$\{-1,1\}$-PERFECT NUMBERS: A CERTIFIED WINDOW TO 10^6, SEVEN NEGATIVE SIGNS, AND THE LEAST COUNTEREXAMPLE TO ROSS'S CONJECTURE 17

KOREA SUPERINTELLIGENCE LABS

ABSTRACT. Ross calls n an $\mathcal{S}$-perfect number of the first kind when the divisors of n strictly between 1 and n can be given coefficients from $\mathcal{S}$ so that $1+\sum_j \lambda_j d_j = n$; for $\mathcal{S}=\{-1,1\}$ this is a signed-divisor, or subset-sum, condition. He conjectured that every odd nonsquare n with $\sigma(n)\geq 2n$ is $\{-1,1\}$-perfect, with computational evidence to 10^4; the conjecture was disproved by Campbell in July 2026, who produced counterexamples ap with a an odd square, $\sigma(a)\geq 2a$, and $p>\sigma(a)$ prime. Neither paper bounds the least counterexample. We prove, by exhibiting and re-checking a signing certificate for each of the 1989 admissible n, that every odd nonsquare $n\leq 10^6$ with $\sigma(n)\geq 2n$ is $\{-1,1\}$-perfect — a hundred times the published range — and, in the same sweep, that seven negative coefficients always suffice there, a refinement neither source considers and one that is not a theorem in waiting: the minimum number of negative coefficients needed is already 29 somewhere below $2.6\cdot 10^8$. We give a self-contained modulus certificate refuting $n_0 = 253\,277\,325 = 3^2 5^2 7^2 \cdot 22973$, which is the least member of Campbell's construction and is about 6810 times smaller than the instance he prints; and we record an exhaustive search, run twice in independent implementations and outside the formal development, which finds n_0 to be the least counterexample of any shape and finds exactly 61 counterexamples below $2.6\cdot 10^8$, all of the form $11025p$ with p prime, $22973\leq p\leq 23581$. Ross's two printed lists are reproduced and completed over $n\leq 80$. All statements below except those explicitly labelled as computations outside the development are machine-checked in Lean 4; §9 says exactly what the machine checks. We also record plainly that the arXiv posting arXiv:2512.04417 of Ross's paper is a republication of the 2024 journal article and that the two texts differ: the journal prints the verification range 10^4, the posting prints no range at all.

1. INTRODUCTION

The objects. Ross [1] introduces the following generalisation of perfect numbers. Verbatim:

> Let $\mathcal{S}\subset\mathbb{Z}$ be any collection of integers, and let $n\in\mathbb{Z}$ with $|n|>1$. Then we call n an *$\mathcal{S}$-perfect number of the first kind* if there exist integers $\lambda_1,\ldots,\lambda_k\in\mathcal{S}$ such that $1+\sum_{j=1}^{k}\lambda_j d_j = n$, where $1=d_0<d_1<\cdots<d_k<d_{k+1}=|n|$ are the positive divisors of n.

Taking $\mathcal{S}=\{1\}$ forces every coefficient to be $+1$ and recovers the perfect numbers, since $1+\sum_{d\in\mathrm{mid}(n)} d = \sigma(n)-n$. The case $\mathcal{S}=\{-1,1\}$ allows cancellation between divisors; it is the case in which both of Ross's conjectures are stated, and it is the subject of this paper. For $n>1$ the definition reads: the divisor $d_0=1$ keeps the coefficient $+1$, the divisor n itself does not occur, and every divisor strictly between 1 and n is given a sign so that the signed sum is $n-1$. Write

$$\mathrm{mid}(n)=\{d:\ d\mid n,\ 1<d<n\},\qquad \sigma(n)=\sum_{d\mid n} d,$$

and let $\mathcal{P}$ denote the set of $\{-1,1\}$-perfect numbers. Splitting a signed sum into its positive and negative parts turns the definition into a subset-sum condition (Lemma 2.2 below): $n\in\mathcal{P}$ if and only if some $T\subseteq\mathrm{mid}(n)$ satisfies $2\sum_{d\in T} d+2n=\sigma(n)$, the set T being exactly the divisors that receive the sign -1. Two consequences are immediate and both are in [1]: $\sigma(n)$ must be even, and $\sigma(n)\geq 2n$. Throughout, following [1], we call n *abundant* when $\sigma(n)\geq 2n$ — the condition others write *nondeficient*, as does [2], which reserves *abundant* for the strict inequality $\sigma(n)>2n$. The two readings differ only on $\sigma(n)=2n$, and never on an odd square, whose σ is odd; so "odd abundant

square", the hypothesis of the theorem of [2] quoted below, means the same thing in either convention. We say a presentation has *width* k when exactly k of the coefficients λ_j equal -1.

The question. The last conjecture of [1] is, verbatim:

Conjecture 17 of [1]**.** Every nonsquare odd abundant number is $\{-1, 1\}$-perfect.

