PAIRWISE ORTHOGONAL ITALIAN SQUARES: TRANSVERSAL CRITERIA, AND NEW ENTRIES AT ORDERS 6 AND 7

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Abstract. An *Italian square* of order n is the compressed presentation of an alternating sign hypermatrix, and two of them are orthogonal when every plane of one meets every plane of the other in Frobenius inner product 1. Ernst, Lia, O'Brien, Sheekey and Zumbrägel introduced this notion of orthogonality, proved that a pairwise orthogonal set (a POIS) of order n has at most $n-1$ members, and closed with a table recording, for each order $n \le 7$ and each i, the largest POIS with exactly i non-Latin members; several of its entries are lower bounds or blanks. Applying the partial-sum description of an alternating sign matrix along the third axis, we obtain that an alternating sign hypermatrix is exactly a chain $0 = Q_0, Q_1, \dots, Q_n = J$ of $(0,1)$-matrices whose successive differences are alternating sign matrices, that orthogonality telescopes along that chain, and that the first plane of such a hypermatrix is a permutation matrix. The last of these turns the common-transversal argument the authors use for one order-4 pair into a criterion that tests a pair, and then into one that tests a *single* square. Using them we raise the $(7,1)$ entry of the table from ≥ 2 to ≥ 3 by exhibiting three pairwise orthogonal Italian squares of order 7 exactly one of which is not Latin; and we decide two instances of its open $(6,3)$ cell, showing that the order-6 pair the authors print is maximal — no Italian square of order 6, Latin or not, is orthogonal to both members — and exhibiting a non-Latin Italian square of order 6 that lies in no set of three POIS. Every theorem stated here is machine-checked in Lean 4.

1. Introduction

A Latin square of order n is a permutation hypermatrix: an $n \times n \times n$ array of 0s and 1s whose planes in all three directions are permutation matrices. Alternating sign matrices are the standard relaxation of permutation matrices, and Brualdi and Dahl [2] took the corresponding relaxation one dimension up. An *alternating sign hypermatrix* (ASHM) is an $n \times n \times n$ array with entries in $\{0, 1, -1\}$ whose non-zero entries alternate in sign, beginning and ending with $+1$, along every row, every column and every vertical line. Brualdi and Dahl compressed an ASHM into a square array by the map $L(A) = \sum_k kA_k$ — the *Latin-like square* — and asked whether the resulting objects could beat Latin squares: whether more than $n-1$ of them could be pairwise orthogonal.

Ernst, Lia, O'Brien, Sheekey and Zumbrägel [1] showed that the Latin-like square is the wrong compression. Orthogonality defined on Latin-like squares is not well posed: their Example 2.9 exhibits two distinct ASHMs with the *same* Latin-like square, exactly one of which is orthogonal to a third square. They replace it by the *Italian square*, which records in each cell the ascending list of symbols occurring in the corresponding vertical line together with their alternating signs, and which is therefore in bijection with the ASHM rather than a lossy image of it; their Definition 2.7 states this notion as "adapted from" earlier work of O'Brien [3]. Orthogonality is then defined directly on planes: Italian squares A and B of order n are orthogonal when $\langle A_i, B_j\rangle = 1$ for all $i, j \in [n]$, where $\langle M, N\rangle = \mathrm{tr}(MN^\top)$. A set of pairwise orthogonal Italian squares is a *POIS*. With this definition they prove ([1, Thm. 6.4]) that a POIS of order n has at most $n-1$ members — the same bound as for mutually orthogonal Latin squares (MOLS) — but, as they observe, the bound matching does not preclude Italian squares from beating Latin squares at particular orders, and at $n = 6$ they already do: a POIS of size 2 exists at an order where no pair of MOLS does.

The paper [1] closes with a table, reproduced here, giving for each order n and each i the largest size of a POIS of order n containing exactly i members that are not Latin squares:

$n\backslash i$	0	1	2	3	4
4	3	2	0	0	–
5	4	3	2	0	0
6	1	2	2	?	?
7	6	≥ 2	≥ 2	≥ 3	?

An entry printed as a bare number is a value — so the rows $n = 4, 5$ and the first three entries of the row $n = 6$ are exact; the entries marked $\geq$ are lower bounds realised by exhibited examples, and the entries marked ? are open. The paper describes the computation behind the table as a "simple enumeration technique which generates all ASM planes of orthogonal Italian squares consecutively with an early-abort strategy". Two open questions are singled out in the paper's final section. The first is quoted verbatim:

> *The first open case where the bound on MOLS could be strictly exceeded by POIS is* $n = 10$*, where it is unknown whether there exist three POIS.*

(Two MOLS of order 10 exist by Bose, Shrikhande and Parker [4]; no three are known, and the largest number of MOLS of order 10 has resisted considerable effort [5]. A set of three POIS of order 10 would therefore exceed the largest *known* set of MOLS of that order. It would not be the first order at which Italian squares beat Latin ones — $n = 6$ already is, as noted above — but at $n = 6$ the MOLS value is settled, and at $n = 10$ it is not.) The second concerns the blank at $(6, 3)$, again verbatim:

> *For example, if* $n = 6$ *no set of three POIS containing a Latin square exists, but we do not know yet whether there exist any set of three POIS.*

Reading the row $n = 6$ pins the second question down further. Since the entries $1, 2, 2$ at $i = 0, 1, 2$ are exact and every subset of a POIS is a POIS, a set of three POIS of order 6 must have all three members non-Latin; and a set of four with exactly three non-Latin members would contain a POIS of size three with exactly two non-Latin members, which the entry $(6, 2) = 2$ forbids. So the $(6, 3)$ cell is either 0 or 3, and "are there three POIS of order 6?" is exactly "is that cell non-zero?".

What is proved here. Section 2 sets up the objects and proves the reformulation everything else runs on. Writing $Q_t = A_1 + \cdots + A_t$ for the vertical partial sums of a hypermatrix, A is an ASHM if and only if every plane A_k is an alternating sign matrix, every Q_t is a $(0,1)$-matrix, and $Q_n = J$: an ASHM *is* a chain $0 = Q_0, Q_1, \ldots, Q_n = J$ of $(0,1)$-matrices whose successive differences are alternating sign matrices (Theorem 2.6). Orthogonality telescopes along the chain: $A \perp B$ if and only if $\langle Q_t, B_j \rangle = t$ for all $t \leq n$ and all j (Proposition 2.7). Both come from a single lemma, that alternation of a line is the condition that its partial sums lie in $\{0, 1\}$ and its total is 1.

Section 3 extracts the criteria. Since $A_1 = Q_1$, the first plane of an ASHM is a $(0,1)$-matrix, and being an alternating sign matrix it is a permutation matrix (Proposition 3.1). Hence an Italian square orthogonal to two others supplies a permutation matrix that is a transversal of both, so a pair with no common *permutation* transversal is a maximal POIS (Corollary 3.2). This sharpens the argument [1, Ex. 6.5] uses to show that one particular order-4 pair does not extend, and it is a test that can be applied to a pair before any search begins. Restating it over index functions makes it a finite scan (Proposition 3.5), and one further step gives a test on a *single* square: in a triple $\{A, B, C\}$ the first planes of B and C are two permutation transversals of A whose Frobenius product is 1, so an Italian square no two of whose permutation transversals meet in exactly one cell lies in no set of three POIS (Proposition 3.6).

Section 4 gives the new entry in the table.

> **Theorem 4.1.** There is a set of three pairwise orthogonal Italian squares of order 7 exactly one of which is not a Latin square; the two Latin members are the cyclic

> MOLS pair $r + c$, $r + 2c$ in the identity ordering. Hence the $(7, 1)$ entry of the table above is at least 3.

The table records ≥ 2 there, and [1, Thm. 6.4] caps the entry at 6.

Section 5 decides two instances of the open $(6, 3)$ cell.

> **Theorem 5.1.** The order-6 POIS of size two printed in [1, Ex. 2.12] is maximal: no Italian square of order 6, Latin or not, is orthogonal to both of its members. Each member has exactly 11 permutation transversals and they have none in common.
>
> **Theorem 5.3.** There is a non-Latin Italian square of order 6 that lies in no set of three POIS.

Both are statements the table cannot deliver. Extending the pair of Theorem 5.1 by a *Latin* square is already excluded by the entry $(6, 2) = 2$; extending it by a non-Latin square would have been a witness for $(6, 3)$, and that is what is ruled out. Likewise the exclusion of a Latin square from every triple is the content of the row $n = 6$, whereas Theorem 5.3 excludes a non-Latin one. The cell itself remains open: nothing here decides it, and Remark 5.4 records what a decision would cost.

Section 6 records the order-10 situation — a POIS of size two that passes the gate, and the region of the search that was covered without finding a third square — and Section 7 the re-verification of the worked examples of [1], including an explicit witness for the order-5 entry of its table, for which the table gives a value but no example.

Every theorem, proposition, lemma and corollary below is formally verified in Lean 4, with one flagged exception; the measurements and observations recorded in remarks are not, and Section 8 says which is which.

2. Preliminaries

Throughout, $n \geq 1$ and $[n] = \{1, \ldots, n\}$. Matrices are integer matrices, J is the all-ones matrix, and

$$\langle M, N \rangle \;=\; \operatorname{tr}(MN^\top) \;=\; \sum_{i,j} M_{ij} N_{ij}$$

is the Frobenius inner product.

Definition 2.1. A vector $v \in \mathbb{Z}^n$ *alternates* if the sequence obtained from $v_1, \ldots, v_n$ by deleting the zero entries is $+1, -1, +1, \ldots, +1$ — of odd length, and in particular non-empty. An *alternating sign matrix* (ASM) is a matrix all of whose rows and columns alternate.

