originals. The order grows by $k|S|$. Whether the diameter is preserved is not automatic; for the graphs below it is checked.

2.4. **Graphs from adjacency tables, and a diameter certificate.** Every graph in this paper is presented as a table $f\colon\{0,\dots,n-1\}\to$ lists of vertices, and the graph is the simple graph on $\{0,\dots,n-1\}$ in which u and v are adjacent when $v\in f(u)$. The table is checked to be symmetric, loop-free, repetition-free and in range, after which $\deg(v) = |f(v)|$ exactly, so statements about maximum and minimum degree are equalities rather than estimates.

Two certificates carry the diameter. For the lower bound, a pair $u\neq v$ that is non-adjacent and has no common neighbour witnesses $\mathrm{d}(u,v)\geq 3$, hence $\operatorname{diam}\geq 3$; the check scans all n candidate common neighbours. For the upper bound we use the following triviality, whose value is that it is cheap to evaluate and easy to trust.

Lemma 2.1. *Let G be a graph on $V=\{0,\dots,n-1\}$. If*

$$\bigcup_{x\in N[v]}\;\bigcup_{y\in N[x]} N[y] \;=\; V \qquad \textit{for every } v\in V,$$

then G is connected and $\operatorname{diam}(G)\leq 3$.

Proof. Given u,v, the hypothesis at u produces $x\in N[u]$, $y\in N[x]$ with $v\in N[y]$, i.e. a walk u,x,y,v of length at most three, each step being an edge or a repetition. □

Encoding each closed neighbourhood as the bitmask $\sum_{u\in N[v]} 2^u\in\mathbb{N}$ turns the hypothesis of Lemma 2.1 into one equation $B(v) = 2^n-1$ per vertex, where B is obtained by two rounds of bitwise OR over closed neighbourhoods. The cost is $n(\Delta+1)$ machine ORs of n-bit integers, which is what makes an all-pairs diameter statement affordable at $n = 9750$; the timings are in Section 8.

3. An explicit Suzuki–Tits polarity

Since [1] names no polarity, we construct one. Let $q = 2^e$ and let $W(q)$ be the symplectic quadrangle: its points are the q^3+q^2+q+1 points of $\mathrm{PG}(3,q)$ and its lines are the totally isotropic lines of the alternating form

$$B(x,y) \;=\; x_0y_3 + x_3y_0 + x_1y_2 + x_2y_1.$$

For a line $\langle u,v\rangle$ put $p_{ij} = u_iv_j + u_jv_i$. The line is totally isotropic if and only if $p_{03} = p_{12}$, so on such lines the Klein quadric $p_{01}p_{23}+p_{02}p_{13}+p_{03}p_{12} = 0$ becomes $p_{01}p_{23}+p_{02}p_{13}+t^2 = 0$ in the coordinates $(p_{01},p_{02},p_{13},p_{23},t)$ with $t = p_{03} = p_{12}$: a parabolic quadric $Q(4,q)$ whose nucleus is the t-axis. Projecting from the nucleus gives a map

$$\delta\colon \langle u,v\rangle \;\longmapsto\; (p_{01},p_{02},p_{13},p_{23})\in\mathrm{PG}(3,q),$$

and dually δ sends a point p to the image under this map of the pencil of totally isotropic lines through p, which is again a line. Thus δ is a duality of $W(q)$. Computation gives $\delta^2 = \mathrm{Frob}$, the collineation induced by $x \mapsto x^2$, when $q = 8$, and $\delta^2 = \mathrm{id}$ when $q = 2$. As Frob has order 3 on $\mathbb{F}_8$ and commutes with δ, the duality

$$\rho \;=\; \delta \circ \mathrm{Frob} \quad (q = 8), \qquad \rho = \delta \quad (q = 2)$$

satisfies $\rho^2 = \mathrm{Frob}^3 = \mathrm{id}$ and is therefore a polarity.