The hypotheses are sharp in an obvious way: $\sigma(n) \geq 2n$ is necessary, and for odd n the number $\sigma(n)$ is odd exactly when n is a square, so odd squares are excluded by the parity condition. The evidence printed for it is a computation. Verbatim from the journal article: "The smallest odd abundant number (OEIS sequence `A005231`) is 945, which is also $\{-1, 1\}$-perfect, as, in fact, is every odd abundant number smaller than 10^4."

The conjecture is false. Campbell [2, Theorem 3.1] proves, verbatim: "Let a be an odd abundant square. If p is a prime such that $p > \sigma(a)$, then $n = ap$ is an odd nonsquare abundant number, but $n \notin \mathcal{P}$." Both of Ross's conjectures are disproved in [2]. What no one has done is *locate* the failure. Ross's evidence stops at 10^4; Campbell's paper is purely structural and prints no search range at all, his stated example being $a = 945^2$. So three quantitative questions are open after [2], and they are what this paper answers, two of them completely and one of them as far as a proof kernel can reach: how far does the conjecture actually hold; how many negative coefficients does a presentation need; and where is the least counterexample.

A republication note. The article [1] appeared in the *Journal of Integer Sequences* in 2024 and was posted to arXiv on 4 December 2025 as arXiv:2512.04417, carrying the journal reference; there is a single version and no withdrawal. The posting is therefore a republication, and we state plainly that the two texts are not interchangeable at the point that matters here. The mathematics — the theorems, the two conjectures, and their numbering — is the same in both. But where the journal prints "every odd abundant number smaller than 10^4", the arXiv posting prints "every other nonsquare odd abundant number that we have been able to check by computer verification", i.e. no range at all. The journal text is the published record, and it is the text against which we measure the improvement below.

Results. Let $W = \{n : \ 1 < n \leq 10^6, \ n \text{ odd}, \ n \text{ not a square}, \ \sigma(n) \geq 2n\}$; there are exactly 1989 such n.

- Every $n \in W$ is $\{-1, 1\}$-perfect (Theorem 3.1). Equivalently, the least counterexample to Conjecture 17 exceeds 10^6, which is 100 times the published verification range.
- Every $n \in W$ has a presentation of width at most 7 (Theorem 3.3). The constant cannot be lowered: width 7 is already forced for some $n < 10^6$. Nor is it a bound waiting to be proved in general — the minimum width reaches 9 below $5 \cdot 10^7$ and 29 below $2.6 \cdot 10^8$ (Remark 3.5). Neither [1] nor [2] counts negative coefficients at all.
- $n_0 = 253\,277\,325 = 3^2 \cdot 5^2 \cdot 7^2 \cdot 22973$ is odd, nonsquare and abundant but not $\{-1, 1\}$-perfect (Theorem 4.2). It is an instance of Campbell's theorem; what is new is that it is the *least* member of his construction (Proposition 4.3 and Remark 4.4), about 6810 times smaller than the instance he prints, and that the refutation is carried by a modulus certificate (Lemma 4.1) which uses neither multiplicativity of σ nor the shape of n, and which is therefore reusable on candidates of any shape.
- Outside the formal development, an exhaustive search over $n < n_0$, run twice in independently written implementations, finds no counterexample; so n_0 is the least counterexample of any shape. Continued to $2.6 \cdot 10^8$ the search finds exactly 61 counterexamples, and every one of them is $11025p$ with p prime, $22973 \leq p \leq 23581$; that is, every counterexample in that range lies in Campbell's construction with the same square $a = 11025$ (§7).
- Ross's two printed lists — the $\{-1, 1\}$-perfect numbers, and the abundant numbers that are not $\{-1, 1\}$-perfect — are printed open-ended in the source and are reproduced here and *completed* over $2 \leq n \leq 80$ (Theorem 5.1).

What is not proved. The window of Theorem 3.1 stops at 10^6 for reasons of cost, not of mathematics; §8 records the measurements and the route past it. The assertion that n_0 is the *least* counterexample is not inside the formal development: what is machine-checked is that no counterexample is $\le 10^6$ and that n_0 is one. The interval $(10^6, n_0)$ is covered by the two searches of §7 only, and we label it as such everywhere it is used. Nothing here touches Ross's other conjecture, on the density of $\mathcal{P}$, which [2] also disproves.

The literature. These objects are new enough that the relevant literature is small, and we checked it before and after the computations of this paper (searches run 3 September 2026). Forward-citation queries on both of [1] and [2] return nothing; a full-text search of the arXiv for *$\mathcal{S}$-perfect*, *Euclid–Euler* and *nondeficient* returns, among papers about these objects, only [1] and [2]; a search of a local mirror of the arXiv mathematics corpus for the same phrases returns the same two papers and one unrelated commutative-algebra usage. Searches of the OEIS [3] for the $\{-1,1\}$-perfect list, for the list of abundant exceptions, and for 253 277 325 itself all return nothing; sequence `A005231` cites [1] but publishes no $\{-1,1\}$-perfect values. Two limits of that check we record rather than read as evidence either way: one citation index was rate-limited on both attempts and returned nothing, and the journal article itself carries no DOI and is not indexed in the index we used for forward citations, so only the arXiv postings could be queried there.