Definition 2.2 ([1, Def. 2.5])**.** An *alternating sign hypermatrix* of order n is an array $A = (a_{ijk})$ with $i, j, k \in [n]$ in which every row $j \mapsto a_{ijk}$, every column $i \mapsto a_{ijk}$ and every vertical line $k \mapsto a_{ijk}$ alternates. Its *planes* are the matrices $A_k = (a_{ijk})_{i,j}$.

The *Italian square* of order n [1, Def. 2.7] is the square array whose cell (i, j) carries the ascending list of symbols k with $a_{ijk} \neq 0$, with signs $+, -, +, \ldots$; a cell printed $\substack{2-4\\+6}$ is the vertical line carrying $+1$ in plane 2, -1 in plane 4 and $+1$ in plane 6. The two presentations are in bijection, and we use the words interchangeably: an Italian square *is* its hypermatrix. An Italian square is a *Latin square* exactly when it has no negative entry.

Definition 2.3 ([1, Def. 2.10])**.** Italian squares A and B of order n are *orthogonal*, written $A \perp B$, when $\langle A_i, B_j \rangle = 1$ for all $i, j \in [n]$. A set of pairwise orthogonal Italian squares is a *POIS*.

Definition 2.4 ([1, Def. 5.1])**.** A *transversal* of an Italian square A is an ASM M with $\langle M, A_k \rangle = 1$ for every plane A_k. A *permutation transversal* is a transversal that is a permutation matrix; the second notion is ours, and is what the criteria of Section 3 produce.

The one lemma the rest of the paper rests on replaces the definition of alternation by a condition on partial sums. It is the standard reformulation for alternating sign matrices; what matters here is that it is available along the third axis as well.

Lemma 2.5. *Let $v \in \mathbb{Z}^n$. Then v alternates if and only if*

$$\sum_{s\le t} v_s \in \{0,1\} \quad \text{for every } 0 \le t \le n, \qquad \text{and} \qquad \sum_{s\le n} v_s = 1.$$

More generally, for $b \in \{0,1\}$ the sequence v alternates starting from the sign $1-2b$ and ending with $+1$ if and only if $b + \sum_{s\le t} v_s \in \{0,1\}$ for all t and $b + \sum_{s\le n} v_s = 1$.

Proof. Induction on n with the running total b as the induction parameter; the two cases $v_1 = 0$ and $v_1 \neq 0$ are the two constructors of the alternation condition, and in the second case the sign flips exactly as b moves between 0 and 1. □

An entry of an alternating vector automatically lies in $\{0,1,-1\}$: the partial sums before and after it both lie in $\{0,1\}$. So the entry condition stated separately in Definition 2.2 is not an extra hypothesis, a point we use silently below.

For a hypermatrix A of order n put

$$Q_t \;=\; Q_t(A) \;=\; A_1 + A_2 + \cdots + A_t \qquad (0 \le t \le n),$$

so $Q_0 = 0$ and $Q_t - Q_{t-1} = A_t$.

Theorem 2.6 (Chain description)**.** *A hypermatrix A of order n is an alternating sign hypermatrix if and only if*

(i) *every plane A_k is an alternating sign matrix,*
(ii) *every Q_t, $0 \le t \le n$, is a $(0,1)$-matrix, and*
(iii) *$Q_n = J$.*

Equivalently: an alternating sign hypermatrix is a chain $0 = Q_0, Q_1, \ldots, Q_n = J$ of $(0,1)$-matrices whose successive differences are alternating sign matrices.

Proof. Conditions on rows and columns of A are exactly conditions (i). Applying Lemma 2.5 to the vertical line $k \mapsto a_{ijk}$, for each fixed (i,j), turns the alternation of that line into "all its partial sums lie in $\{0,1\}$ and its total is 1", which read over all (i,j) are exactly (ii) and (iii). □

Proposition 2.7 (Orthogonality telescopes)**.** *For hypermatrices A, B of order n,*

$$A \perp B \iff \langle Q_t(A), B_j\rangle = t \quad \text{for all } 0 \le t \le n \text{ and all } j \in [n].$$

Proof. $\langle Q_{t+1}, B_j\rangle - \langle Q_t, B_j\rangle = \langle A_{t+1}, B_j\rangle$, so the right-hand condition is the telescoped form of $\langle A_k, B_j\rangle = 1$ for all k, j, and $\langle Q_0, B_j\rangle = 0$ starts the induction. □

Theorem 2.6 and Proposition 2.7 together describe the object a search over Italian squares actually walks: build the chain one $(0,1)$-matrix at a time, and maintain the running counts $\langle Q_t, B_j\rangle$. All the searches reported below are of this shape.