Four checks were run inside the construction: that δ preserves incidence; that the image of every pencil is a line; that $p' \in \rho(p) \Leftrightarrow p \in \rho(p')$, which is exactly involutivity; and that the number of absolute points is $q^2 + 1$, namely 65 for $q = 8$ and 5 for $q = 2$ — the size of the Suzuki–Tits ovoid, and the strongest available sanity check on the whole construction.

Nothing below depends on this provenance. What enters the formal development is the resulting adjacency table of $\widehat{\Gamma}_8$ — 585 vertices, nine entries each — and the table is verified there, independently of where it came from, to be symmetric, repetition-free and in range, with the parameters of Proposition 4.1.

4. The graphs

Proposition 4.1. (i) $\Gamma_2 = \bar{P}(G_4(2,2))$ *has* 15 *vertices, maximum degree* 3*, a vertex of degree exactly* 3*, is connected, and has diameter exactly* 3*.*

(ii) $\Gamma_8 = \bar{P}(W(8))$*, for the polarity* ρ *of Section 3, has* 585 *vertices, maximum degree* 9*, minimum degree* 8*, a vertex of degree exactly* 9*, is connected, and has diameter exactly* 3*.*

These are Delorme's graphs [3], the degree-3 and degree-9 rows of [1, Table 2], and neither is a new record: 585 at $(9, 3)$ is exactly the published entry [8], and [1] claims no record at degree 3 — its Table 2 is a design-space table. They are the positive controls for the construction and the two factors of everything that follows.

Proposition 4.2. $\Gamma(1, 2)$ *has* 60 *vertices, maximum degree* 6*, minimum degree* 5*, a vertex of degree exactly* 6*, is connected, and has diameter exactly* 3*.*

This is the degree-6 row of [1, Table 2], which is a design-space table and not a records claim; 60 is far below the published 111 at $(6, 3)$ [9, 8]. It is included because it is the smallest instance of the product construction and exercises the same code path as the record graphs.

Now let $G_{18} := \Gamma(1, 8)$, on $2340 = 4 \cdot 585$ vertices, and let $A \subseteq V(G_{18})$ be its set of 130 absolute vertices. Let G_{19} and G_{20} be the 1- and 2-fold Delorme replications of A in G_{18}, on 2470 and 2600 vertices.

Proposition 4.3. (i) G_{18} *has* 2340 *vertices, maximum degree* 18*, minimum degree* 17*, a vertex of degree exactly* 18*, is connected, and has diameter exactly* 3*.*

(ii) G_{19} *has* 2470 *vertices, maximum degree* 19*, minimum degree* 17*, a vertex of degree exactly* 19*, is connected, and has diameter exactly* 3*.*

(iii) G_{20} *has* 2600 *vertices, maximum degree* 20*, minimum degree* 17*, a vertex of degree exactly* 20*, is connected, and has diameter exactly* 3*.*

Proof sketch. Each of the three graphs is a table, and each of the six assertions per graph is a finite check run exhaustively over that table: symmetry (every listed neighbour lists back), looplessness, absence of repeated entries, membership in range, $|f(v)| \leq \Delta$ and $|f(v)| \geq \delta$ for all v, and $|f(1)| = \Delta$ for the witness of the maximum. The diameter is bracketed from above by Lemma 2.1 — n bitmask equations, each an OR of $\Delta + 1$ masks of n bits, over $n = 2340$, 2470, 2600 — and from below by the certificate that vertices 0 and 73 are distinct, non-adjacent and have no common neighbour, checked by a scan of all n vertices. Connectedness follows from the same upper bound. No step is a sampling argument; every quantifier is over the whole vertex set. □

The orders in Proposition 4.3 are those announced in [1, Table 1]; the existence claim is theirs, and, as Section 1 records, the largest previously published orders at these degrees are 1620, 1638 and 1958. Against the Moore bound the three graphs reach $2340/5527 = 42.3\%$, $2470/6518 = 37.9\%$ and $2600/7621 = 34.1\%$.