2. Preliminaries

Throughout, $n > 1$ is a positive integer, σ is the sum-of-divisors function, and $\mathrm{mid}(n)$ is the set of divisors of n strictly between 1 and n, so that $\sigma(n) = 1 + n + \sum_{d\in\mathrm{mid}(n)} d$.

Definition 2.1. n is *$\{-1,1\}$-perfect*, written $n \in \mathcal{P}$, if there are signs $\lambda_d \in \{-1,1\}$, one for each $d \in \mathrm{mid}(n)$, with

$$1 + \sum_{d\in\mathrm{mid}(n)} \lambda_d\, d \;=\; n.$$

A presentation has *width* k if $\#\{d : \lambda_d = -1\} = k$; we write $\mathrm{w}(n) \le k$ if a presentation of width at most k exists.

This is the definition of [1] restricted to $\mathcal{S} = \{-1,1\}$ and $n > 1$, with the enumeration $d_1 < \cdots < d_k$ of $\mathrm{mid}(n)$ left implicit; nothing depends on the order of the d_j.

Lemma 2.2 (Campbell [2, Lemma 2.1])**.** *For $n > 1$,*

$$n \in \mathcal{P} \quad\Longleftrightarrow\quad \exists\, T \subseteq \mathrm{mid}(n) \;\; \textit{with} \;\; 2\sum_{d\in T} d + 2n = \sigma(n),$$

and T may be taken to be exactly the set of divisors receiving the coefficient -1. Consequently $\mathrm{w}(n) \le k$ if and only if such a T exists with $\#T \le k$.

Proof. Write $M = \sum_{d\in\mathrm{mid}(n)} d = \sigma(n) - 1 - n$. Given signs λ, let T be the set of d with $\lambda_d = -1$; then $\sum \lambda_d d = M - 2\sum_{d\in T} d$, and the defining equation $1 + M - 2\sum_T d = n$ is $2\sum_T d + 2n = \sigma(n)$. Conversely, given such a T, the signs $\lambda_d = -1$ for $d \in T$ and $\lambda_d = +1$ otherwise reverse the computation. Both directions preserve T, whence the statement about widths. □

The criterion is stated in [2] in the equivalent divided form; verbatim: "$n \in \mathcal{P}$ if and only if $\frac{\sigma(n)-2n}{2}$ is a sum of distinct divisors chosen from $d_1, d_2, \ldots, d_k$, allowing for the possibility of an empty sum." We keep the undivided form because it makes the parity condition visible and needs no side condition. We call

$$H(n) = \tfrac{1}{2}\big(\sigma(n) - 2n\big)$$

the *target*; it is half the abundance, hence small for barely abundant n. For $n = 945$, $\sigma(945) = 1920$ and $H(945) = 15$, a divisor of 945: width 1.

Lemma 2.3 (Ross [1])**.** *If $n > 1$ is $\{-1,1\}$-perfect then $\sigma(n)$ is even and $\sigma(n) \ge 2n$.*

Proof. Both are read off the equation $\sigma(n) = 2\sum_{d\in T} d + 2n$ of Lemma 2.2. □

The first of these is the step that proves the converse half of Theorem 15 of [1] (verbatim: $\sigma(n) = 2 + \sum_{j=1}^{k}(1+\lambda_j)d_j$ "is even, since every $1+\lambda_j = 0$ or 2"), the second the remark "Evidently, every positive $\{-1,1\}$-perfect number is abundant". Together they explain the shape of Conjecture 17: for odd n, $\sigma(n)$ is odd exactly when n is a square, so "odd, nonsquare, abundant" is precisely "odd and not excluded by Lemma 2.3".

Finally, a computational remark that is invisible in the mathematics and decisive for the size of the computation below. A window sweep must evaluate $\sigma(n)$ and the divisor set of n for every n in the window; done by the definition, i.e. by filtering $\{1,\dots,n\}$, this costs $\Theta(n)$ per n and $\Theta(N^2)$ over $n \le N$.

Lemma 2.4. *Let* $\mathrm{dl}(n)$ *be the list obtained by trial division: for each* $d \le \lfloor\sqrt{n}\rfloor$ *with* $d \mid n$*, record* d *and* n/d*. Then the set of entries of* $\mathrm{dl}(n)$ *is exactly the set of divisors of* n*.*

Proof. Every entry divides n. Conversely let $e \mid n$. If $e \le \sqrt{n}$ then e is recorded directly; otherwise $n/e \le \sqrt{n}$, since $e > \sqrt{n}$ and $n/e > \sqrt{n}$ would give $n = e\cdot(n/e) > n$, and $e = n/(n/e)$ is recorded as the cofactor of n/e. □

Evaluating σ and mid through Lemma 2.4 costs $\Theta(\sqrt{n})$, so a sweep over $n \le N$ costs $\Theta(N^{3/2})$: at $N = 10^6$ this is the difference between 10^9 and 10^{12} elementary steps, and it is what makes Theorem 3.1 affordable at all.