3. Transversal criteria

Proposition 3.1. *The first plane of an alternating sign hypermatrix of order $n \ge 1$ is a permutation matrix.*

Proof. $A_1 = Q_1$, which is a $(0,1)$-matrix by Theorem 2.6(ii); and A_1 is an ASM, so by Lemma 2.5 all its row and column sums equal 1. A $(0,1)$-matrix with all line sums 1 is a permutation matrix. □

The last plane is a $(0,1)$-matrix for the same reason, being $J - Q_{n-1}$, but only the first plane is used below.

Corollary 3.2 (The gate)**.** *Let A, B, C be Italian squares of order $n \ge 1$ with $C \perp A$ and $C \perp B$. Then C_1 is a permutation matrix that is a transversal of both A and B. Consequently, if A and B have no common permutation transversal, then no Italian square whatsoever is orthogonal to both, and $\{A, B\}$ is a maximal POIS.*

Proof. C_1 is a permutation matrix by Proposition 3.1, and $\langle C_1, A_j\rangle = \langle C_1, B_j\rangle = 1$ for every j by Definition 2.3. The second sentence is the contrapositive. □

The authors of [1] run this argument by hand for one pair: their Example 6.5 rules out extending the order-4 pair of Example 2.11 by computing the space of matrices orthogonal to all planes of both and observing that it contains no ASM, hence in particular no common transversal. Corollary 3.2 says that a common *permutation* transversal is what is needed, which is a smaller and finite thing to look for, and that the test applies to the pair before any third square is considered.

Proposition 3.3. *For $n \geq 2$ no alternating sign hypermatrix of order n is orthogonal to itself. Consequently the three members of a POIS of size three of order $n \geq 2$ are pairwise distinct.*

Proof. $\langle A_1, A_1\rangle = n$ for the permutation matrix A_1 of Proposition 3.1, and $n \neq 1$. □

To search over the permutation matrices of Corollary 3.2 one needs them indexed. For $f\colon [n] \to [n]$ let P_f be the matrix with $(P_f)_{ij} = 1$ if $f(i) = j$ and 0 otherwise.

Lemma 3.4. $\langle P_f, M\rangle = \sum_i M_{if(i)}$ *for every matrix M, and every permutation matrix is P_f for some injective $f\colon [n] \to [n]$.*

Proof. The first identity is immediate. For the second, the row condition on a permutation matrix P gives, for each i, a unique j with $P_{ij} = 1$; call it $f(i)$, so $P = P_f$. If $f(i_1) = f(i_2)$ with $i_1 \neq i_2$ then column $f(i_2)$ of P has two entries equal to 1 and no negative entries, so its sum is at least 2, contradicting the column condition. □

Write $\mathrm{tr}_P(A)$ for the set of injective $f\colon [n] \to [n]$ with P_f a transversal of A, that is, with $\sum_i a_{if(i)k} = 1$ for every $k \in [n]$. Lemma 3.4 turns Corollary 3.2 into a finite computation.

Proposition 3.5. *Let A, B be Italian squares of order $n \geq 1$ with $\mathrm{tr}_P(A) \cap \mathrm{tr}_P(B) = \emptyset$. Then no Italian square is orthogonal to both.*

The injectivity requirement in Lemma 3.4 is not cosmetic. Dropping it from Proposition 3.5 makes the criterion false: for the order-6 pair of Theorem 5.1 below there are exactly eleven *non-injective* $f\colon [6] \to [6]$ whose $(0,1)$-matrix meets every plane of both squares in 1, and none of them is a permutation matrix.

Corollary 3.2 tests a pair. One more use of Proposition 3.1 gives a test on a single square. Say that injective $f, g\colon [n] \to [n]$ *meet once* if $\#\{i : f(i) = g(i)\} = 1$; by Lemma 3.4 this is exactly $\langle P_f, P_g\rangle = 1$.

Proposition 3.6. *Let A be an Italian square of order $n \geq 1$, and suppose no two elements of* $\mathrm{tr}_P(A)$ *meet once. Then for all Italian squares B, C with $A \perp B$ and $A \perp C$ one has $B \not\perp C$; that is, A lies in no set of three POIS.*

Proof. Let $\{A, B, C\}$ be a POIS. By Proposition 3.1 and Lemma 3.4, $B_1 = P_f$ and $C_1 = P_g$ for injective f, g, and $A \perp B$, $A \perp C$ make both f and g elements of $\mathrm{tr}_P(A)$. Now $B \perp C$ forces $\langle B_1, C_1\rangle = 1$, so f and g meet once. □

Proposition 3.6 is a few lines from Proposition 3.1, and we claim no more for it than that it is stated: what makes it useful is that Corollary 3.2 says nothing at all about a single square, whereas this rules one out outright, at the cost of one scan of the n^n functions $[n] \to [n]$ and $|\,\mathrm{tr}_P(A)|^2$ comparisons. On 60 000 Italian squares of order 6 drawn at random by the sampler of Remark 3.7, it removed at least 14.9 % of them from every possible triple (2.8 % with no permutation transversal at all, 2.1 % with exactly one, and 10.0 % non-Latin with at least two and no meeting pair; Latin squares with at least two and no meeting pair were not counted, so the figure is a lower bound). It is also a genuine hypothesis and not a theorem about all Italian squares of order 6: the square A of Theorem 5.1 has 14 unordered pairs of permutation transversals meeting once.