The same recipe at $(q, r) = (2, 8)$ is Example 2 of [1], and we check it as well; no published table reaches degree 27, so these are not records and are not claimed as such.

Proposition 4.4. *$\Gamma(2, 8)$ has 8775 vertices, maximum degree 27, minimum degree 26, a vertex of degree exactly 27, is connected, and has diameter exactly 3. Its k-fold Delorme replications of the 325 absolute vertices, for $k = 1, 2, 3$, have 9100, 9425 and 9750 vertices, maximum degree 28, 29 and 30 respectively, a vertex of that degree, are connected, and have diameter exactly 3.*

Proof sketch. As for Proposition 4.3. The largest check is the diameter bound at $n = 9750$, $\Delta = 30$: 9750 equations between 9750-bit integers, each the OR of 31 masks that are themselves ORs of 31 masks. □

5. The absolute vertices form a perfect code

[1] quotes Delorme for the packing half of the following: "Delorme observes that the absolute vertices … are at distance at least 3 from each other … and further observes that … the absolute vertices of the polarity quotient can be replicated", citing [3, Section 8(c,d)]. We were not able to consult [3] itself, and nothing in this paper is quoted from it: what is attributed to Delorme here is attributed on the authority of [1] and of the table [8]. The words *perfect code*, *dominating* and *partition* do not occur in [1], and we did not find the following recorded for these graphs, or for any polarity quotient of a generalized quadrangle, elsewhere. The nearest published title we are aware of is Cameron, Thas and Payne [12], on polarities of generalized hexagons and perfect codes; that paper was not readable to us either, and we record only that its subject is hexagons rather than quadrangles. What follows is nonetheless a two-line consequence of the published facts, and we state it as such rather than as a new theorem.

Proposition 5.1. *Let $A_8 \subseteq V(\Gamma_8)$ be the 65-element set of absolute vertices of Γ_8 — the Suzuki–Tits ovoid. Then A_8 is a perfect 1-code: every one of the 585 vertices has exactly one member of A_8 in its closed neighbourhood.*

Proposition 5.2. *The 130-element absolute set A of G_{18} is a perfect 1-code: the closed neighbourhoods $N[a]$, $a \in A$, partition the 2340 vertices.*

Proof sketch. Both statements are established by one exhaustive check: for each vertex v of the graph, count the members of the code set lying in $N[v]$ and verify that the count is exactly 1 — 585 counts against a 65-element set, and 2340 counts against a 130-element set.

The arithmetic behind both is a single line. An absolute vertex loses its loop and so has degree $\Delta - 1$, hence $|N[a]| = \Delta$; distinct absolute vertices are at distance at least three, so these closed neighbourhoods are pairwise disjoint; and $|A| \cdot \Delta = |V|$ on the nose, since $(q^2+1)(r^2+1) \cdot (q+1)(r+1)$ is $(q^3 + q^2 + q + 1)(r^3 + r^2 + r + 1)$ identically — $65 \cdot 9 = 585$ and $130 \cdot 18 = 2340$. A disjoint family of sets exhausting the vertex count by cardinality is a partition. □

A second exhaustive check, over all 2340 vertices, establishes that $\deg(v) < 18$ holds if and only if $v \in A$: the absolute set is exactly the set of degree-17 vertices. That check is Check 1 in the proof of Theorem 6.1 below, where it is what the argument needs; we separate it from Proposition 5.2 because the two are independent finite checks and only the first is the perfect-code statement.

Remark 5.3. The arithmetic in the proof of Proposition 5.2 uses nothing but the parameters, so the same argument applies verbatim to $\Gamma(q, r)$ for every pair (q, r) for which the propositions of [1] hold. We verify it only as a computation, and only at $(q, r) = (1, 2)$ and $(1, 8)$: at $(2, 8)$ the per-vertex code count is too expensive in the form we use and was not run. A uniform statement is offered here as a scope observation and is not part of the verified development.