3. The certified window

Theorem 3.1. *Every odd nonsquare* n *with* $1 < n \le 10^6$ *and* $\sigma(n) \ge 2n$ *is* $\{-1,1\}$*-perfect.*

Corollary 3.2. *The least counterexample to Conjecture 17 of* [1] *exceeds* 10^6*.*

The published verification range is 10^4, so Theorem 3.1 extends it by a factor of 100. The theorem is proved with a certificate, and the certificate carries more than the theorem states:

Theorem 3.3. *Every odd nonsquare* n *with* $1 < n \le 10^6$ *and* $\sigma(n) \ge 2n$ *has a* $\{-1,1\}$*-presentation in which at most seven of the coefficients* λ_j *equal* -1*; that is,* $\mathrm{w}(n) \le 7$*.*

Theorem 3.1 is Theorem 3.3 followed by Lemma 2.2, so we sketch the proof of the latter.

Proof sketch. The proof is exhaustive over the window, and we say precisely what is enumerated. Let W be the set of n with $1 < n \le 10^6$, n odd, n not a square and $\sigma(n) \ge 2n$; $\#W = 1989$. Outside the development, a search assembles for each $n \in W$ a minimum-cardinality set $T_n \subseteq \mathrm{mid}(n)$ with $\sum_{d\in T_n} d = H(n)$, by iterative-deepening depth-first search with suffix-sum pruning over the divisors of n; the resulting table of 1989 records $(n, \#T_n, T_n)$, in descending order of n, is the only input.

Inside the development the table is not trusted. A checker walks $n = 10^6, 10^6-1, \dots, 1$ and, for each n that passes the Boolean test "n odd, $\lfloor\sqrt{n}\rfloor^2 \ne n$, $\sigma(n) \ge 2n$", consumes one record and verifies it from n alone: that the record's first entry is n; that it lists at most seven numbers; that they are distinct; that each of them divides n and lies strictly between 1 and n; and that $2\sum_{d\in T_n} d + 2n = \sigma(n)$, where $\sigma(n)$ is recomputed by trial division (Lemma 2.4). A record that is missing, misplaced, corrupt, longer than seven entries, or attached to the wrong n makes the checker fail. A monotone induction on the descending walk turns a successful run into the statement: for every $n \le 10^6$ passing the test there is a $T \subseteq \mathrm{mid}(n)$ with $\#T \le 7$ and $2\sum_T d + 2n = \sigma(n)$. Lemma 2.2 converts this to the width statement, and a further check reconciles the Boolean test with the hypotheses as stated: n odd, n not a square, $\sigma(n) \ge 2n$. The generator of the table is therefore not part of the trusted base — only the checker and σ are. □

Remark 3.4 (the constant 7 is attained). Seven cannot be replaced by six. The search reports, as a computation outside the formal development, the exact distribution of minimum widths over the 1989

elements of W:

width	1	2	3	4	5	6	7
$\#\{n \in W\}$	46	3	1079	18	821	5	17

so 17 of them need seven negative coefficients. Widths are constrained modulo 2: for odd n every divisor is odd, so the width of any presentation has the parity of $H(n)$, which is why odd widths dominate the table. Below $3 \cdot 10^5$ the maximum is 6 (over 590 admissible n), so the constant 7 is a fact about the window 10^6 and is best possible for it.

Remark 3.5 (width is not bounded in general). Nothing suggests that a constant works for all n, and the same search, continued outside the formal development, shows that no constant near 7 can: the maximum of the minimum width over admissible n is 7 still at $5 \cdot 10^6$ (10159 admissible n), but 9 at $5 \cdot 10^7$ (103001), and 29 at $2.6 \cdot 10^8$ (532226), the value 29 being attained at $n = 253\,167\,075$. So Theorem 3.3 is a statement about its window and not an approximation to a general theorem; how $\max_{n \le N} \mathrm{w}(n)$ grows is, as far as we know, untouched.

4. The least counterexample

Campbell's disproof of Conjecture 17 runs through the multiplicativity of σ and the shape of the divisor sums of *ap*. For a single number one can do without both, by a certificate that only looks at residues.

Lemma 4.1 (modulus certificate)**.** *Let $n > 1$, let p be a positive integer, and put $H = H(n) = \frac{1}{2}(\sigma(n) - 2n)$. If*

$$R := \sum_{d \in \mathrm{mid}(n)} (d \bmod p) \;<\; p \qquad \text{and} \qquad H \bmod p \;>\; R,$$

then $n \notin \mathcal{P}$.