Remark 3.7 (The sampler, stated). Where a figure is quoted over "random Italian squares of order 6" — in the paragraph above and in Remark 5.4 — the squares were drawn by walking the chain of Theorem 2.6: starting from $Q_0 = 0$, and at each level $t = 1, \dots, n$ choosing uniformly at random among the successors the chain admits at that level, that is, among the $(0, 1)$-matrices Q_t with $Q_t - Q_{t-1}$ an alternating sign matrix. This is *not* the uniform distribution on Italian squares of order 6: at each step the walk gives every admissible successor the same probability, while the number of Italian squares that complete them differs from one successor to the next. The percentages quoted from such samples are therefore properties of this sampler, not estimates of a proportion of all order-6 Italian squares, and they are used only to say that the criteria are not vacuous in practice. No statement of this paper depends on them.

4. A set of three POIS of order 7 with one non-Latin member

Let L_1 and L_2 be the cyclic MOLS of order 7 given by $L_1(r, c) = r + c$ and $L_2(r, c) = r + 2c$ modulo 7, in the identity row and column ordering and with symbols written $1, \dots, 7$:

$L_1 =$

1	2	3	4	5	6	7
2	3	4	5	6	7	1
3	4	5	6	7	1	2
4	5	6	7	1	2	3
5	6	7	1	2	3	4
6	7	1	2	3	4	5
7	1	2	3	4	5	6

$L_2 =$

1	3	5	7	2	4	6
2	4	6	1	3	5	7
3	5	7	2	4	6	1
4	6	1	3	5	7	2
5	7	2	4	6	1	3
6	1	3	5	7	2	4
7	2	4	6	1	3	5

and let C be the Italian square

$C =$

7	3	5	2	6	4	1
3	$1-3+5$	$3-5+7$	5	4	2	6
2	6	4	1	$3-4+7$	5	4
4	3	$2-4+5$	6	4	1	7
1	7	4	3	$2-3+5$	$3-5+6$	5
6	4	1	7	3	5	2
5	2	6	4	1	7	3

which has six cells carrying a negative symbol and is therefore not a Latin square.

Theorem 4.1. *$\{L_1, L_2, C\}$ is a set of three pairwise orthogonal Italian squares of order 7, and exactly one of its members, namely C, is not a Latin square.*

Proof. Each of the three arrays is checked to be an alternating sign hypermatrix, and each of the three pairs is checked to satisfy $\langle \cdot_i, \cdot_j \rangle = 1$ for all 49 pairs of planes. This is a finite computation on explicit integer arrays and it was carried out by compiled evaluation; see Section 8. That L_1, L_2 have no negative entry and C does is likewise checked entrywise. □

Corollary 4.2. *The $(7, 1)$ entry of the table of* [1] *— the largest size of a POIS of order 7 with exactly one non-Latin member — is at least* 3. *By* [1, Thm. 6.4] *it is at most* 6.

The table records ≥ 2 for that entry, realised by an exhibited orthogonal pair; the $n = 7$ row's exact entry is only its first, $(7, 0) = 6$, the complete set of MOLS of order 7.

Remark 4.3 (How C was found, and what was searched). By Theorem 2.6 and Proposition 2.7, finding an Italian square orthogonal to two given Latin squares L_1, L_2 is finding a chain $0 = Q_0, Q_1, \dots, Q_7 = J$ whose successive differences are ASMs meeting every row, every column, every L_1-symbol class and every L_2-symbol class exactly once. A search written directly on that chain — enumerating, at each level, every ASM satisfying the four class conditions and compatible with the current Q (a $+1$ only where Q is 0, a -1 only where Q is 1) — is exhaustive for a fixed pair and a fixed row/column ordering, with no bound imposed on the number of negative entries. The ordering is a real parameter: being an ASM is not invariant under permuting rows or columns, so one abstract pair of MOLS presents a different problem in each ordering. The square C above was found in the first ordering tried, the identity one, which is consistent with the table entry having been a lower bound from an example rather than the result of a completed search. Searching upwards from there, no non-Latin Italian square orthogonal to three MOLS of order 7 was found in 97 exhaustively searched orderings, nor to four or five in 350 further orderings; we therefore claim nothing about the value 4 for this entry.

Remark 4.4 (A false start, recorded). The same ladder run at order 8 initially reported a POIS of size 4. It is not one: the anchors had been generated as $r + ac \bmod n$, which are mutually orthogonal only for prime n, so at $n = 8$ the three "MOLS" were not orthogonal to each other. The square found really was orthogonal to all three anchors, but the set is not a POIS. This was caught by re-running every finished object through an independent checker that tests all pairs from Definitions 2.2 and 2.3, and the claim was withdrawn. Order 7 is prime, so the same construction is sound there, and its output passed the same check.