Two consequences are worth isolating. The first is that the perfect code makes the replication bookkeeping exact: every non-absolute vertex of G_{18} has *exactly* one absolute neighbour, so each round of replication raises the degree of every non-absolute vertex by exactly one and leaves the absolute vertices, and the new copies, at degree 17. That is the degree sequence recorded in Proposition 4.3(ii),(iii). The second is that the replication is as large as it could be.

Proposition 5.4. *The* 2*-packing number of* G_{18} *is exactly* 130*: every set of vertices at pairwise distance at least three has at most* 130 *elements, and the absolute set* A *attains the bound. Consequently, Delorme replication applied to* G_{18} *cannot produce more than* $2340+130k$ *vertices at degree* $18+k$*; the orders* 2470 *and* 2600 *of Proposition 4.3 are optimal for the method that produced them.*

Proof. For the upper bound, let S be a set at pairwise distance at least three. The closed neighbourhoods $N[s]$, $s \in S$, are pairwise disjoint: a common element w would give $\mathrm{d}(s,s') \leq \mathrm{d}(s,w) + \mathrm{d}(w,s') \leq 2$. Each has at least $\delta + 1 = 18$ elements by Proposition 4.3(i), so $18\,|S| \leq 2340$ and $|S| \leq 130$. For the lower bound, A has 130 elements and is a 2-packing: if $a \neq b$ in A were adjacent, or had a common neighbour w, then $N[b]$ — respectively $N[w]$ — would contain two members of the code A, contradicting Proposition 5.2. The final sentence is the definition of the replication: it adds k copies of each member of a 2-packing. □

The only computational ingredients here are the minimum degree of Proposition 4.3(i) and the code check of Proposition 5.2; the rest is counting. Note also what the statement does *not* say: it bounds what replication can extract from this graph, not what $N(19,3)$ or $N(20,3)$ may be.

6. Maximality

We come to the new results. Call a graph G with $\Delta(G) \leq \Delta$ and $\mathrm{diam}(G) \leq 3$ *maximal* for $(\Delta, 3)$ if no graph G' obtained from G by adding a single vertex satisfies $\Delta(G') \leq \Delta$ and $\mathrm{diam}(G') \leq 3$. Adding a vertex x with neighbourhood $S \subseteq V(G)$ keeps the degree bound exactly when $|S| \leq \Delta$ and every $s \in S$ has $\deg_G(s) < \Delta$, and keeps the diameter bound exactly when every old vertex w has some $s \in S$ with $\mathrm{d}_G(s,w) \leq 2$, because every path from x leaves through S and cannot return. So maximality is the statement that no admissible S two-step-dominates the graph, and it suffices to exhibit, for each admissible S, one old vertex that S misses.

Theorem 6.1. *Let* A *be the* 130*-element absolute set of* G_{18}*, viewed inside each of the three graphs.*

- (i) *If* $S \subseteq V(G_{18})$ *satisfies* $|S| \leq 18$ *and* $\deg(v) < 18$ *for every* $v \in S$*, then there is a* $w \in A$ *with* $\mathrm{d}_{G_{18}}(s,w) \geq 3$ *for every* $s \in S$*.*
- (ii) *If* $S \subseteq V(G_{19})$ *satisfies* $|S| \leq 19$ *and* $\deg(v) < 19$ *for every* $v \in S$*, then there is a* $w \in A$ *with* $\mathrm{d}_{G_{19}}(s,w) \geq 3$ *for every* $s \in S$*.*
- (iii) *If* $S \subseteq V(G_{20})$ *satisfies* $|S| \leq 20$ *and* $\deg(v) < 20$ *for every* $v \in S$*, then there is a* $w \in A$ *with* $\mathrm{d}_{G_{20}}(s,w) \geq 3$ *for every* $s \in S$*.*

Corollary 6.2. G_{18}*,* G_{19} *and* G_{20} *are maximal for* $(18,3)$*,* $(19,3)$ *and* $(20,3)$ *respectively: no vertex can be added to any of them without raising the maximum degree or the diameter.*

This corollary is the one step of the paper that is argued by hand rather than machine-checked; its proof is the following paragraph, and the statement it rests on, Theorem 6.1, is verified.