Proof. Suppose $T \subseteq \mathrm{mid}(n)$ has $\sum_{d \in T} d = H$, as Lemma 2.2 would provide. Then $\sum_{d \in T}(d \bmod p) \le R < p$, so that sum is its own residue, and $H \bmod p = \left(\sum_{d \in T} d\right) \bmod p = \sum_{d \in T}(d \bmod p) \le R$, contradicting $H \bmod p > R$. □

Theorem 4.2. $n_0 = 253\,277\,325 = 3^2 \cdot 5^2 \cdot 7^2 \cdot 22973$ *is odd, is not a square, satisfies $\sigma(n_0) = 527\,735\,754 \ge 2n_0$, and is* not *$\{-1, 1\}$-perfect. In particular it is a counterexample to Conjecture 17 of* [1].

Proof sketch. Apply Lemma 4.1 with $p = 22973$. The divisors of n_0 are the numbers d and $22973d$ for $d \mid 11025$, so the residues mod p of the elements of $\mathrm{mid}(n_0)$ are the 26 divisors of 11025 other than 1, together with 26 zeros; hence $R = 22970 < 22973 = p$. From $\sigma(n_0) = 527\,735\,754$ one gets $H(n_0) = 10\,590\,552 = 460 \cdot 22973 + 22972$, so $H(n_0) \bmod p = 22972 > 22970 = R$. The hypotheses of the lemma hold — with two to spare — and $n_0 \notin \mathcal{P}$. The only step that is a computation is the evaluation of the divisor list of n_0 by trial division to $\lfloor \sqrt{n_0} \rfloor = 15914$, which yields both $\sigma(n_0)$ and the residue sum R; the rest is arithmetic on five integers. □

The number n_0 is an instance of [2, Theorem 3.1]: $a = 11025$ is an odd abundant square and $p = 22973$ is a prime exceeding $\sigma(a)$. It is the least such instance.

Proposition 4.3. $11025 = 105^2$ *is an odd square with $\sigma(11025) = 22971 > 2 \cdot 11025$; 22973 is prime and exceeds $\sigma(11025)$; and $11025 \cdot 22973 = 253\,277\,325 = n_0$.*

Remark 4.4 (why n_0 is the least member, and how much smaller it is). Let *ap* be any member of Campbell's construction, a an odd abundant square and $p > \sigma(a)$ prime. Then $p > \sigma(a) \ge 2a$, so $ap > 2a^2$: the member is controlled by its square alone. As a computation outside the formal development, the two smallest odd abundant squares are $11025 = 105^2$ and $99225 = 315^2$ (these are the squares among the odd abundant numbers; for odd n, $\sigma(n)$ is odd exactly when n is a square, so they are the odd abundant numbers of odd abundance, sequence `A156942` of [3]). For $a = 99225$ the bound gives $ap > 2 \cdot 99225^2 \approx 1.97 \cdot 10^{10}$, far above n_0, and for every larger odd abundant square more

so; and for $a = 11025$ the least admissible prime is 22973, the least prime above $\sigma(11025) = 22971$. So the least member of the construction is exactly $n_0 = 11025 \cdot 22973$. By contrast the example printed in [2] is $a = 945^2 = 893025$; since $\sigma(893025) = 1\,931\,331$, its least admissible prime is $1\,931\,339$, giving the counterexample $893025 \cdot 1\,931\,339 = 1\,724\,734\,010\,475 \approx 1.72 \cdot 10^{12}$, which is $6809.6\ldots$ times n_0. Locating the least member is what makes the search of §7 finite: it fixes the range in which a smaller counterexample of some other shape would have to lie.

5. Ross's printed lists, completed

The source prints two lists of small examples, both open-ended. Verbatim: "$\mathcal{S} = \{-1, 1\}$: $6, 12, 24, 28, 30, 40, 42, 48, 54, 56, 60, 66, 70, 78, 80, \ldots$", and "the first few abundant numbers that are not also $\{-1, 1\}$-perfect are $18, 20, 36, 72, \ldots$". We reproduce both and, over $2 \le n \le 80$, close them: the reproduction certifies not only that the printed values are right but that nothing is missing between them.

Theorem 5.1. *For $2 \le n \le 80$:*

(1) *$n \in \mathcal{P}$ if and only if $n \in \{6, 12, 24, 28, 30, 40, 42, 48, 54, 56, 60, 66, 70, 78, 80\}$;*
(2) *$\sigma(n) \ge 2n$ and $n \notin \mathcal{P}$ if and only if $n \in \{18, 20, 36, 72\}$.*

Proof sketch. Exhaustive. For each n in $[2, 80]$ the full power set of $\mathrm{mid}(n)$ is enumerated and tested against Lemma 2.2; $\#\,\mathrm{mid}(n) \le 10$ over this range, so the search is 79 power sets of at most 2^{10} elements. □

Both printed lists are correct as far as they go, and (1) and (2) show that the fifteen and the four values are all there are below 81. The next few exceptions after 72 are $100, 104, 144, 162, 196$.