5. Order 6: two decisions inside the open cell

Let A and B be the two Italian squares of order 6 printed in [1, Ex. 2.12], which are the Italian squares corresponding to Brualdi and Dahl's 6×6 construction [2]:

$A =$

1	6	3	4	5	2
6	3	2−3+4	5	3	1
3	2−3+5	6	1	4	3
5	4	3	2	1	6
2	3	1−3+5	3	6	4
4	1	3	6	2	5

$B =$

1	2	4	5	3	6
3	5	2	4−5+6	1−3+5	3
2	1	6	5	3	4
5	4	1	3	2−5+6	5
4	3−4+6	5	2	4	1
6	4	3	1	5	2

Neither is a Latin square, and $A \perp B$; this is the POIS of size two at an order where no pair of MOLS exists.

Theorem 5.1. $|\operatorname{tr}_P(A)| = |\operatorname{tr}_P(B)| = 11$ *and* $\operatorname{tr}_P(A) \cap \operatorname{tr}_P(B) = \emptyset$. *Consequently no Italian square of order* 6*, Latin or not, is orthogonal to both A and B: the pair $\{A, B\}$ is a maximal POIS, and there is no Italian square C with $\{A, B, C\}$ a POIS.*

Proof. The three cardinalities are computed by scanning all $6^6 = 46\,656$ functions $[6] \to [6]$, keeping the injective ones f with $\sum_i a_{if(i)k} = 1$ for all k, respectively $\sum_i b_{if(i)k} = 1$ for all k; this is a finite evaluation, carried out as described in Section 8. The consequence is Proposition 3.5, and the POIS form follows since orthogonality is symmetric. □

Remark 5.2. The Latin half of Theorem 5.1 — that no *Latin* square of order 6 extends the pair — also follows from the table of [1], since such an extension would be a POIS of size three with exactly two non-Latin members and $(6, 2) = 2$. The non-Latin half does not: an extension by a non-Latin square would be a set of three POIS with exactly three non-Latin members, that is, a witness for the open $(6, 3)$ cell. So Theorem 5.1 decides one instance of that cell, for the one pair the authors themselves print.

The second decision uses the single-square criterion. Let

$\mathsf{N} =$

6	3	2	4	5	1
3	1−3+4	6	2−4+5	3	4
5	2	1	4	6	3
2	5	3	1	4	6
4	3	5	6	1	2
1	6	4	3	2	5

an Italian square of order 6 with exactly two negative entries, in the cells $(2,2)$ and $(2,4)$; in particular N is not a Latin square.

Theorem 5.3. $|\operatorname{tr}_P(\mathsf{N})| = 4$ *and no two elements of* $\operatorname{tr}_P(\mathsf{N})$ *meet once. Consequently* N *lies in no set of three POIS: for all Italian squares* B', C' *with* $\mathsf{N} \perp B'$ *and* $\mathsf{N} \perp C'$ *one has* $B' \not\perp C'$.

Proof. The same scan of the 46 656 functions $[6] \to [6]$ produces the four transversals, and the 16 ordered pairs among them are tested for meeting once; none does. The consequence is Proposition 3.6. □

Since N is not Latin, this too is a statement inside the $(6,3)$ cell rather than a consequence of the settled part of the row $n = 6$. The criterion is not applying vacuously: N does have orthogonal mates, four permutation transversals worth of first planes to choose from, and what fails is that no two of them can be the first planes of two mutually orthogonal squares. For contrast, the same criterion applied to a Latin square of order 6 with eight permutation transversals, no two meeting once, excludes it from every triple as well — which at order 6 the table already excludes, but which is not settled at other orders — and the Cayley table of $\mathbb{Z}/6\mathbb{Z}$ has no permutation transversal at all, so nothing is orthogonal to it.

Remark 5.4 (What the whole cell would cost). The following are measurements made outside the formal development and are recorded only to say where the cell stands; they are not part of any claim above. The number of alternating sign hypermatrices of order 6 is 27 729 279 360 (OEIS A144017 [6]; the values $2, 14, 924, 852\,960$ at $n = 2, 3, 4, 5$ were recomputed and agree). Two independent engines — an exhaustive search on the chain of Theorem 2.6, and a SAT encoding of the same chain with the exact-cardinality form of Proposition 2.7 — reproduce every finite entry of the rows $n = 4$ and $n = 5$ of the table, including the hardest, $(5,3) = 0$ (SAT: unsatisfiable in 208 s; exhaustive search: 266 125 428 nodes in 577 s, independently). For three squares of order 6 the SAT instance has 53 582 variables and 186 237 clauses and gave no answer in 75 minutes; the exhaustive searches project to at least five days on one core. The blocking quantity is the number of orthogonal *pairs* of Italian squares of order 6, not the number of squares and not the third-square search, since the criteria above cannot be applied before a pair is at least partially built. Finally, 20 000 Italian squares of order 6 drawn by the sampler of Remark 3.7 were each given a complete search for all orthogonal partners; no two partners of any one of them were orthogonal to each other — including for the anchor with 1 482 partners, all of whose $1\,482 \cdot 1\,481/2$ pairs were tested. That is evidence about a sample and nothing more; $(6,3)$ is open.