Proof. Let G be one of the three, with degree bound Δ, and let G' add a vertex x with $N_{G'}(x) = S$. If $\Delta(G') \leq \Delta$ then $|S| = \deg_{G'}(x) \leq \Delta$ and every $s \in S$ has $\deg_G(s) = \deg_{G'}(s) - 1 < \Delta$, so Theorem 6.1 supplies $w \in A$ with $\mathrm{d}_G(s,w) \geq 3$ for all $s \in S$. Every path in G' from x to w begins with an edge xs for some $s \in S$ and continues inside G, so $\mathrm{d}_{G'}(x,w) = 1 + \min_{s \in S} \mathrm{d}_G(s,w) \geq 4$, taking the minimum over the empty set to be infinite. Hence $\mathrm{diam}(G') \geq 4$. □

Proof of Theorem 6.1, sketch. Two exhaustive checks and a pigeonhole. Fix one of the three graphs, with degree bound Δ and vertex count n, and let L be the set of its vertices of degree less than Δ.

Check 1 (the low-degree set). Over all n vertices, $\deg(v) < \Delta$ if and only if v belongs to an explicitly listed set L. For G_{18}, $L = A$ has 130 elements; for G_{19}, L is A together with the 130 replicas, 260 elements; for G_{20}, L is A together with the 260 replicas, 390 elements. This is the degree-sequence check announced after Proposition 5.2, and its analogues at degrees 19 and 20.

Check 2 (separation). For each $x \in L$ — 130, 260, respectively 390 vertices — the ball $B_2(x)$ of vertices at distance at most two from x is computed as a bitmask and intersected with A; the check is that the intersection has at most one element. In words: a vertex with spare degree is within distance two of at most one absolute vertex.

Pigeonhole. Suppose S is as in the statement and every $w \in A$ has some $s \in S$ with $\mathrm{d}(s, w) \leq 2$; choose one such $\varphi(w) \in S$ for each w. Each $s \in S$ has $\deg(s) < \Delta$, so $s \in L$ by Check 1, so by Check 2 at most one $w \in A$ has $\mathrm{d}(s, w) \leq 2$; hence φ is injective. Then $130 = |A| \leq |S| \leq \Delta \leq 20$, a contradiction. □

The mechanism is worth stating in words, because it is not special to these three graphs. A vertex may only be attached where there is spare degree; in a graph whose degree deficiency is concentrated on a 2-packing, the spare degree sits on vertices that are mutually far apart and therefore cover almost nothing in common; so if the packing is larger than Δ, no admissible neighbourhood can reach all of it. Here the packing is the perfect code of Section 5, of size 130, against $\Delta \leq 20$ — a factor of six. The conclusion is robust rather than delicate, which is perhaps why nobody has needed to check it; what makes it worth recording is that the three graphs it applies to are the current lower-bound witnesses for $N(18, 3)$, $N(19, 3)$ and $N(20, 3)$.

Finally, a formalized universal statement can be true because it is empty, so the hypotheses were checked to be inhabited, along with two controls on the parameters of G_{18}.

Proposition 6.3. *For G_{18}: the diameter is not at most 2; the maximum degree is not at most 17; the order is not 2341; and there is a set of 18 vertices each of degree less than 18, so the hypothesis of Theorem 6.1(i) is satisfiable.*

The first control is the interesting one. A diameter-two graph of maximum degree 18 has at most $\mathrm{MB}(18, 2) = 325$ vertices, so a proof that produced diameter 2 on 2340 vertices would be proving something false; the control pins that the lower-bound certificate of Proposition 4.3 is doing real work.

7. What is not settled

Question 7.1. Is $N(18, 3) = 2340$?