6. Controls

A window theorem proved by machine is only as informative as the falsity of its neighbours. We record the checks that each hypothesis of Theorem 3.1 is load-bearing, that the theorem is not vacuous, and that the checker of §3 rejects a wrong certificate. These are verifications, not new mathematics.

Proposition 6.1. (1) *(Nonsquare cannot be dropped.)* $893025 = 945^2$ *is odd and abundant, with* $\sigma(893025) = 1\,931\,331$ *odd, hence* $893025 \notin \mathcal{P}$ *by Lemma 2.3.*
(2) *(Abundant cannot be dropped.)* 15 *is odd, nonsquare and inside the window, but* $\sigma(15) = 24 < 30$*, so* $15 \notin \mathcal{P}$*.*
(3) *(The two necessary conditions are not sufficient.)* $\sigma(20) = 42$ *is even and* ≥ 40*, yet* $20 \notin \mathcal{P}$*: the target* $H(20) = 1$ *is smaller than every element of* $\mathrm{mid}(20)$*.*
(4) *(Non-vacuity.)* 945 *is odd, nonsquare and abundant, and* $\mathrm{w}(945) \le 1$*, the certificate being* $T = \{15\}$*.*
(5) *(The checker rejects a wrong certificate.)* $T = \{5\}$ *fails the check for* 945*, while* $T = \{15\}$ *passes.*

Item (1) is the example Ross gives for the necessity of "nonsquare", verbatim: "So, for example, $945^2 = 893025$ is odd and abundant, but not $\{-1, 1\}$-perfect." Items (1) and (2) show that neither "nonsquare" nor "abundant" can be dropped from Theorem 3.1; item (3) shows that the two conditions of Lemma 2.3 are not sufficient in general; and (4)–(5) exclude the two ways a machine-checked window can be true for the wrong reason — a hypothesis satisfied by nothing, and a checker that accepts anything.

7. The search beyond the formal development

Everything in this section is a computation *outside* the formal development. We state it as measurement, with the implementations and the cross-checks named, and we do not use it in the proof of any statement above.

An exhaustive search over odd n sieves σ, keeps the nonsquares with $\sigma(n) \geq 2n$, and decides each by iterative-deepening depth-first search for a minimum-cardinality $T \subseteq \mathrm{mid}(n)$ with $\sum_T d = H(n)$, pruned by suffix sums. Two independently written implementations were used: the primary one sieves 32-bit σ values over odd n only and enumerates odd trial divisors; the cross-check sieves 64-bit σ over all n, enumerates all trial divisors, and has no node cap. A third implementation, in Python with its own σ, agrees on the small ranges and re-verified the certificate table of §3 end to end, including that no required record is missing.

Measurement 1. Every odd nonsquare $n < 253\,277\,325$ with $\sigma(n) \geq 2n$ is $\{-1, 1\}$-perfect. Together with Theorem 4.2, the least counterexample to Conjecture 17 of [1] is $n_0 = 253\,277\,325$. The cross-check implementation reports 518 476 admissible n below n_0 and no failure.

Measurement 2. Of the 532 226 odd nonsquare $n \leq 2.6 \cdot 10^8$ with $\sigma(n) \geq 2n$, exactly 61 are not $\{-1, 1\}$-perfect, and each of these is $11025p$ for a prime p with $22973 \leq p \leq 23581$. There are exactly 61 primes in that interval, and each gives such an n by [2, Theorem 3.1], so the 61 counterexamples here are precisely the numbers $11025p$, p prime, $22973 \leq p \leq 23581$; the largest is $11025 \cdot 23581 = 259\,980\,525$ and the next, $11025 \cdot 23593 = 260\,112\,825$, is just above the search bound.

The search is sound in both directions, which is what makes these measurements claims rather than impressions. A positive verdict carries a subset that is re-summed and re-checked, so it cannot be wrong. A negative verdict could in principle be an artefact of a search cap (the primary implementation caps at $4 \cdot 10^7$ nodes and depth 30, and reports an overflow as a failure); so each of the 61 reported failures was checked against [2, Theorem 3.1] instead, and all 61 satisfy its hypotheses with $a = 11025$ an odd abundant square and prime cofactor exceeding $\sigma(a) = 22971$. Hence no reported counterexample is spurious, and no genuine one was skipped. Independently, the cross-check implementation — a different sieve, a different divisor enumeration and no node cap — traversed the whole range $3 \leq n < n_0$ and agrees that there is no failure below n_0.

We emphasise the boundary. The machine-checked part of the interval $[2, n_0]$ is $[2, 10^6]$ (Theorem 3.1) together with the endpoint (Theorem 4.2); the rest, $(10^6, n_0)$, rests on Measurement 1 alone.

Remark 7.1. All 61 counterexamples below $2.6 \cdot 10^8$ have the *same* square, $a = 11025$. By Remark 4.4 the next square in Campbell's construction contributes nothing below $\approx 2.1 \cdot 10^{10}$, so between $2.6 \cdot 10^8$ and $2.1 \cdot 10^{10}$ the construction predicts only further multiples $11025p$. Whether every counterexample has this shape is open, and testing the prediction is a computation of about an hour of processor time (§8).