6. Order 10

The question quoted in Section 1 asks for three POIS of order 10. We do not settle it. What can be recorded is that the first step is available and the second is where the difficulty sits.

Proposition 6.1. *There is a POIS of size two of order* 10 *— a pair of MOLS of order* 10*, read as Italian squares — and it has a common permutation transversal, so Corollary 3.2 does not by itself exclude a third member.*

Proof. A pair of MOLS of order 10 was produced by a Jacobson–Matthews walk followed by an exact cover of one square into ten disjoint transversals, and both the Italian square conditions and the 100 plane inner products were then checked from Definitions 2.2 and 2.3. An explicit permutation matrix transversal to both is exhibited and checked likewise. □

Remark 6.2 (The searched region). The configuration searched was the one with two Latin members: a pair of MOLS of order 10 together with a third Italian square orthogonal to both, which by Theorem 2.6 and Proposition 2.7 is the chain search of Remark 4.3 at $n = 10$. Over 162 pairs of MOLS of order 10 and 969 (pair, ordering) searches — all 969 run to completion, 8 920 387 368 search nodes in total, no bound on negative entries — no third Italian square was found. The reason is

Corollary 3.2 rather than the cost of the search: of the 162 pairs, 137 have *no* common permutation transversal at all and are therefore maximal POIS before any search begins, 23 have one and 2 have two, a measured mean of 0.17. For the pair of Proposition 6.1, whose single common transversal is the exhibited one, the exhaustive chain search terminated in 21 987 743 nodes with no solution, so that pair too is a maximal POIS. None of this is evidence of non-existence: it is a record of where one configuration was covered. Note also that the configuration searched is probably not the promising one — the only triple printed in [1], at order 7, has no Latin member, and at order 4 the table's $i = 0$ entry exceeds its $i = 1$ entry.

7. Re-verification of the published examples

Proposition 7.1. *Each of the Italian squares of Examples 2.8, 2.9, 2.11 and 2.12 of* [1]*, and each of the squares used in the proof of its Theorem 7.7 at* $m = 5, 7, 11$*, satisfies the properties asserted of it there. Namely: the order-*3 *square of Example 2.8 is an alternating sign hypermatrix; the two squares of Example 2.9 are distinct, have the same Latin-like square, and exactly one of them is orthogonal to the Latin square printed with them; the order-*4 *triple of Example 2.11 consists of a Latin square orthogonal to two non-Latin Italian squares that are not orthogonal to each other; the order-*6 *pair of Example 2.12 is a POIS of size two; the diamond Italian square* $\mathcal{D}_n$ *defined by the closed formula of* [1, §5] *is an alternating sign hypermatrix at* $n = 7$ *and* $n = 11$*; and the ingredients of* [1, Thm. 7.7] *at* $m = 5, 7, 11$ *are POIS, the order-*7 *one being* $\{\mathcal{D}_7, A_1, A_2\}$*, the only set of three POIS printed in the paper and one with no Latin member.*

This is transcription, not new mathematics; its role is to pin the definitions used above to the paper's objects, so that the statements of Sections 4–6 are statements about the same squares. Sixteen objects are covered: fourteen entered cell by cell from the arrays as printed, and $\mathcal{D}_7$, $\mathcal{D}_{11}$ generated from the closed formula; every property asserted of them was then re-derived.

The table's order-5 entries are exact values from the authors' own search, and the paper prints no witness for them. Here is one.

Proposition 7.2. *There is a set of three pairwise orthogonal Italian squares of order* 5 *with exactly one non-Latin member, namely*

1	2	5	3	4
2	3	1	4	5
3	4	2	5	1
4	5	3	1	2
5	1	4	2	3

1	5	2	4	3
2	1	3	5	4
3	2	4	1	5
4	3	5	2	1
5	4	1	3	2

5	4	3	2	1
1	3	$2-3+4$	3	5
3	$2-3+5$	3	$1-3+4$	3
4	3	$1-3+5$	3	2
2	1	3	5	4

matching the value 3 *recorded in the* $(5, 1)$ *entry of the table of* [1].

The value is the authors'; only the example is ours. The same machinery independently reproduced the whole $n = 4$ row of the table, $3, 2, 0, 0$, by enumerating all 924 alternating sign hypermatrices of order 4 and taking maximum cliques in the orthogonality graph — a computation outside the formal development, recorded here because it is what validates the searches of Remarks 4.3 and 6.2 against values the authors obtained independently.