Corollary 6.2 says only that G_{18} cannot be grown one vertex at a time. It says nothing about other graphs, and the Moore bound leaves a factor of 2.4.

Question 7.2. Is there a diameter-three graph of maximum degree 19 on more than 2470 vertices obtained from G_{18} by adding several vertices at once?

This is the one route to beating [1] at degree 19 that does not need a new algebraic idea, and we priced it before deciding not to pursue it. The following numbers were produced by an independent implementation outside the formal development and are reported as measurements, not as theorems. At $\Delta = 19$ every vertex of G_{18} has spare capacity — two at each of the 130 absolute vertices and one at each of the other 2210, for 2470 free edge-endpoints in total — and a new vertex may be attached to any set whose radius-two balls cover $V(G_{18})$. The radius-two balls of G_{18} have sizes $|B_2(v)| \in \{219, 228, 235, 236\}$, so any such covering set has at least $\lceil 2340/236 \rceil = 10$ members; a greedy search found one with 17, so the covering number c lies between 10 and 17, and we did not compute it exactly. A greedy attempt to pack disjoint covering sets found only 7 of them, giving order 2347, which is worse than 2470. Suppose every added vertex is joined only to old vertices. Then the free capacity admits at most $\lfloor 2470/c \rfloor$ new vertices, so order at most $2340 + \lfloor 2470/c \rfloor$. With the proved bound $c \geq 10$ that is $2340 + 247 = 2587$, a gain of 4.7% over 2470; with the greedy value $c = 17$ — which we have *not* proved optimal — it is $2340 + 145 = 2485$, a gain of 0.6%. Only the first is a bound. Either way the prize is bought with an open-ended set-packing search.

Question 7.3. Are the polarity quotients $\Gamma(q, r)$ maximal for $((q+1)(r+1), 3)$ in general?

The proof of Theorem 6.1 needs only that the absolute set is a 2-packing of size $(q^2+1)(r^2+1) > (q+1)(r+1)$ and that each low-degree vertex is within distance two of at most one absolute vertex. The first inequality holds for every pair $(q, r) \neq (1, 1)$, with equality at $(1, 1)$; the second is the part that is checked here by computation, at $(q, r) = (1, 2)$ and $(1, 8)$ only, and we do not know a proof of it from the geometry.

Question 7.4. Is any other largest-known graph in the degree/diameter table maximal?

The question can be asked of every cell, and the certificate that answers it here — a low-degree set that is a large 2-packing — is cheap to look for in any graph whose adjacency list is published.

8. Verification

Every theorem and proposition in this paper is formally verified in Lean 4 against *Mathlib* — the single exception being the elementary reduction of Corollary 6.2 from Theorem 6.1, which is carried out above by hand — and the development contains no `sorry`. The graphs enter as literal adjacency tables; the bridge from a table to a `SimpleGraph`, the degree equalities, Lemma 2.1, the far-pair certificate, and all of Sections 5–6 are kernel-checked, depending on no axioms beyond `propext`, `Classical.choice` and `Quot.sound` together with one axiom per compiled Boolean evaluation. What is delegated to compiled evaluation, through Lean's `native_decide`, is exactly the finite searches named in the proof sketches: for each graph, the table checks (symmetry, looplessness, no repeats, range, degree bounds and one degree witness), the diameter bitmask equations and the far-pair certificate; for Section 5, the per-vertex code counts; and for Section 6, the low-degree identification and the separation check. Each such search is a single Boolean, evaluated over the entire vertex set — 112 Booleans in total, covering graphs up to 9750 vertices and degree 30. These Boolean modules import nothing, not even the standard library, which holds their peak memory near the bare-compiler baseline: the six graphs up to 2600 vertices are checked in 18–27 seconds and the four up to 9750 vertices in 74–81 seconds, both at about 0.8 GB, against 7 GB for the Mathlib layer that consumes them. Every numerical value quoted was first produced by an independent implementation in Python, and the formal statements were written against it. Repository reference to be added at submission.

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