8. The extent of the computation

We state precisely which assertions rest on exhaustive computation, how large each search was, and — because they are the frontier for anyone extending this — what stopped the larger ones.

- Lemmas 2.2, 2.3, 2.4 and 4.1, and Corollary 3.2, rest on no computation: they are uniform in n.
- Theorems 3.1 and 3.3 are the exhaustive claims. The search is the complete set of 1989 integers n with $1 < n \leq 10^6$, n odd, n nonsquare and $\sigma(n) \geq 2n$, i.e. the whole hypothesis range of Conjecture 17 up to 10^6; for each the checker recomputes $\sigma(n)$ and every divisibility from n. The certificate table holds 1989 records — 11 593 integers in all — and is stored as a single whitespace-separated string of 48 449 characters, decoded inside the development, because a nested list literal of the same data elaborates far more slowly.
- Theorem 4.2 rests on one evaluation: the divisor list of $n_0 = 253\,277\,325$ by trial division to 15914, from which $\sigma(n_0)$ and the residue sum $R = 22970$ are formed. Proposition 4.3 rests on the divisor list of 11025 and a primality test of 22973.

- Theorem 5.1 is 79 power-set enumerations, of at most 2^{10} subsets each. Proposition 6.1 is six evaluations at fixed integers.
- Outside the formal development: the searches of §7 and the width data of Remarks 3.4 and 3.5. The primary search to $2.6 \cdot 10^8$ took about 15 minutes of wall clock, roughly 10 minutes of processor time, in 0.51 GB; the cross-check over $3 \le n < 253\,277\,325$ took about 20 minutes of processor time in 2.0 GB.

The window stops at 10^6 for a reason that has nothing to do with the arithmetic. The in-kernel sweep costs $\Theta(N^{3/2})$ evaluation steps: measured, 65 seconds of wall clock at $N = 3 \cdot 10^5$ and 678 seconds at $N = 10^6$ on a shared machine. A single run at $N = 5 \cdot 10^6$ (10159 certificates, a 298 KB table) was stopped at thirty minutes of wall clock with no result, under our own budget for one file; the arithmetic is unchanged, so what that measures is the machine, not the method. The route past it is to split the range: a checker for $n \in (\ell, \ell + F]$ with the same soundness argument and a three-line case split to glue the pieces makes $5 \cdot 10^6$ four runs of about eight minutes and 10^7 eight of them. Closing the whole interval up to n_0 in the kernel is a different matter and we price it rather than attempt it: $\sum_{n \le 2.5 \cdot 10^8} \sqrt{n} \approx 2.6 \cdot 10^{12}$ evaluation steps, four orders of magnitude beyond what we are willing to spend. The only route that reaches that far is a sieve for σ inside the development, with a proved invariant, which would change the exponent from $N^{3/2}$ to $N \log N$; that is a separate development and we have not attempted it. Outside the development, extending the search from $2.6 \cdot 10^8$ to 10^9 costs about four times the reported run — roughly one hour of processor time and 4 GB — and would test the prediction of Remark 7.1.

9. Verification

Every statement above except those explicitly labelled as computations outside the development — that is, Lemmas 2.2, 2.3, 2.4 and 4.1, Theorems 3.1, 3.3, 4.2 and 5.1, and Propositions 4.3 and 6.1 (Corollary 3.2 being a restatement of Theorem 3.1) — has been machine-checked in Lean 4 [4] against Mathlib [5], in six files: the definition and the criterion; the trial-division layer; the width notion, the checker and its soundness lemma; the certificate table and the two window theorems; the modulus certificate and n_0; and the lists with the controls. There is no `sorry` anywhere. The whole reasoning layer — the criterion of Lemma 2.2, the two necessary conditions of Lemma 2.3, the divisor-list identification of Lemma 2.4, the identification of a certificate set with the set of negative coefficients, the modulus certificate of Lemma 4.1, and the soundness of the checker — depends on nothing beyond the three standard axioms of the system (propositional extensionality, quotient soundness, choice). Evaluation of finite data by the compiled evaluator is confined to the tables, one constant per table, and we list them exactly: Theorems 3.1 and 3.3 share a single such constant, the one asserting that the checker accepts the 1989-record table; Theorem 4.2 depends on one, the divisor list of n_0; Proposition 4.3 on two; Theorem 5.1(1) and (2) share one, the power-set sweep over $[2, 80]$; and the five parts of Proposition 6.1 on one each, except the non-vacuity check at 945, which depends on two. No other statement in the development depends on such a constant. Repository reference to be added at submission.

10. What remains

(1) *Is every counterexample of Campbell's shape?* All 61 below $2.6 \cdot 10^8$ are $11025p$. A counterexample not of the form ap with a an odd abundant square and $p > \sigma(a)$ prime would be a genuinely new object; the search to 10^9 priced in §8 is the cheapest test.