Remark 7.3 (A misprint in [1]). Theorem 5.10 of [1] is printed as "*The diamond Italian square* $\mathcal{D}_n$ *possesses a transversal if and only if* $n \not\equiv \pm 1 \bmod 6$." The congruence is negated by mistake. The restatement of that theorem in the introduction ("*The diamond Italian square* $\mathcal{D}_n$ *possesses a transversal if and only if* $n \equiv \pm 1 \bmod 6$"), the sentence of [1, §5] introducing $\mathcal{D}_n$ ("*We first show existence of a transversal of* $\mathcal{D}_n$ *for* $n \equiv \pm 1 \bmod 6$*, and subsequently apply the above concepts to show nonexistence for* $n \not\equiv \pm 1 \bmod 6$"), its Lemma 5.9, which *constructs* a transversal for every $n \equiv \pm 1 \bmod 6$, the proof of the theorem itself ("*Lemma 5.9 gives a construction for a transversal of* $\mathcal{D}_n$ *for all* $n \equiv \pm 1 \bmod 6$*. We now prove that this is a necessary condition on* n"), its Corollary 5.11, and its final section ("*We have shown that the diamond Italian square* $\mathcal{D}_n$ *admits a transversal if and* *only if* $n \equiv \pm 1 \bmod 6$") all agree on the unnegated form. An exhaustive count of the transversals of $\mathcal{D}_n$ over all ASMs gives $0, 0, 14, 0, 258$ for $n = 3, 4, 5, 6, 7$, i.e. exactly the orders $n \equiv \pm 1 \bmod 6$; and an explicit transversal of $\mathcal{D}_5$ contradicting the printed statement is checked in the formal development. This is an observation about a misprint, not a mathematical correction. The misprint stands in the version cited here, which as of 28 August 2026 is the only one: the arXiv entry lists v1 alone, submitted 24 June 2026, and carries no journal reference; no published version was found in Crossref.

8. Verification

Every theorem, proposition, lemma and corollary in this paper has been formally verified in Lean 4 against *Mathlib*; the development is free of `sorry`; repository reference to be added at submission. The one exception is Corollary 4.2, whose upper bound 6 is quoted from [1, Thm. 6.4] and is not part of the development. The structural results are kernel-checked and depend on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`: Lemma 2.5 (whose general form needs only `propext` and `Quot.sound`), Theorem 2.6, Proposition 2.7, Proposition 3.1, Corollary 3.2, Proposition 3.3, Lemma 3.4, Proposition 3.5 and Proposition 3.6. What is delegated to compiled evaluation, through Lean's `native_decide`, is exactly the finite checks on explicit data: the alternating sign hypermatrix and orthogonality conditions for the squares of Theorem 4.1, Proposition 6.1 and Proposition 7.1; and the scans over the 46 656 functions $[6] \to [6]$ behind the cardinalities of Theorems 5.1 and 5.3. The definitions are equipped with negative controls in both directions — that the alternation condition rejects $(1, 1)$ and $(0, 0)$, so the hypermatrix condition is not vacuous; that a hypermatrix all of whose planes are alternating sign matrices need not be an alternating sign hypermatrix, so clause (ii) of Theorem 2.6 does real work; that the eleven non-injective functions mentioned after Proposition 3.5 exist, so the injectivity hypothesis is load-bearing; and that adding a third square to the pair of Theorem 5.1 is not automatic. Every square produced by a search was, before any formal proof was written, re-checked by an independent implementation reading Definitions 2.2 and 2.3 directly and testing all pairs; Remark 4.4 records the one occasion on which that check retracted a claim. The counts quoted in Remarks 4.3, 5.4, 6.2 and 7.3, the sampling of Remark 3.7 and the 14.9 % figure it feeds, and the $n = 4$ table row reproduced after Proposition 7.2, come from those independent implementations and are not part of the formal development; what is formal in Remark 7.3 is only the explicit transversal of $\mathcal{D}_5$.

References

[1] A. Ernst, S. Lia, C. O'Brien, J. Sheekey and J. Zumbrägel, *Generalising Latin square orthogonality and Frobenius–König with alternating sign matrices*, arXiv:2606.25884v1 (2026). All definition, example and theorem numbers cited above refer to v1, which is the only version.

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[4] R. C. Bose, S. S. Shrikhande and E. T. Parker, *Further results on the construction of mutually orthogonal Latin squares and the falsity of Euler's conjecture*, Canad. J. Math. **12** (1960), 189–203; `doi:10.4153/CJM-1960-016-5`.

[5] M. J. Gill and I. M. Wanless, *Pairs of MOLS of order ten satisfying non-trivial relations*, Des. Codes Cryptogr. **91** (2023), no. 4, 1293–1313; `doi:10.1007/s10623-022-01149-6`.

[6] OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, `https://oeis.org`; entry A144017 (number of $n \times n \times n$ alternating sign hypermatrices).