(2) *How does the width grow?* $\max\{\mathrm{w}(n) : n \le N,\ n \text{ admissible}\}$ is 7 at $N = 10^6$, 9 at $5 \cdot 10^7$ and 29 at $2.6 \cdot 10^8$ (Remark 3.5). Even the order of growth is open, and so is the natural refinement of Conjecture 17: is $\mathrm{w}(n)$ bounded in terms of the number of distinct prime factors of n, or is $\mathrm{w}(n) = O(\log n)$?

(3) *A kernel-verified window past* 10^6. The range-split checker of §8 reaches 10^7 at a cost we have measured; reaching n_0 needs a verified σ sieve.

(4) *The density conjecture.* Ross's other conjecture is also disproved in [2]. Verbatim, Conjecture 16 of [1]: "The positive $\{-1, 1\}$-perfect numbers have a density equal to the density $\mathcal{A}$ of the abundant numbers, for which we have the bounds $0.2474 < \mathcal{A} < 0.2480$, obtained by Deléglise." What [2, Theorem 2.4] proves is, verbatim, that "there exists a set $\mathcal{E} \subseteq \mathcal{A} \setminus \mathcal{P}$ satisfying $\underline{d}(\mathcal{E}) > 0$. Consequently, the strict inequality $\overline{d}(\mathcal{P}) < d(\mathcal{A})$ holds"; in [2] the letter $\mathcal{A}$ denotes the *set* $\{n \geq 1 : \sigma(n) \geq 2n\}$, not the density that [1] writes with the same letter. So the disproof exhibits a subfamily of the abundant non-$\{-1, 1\}$-perfect numbers of positive lower density, which separates the two densities. Nothing here touches it; for the record, of the 247 549 nondeficient $n \leq 10^6$ we count 228 789 that are $\{-1, 1\}$-perfect and 18 760 that are not, of which 733 fail by the parity condition of Lemma 2.3 and 18 027 by the subset-sum condition (a computation outside the development).

References

[1] Tyler Ross, *A Perfect Number Generalization and Some Euclid–Euler Type Results*, J. Integer Seq. **27** (2024), Article 24.7.5, 11 pp.; published 8 September 2024; `https://cs.uwaterloo.ca/journals/JIS/VOL27/Ross/`. Also posted as arXiv:2512.04417 [math.NT] (4 December 2025), a republication carrying the journal reference, with one version and no withdrawal. Theorem 15, Conjecture 16 and Conjecture 17 are the numbers printed in the journal article; the definition we quote is on p. 2 and the list of $\{-1, 1\}$-perfect numbers on p. 3, while Theorem 15 is on p. 9 and everything else we quote — the proof step giving $\sigma(n)$ even, the remark on abundance, the list of exceptions, the 945 and 945^2 sentences, and both conjectures — is on p. 10, the page that the reference to this article in `A005231` points at. The two texts were compared: the mathematics agrees, and so does the numbering (both share one `amsthm` counter across theorems, lemmas, examples and conjectures, and both run it to 17), but the journal's verification range "every odd abundant number smaller than 10^{4}" is replaced in the posting by "every other nonsquare odd abundant number that we have been able to check by computer verification". Statement numbers cited here are those of the journal article, which is the text we quote throughout.

[2] John M. Campbell, *Disproofs of two conjectures concerning nondeficient numbers*, arXiv:2607.11043 [math.NT] (13 July 2026). One version (v1); the posting carries no journal reference and is marked "Submitted for publication". The numbers we cite are those of v1, whose statements are numbered within sections: the subset-sum criterion is Lemma 2.1, the density disproof is Theorem 2.4 (§2), and the family of counterexamples to Conjecture 17 of [1] is Theorem 3.1 (§3).

[3] The On-Line Encyclopedia of Integer Sequences, `https://oeis.org`; sequences `A000396` ("Perfect numbers k: k is equal to the sum of the proper divisors of k"), `A005101` ("Abundant numbers (sum of divisors of m exceeds $2m$)"), `A005231` ("Odd abundant numbers (odd numbers m whose sum of divisors exceeds $2m$)", first term 945) and `A156942` ("Odd abundant numbers whose abundance is odd", first terms $11025, 99225, 245025, \ldots$; an entry of it must be an odd square, so this is the sequence of odd abundant squares). The last two are defined there by the strict inequality $\sigma(m) > 2m$, which for odd m differs from Ross's $\sigma(m) \geq 2m$ only at an odd perfect number, and never at an odd square. The entry `A005231` links [1] ("See p. 10.") but publishes no $\{-1, 1\}$-perfect values; neither entry mentions $\{-1, 1\}$-perfectness.

[4] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, in: Automated Deduction — CADE 28, Lecture Notes in Computer Science 12699, Springer, 2021, 625–635; DOI 10.1007/978-3-030-79876-5_37.

[5] The mathlib Community, *The Lean mathematical library*, in: Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020), ACM, 2020, 367–381; DOI 10.1145/3372885.3373824.