REAL ROOTS, PRINTED DECIMALS AND ALGEBRAIC DEGREE OF THE CHEN–EDELMAN–URSCHEL GROWTH-FACTOR POLYNOMIALS

KOREA SUPERINTELLIGENCE LABS

Abstract. Chen, Edelman and Urschel have recently attached to the maximal growth factor of Gaussian elimination with complete pivoting on 5×5 matrices an explicit integer polynomial P_5 of degree 61, conjecturing that the growth factor is its unique real root in $[4,5]$; their method also produces a sextic P_7 at $n=7$. We determine the complete real-root structure of both polynomials: P_5 has exactly seven real roots and P_7 exactly four, each isolated inside an interval with integer endpoints by a one-sign-change certificate. We show that P_7 is irreducible over $\mathbb{Q}$, so that the value their method produces at $n=7$ is an algebraic number of degree exactly six with minimal polynomial $P_7/3$; the source exhibits the sextic but leaves minimality open, and the general bound it proves for the algebraic degree of the $n\times n$ growth factor reads 14^{48} at $n=7$. We also record two errata to the source, neither of which affects any of its conclusions: the value printed "to 200 decimal places" for the $n=5$ root is correct to 182 places and exceeds the root by between $2\cdot10^{-183}$ and $3\cdot10^{-183}$ (the 78 decimals printed at $n=7$ are, by contrast, all correct), and the index at which the sign block of the source's own Descartes computation turns over is 9, not 10. Every theorem and proposition below is checked by the Lean 4 kernel on exact integer arithmetic; no floating point and no compiled evaluation enters any proof.

1. Introduction

1.1. **Growth in Gaussian elimination.** Let $A\in\mathbb{R}^{n\times n}$ be nonsingular and let $A^{(1)}=A,A^{(2)},\dots,A^{(n)}$ be the intermediate matrices produced by Gaussian elimination with complete pivoting, in which the entry of largest modulus of the active submatrix is brought to the pivot position at every step. The *growth factor* of A is

$$\rho(A)\;=\;\frac{\max_{i,j,k}\bigl|A^{(k)}_{ij}\bigr|}{\max_{i,j}\bigl|A_{ij}\bigr|},\qquad g_n\;=\;\sup\{\rho(A):A\in\mathbb{R}^{n\times n}\text{ nonsingular}\},$$

and g_n is the constant that controls the backward error of the algorithm. The values $g_1=1$, $g_2=2$, $g_3=\frac{9}{4}$ and $g_4=4$ are classical; the growth-factor table of Edelman–Urschel [4] marks only $n\le4$ as known exactly — its caption reserves that status for the four boldface entries 1, 2, 2.25, 4 — and lists $n=5$ as the plain decimal 4.1325.

At $n=5$ the number $4.1325\ldots$ has been produced numerically, and only numerically, four times on the account [1] gives of it: by Day and Peterson [3] with NPSOL in 1988, by Gould [5] with LANCELOT in 1991, by Edelman and Urschel [4] with

Date: August 30, 2026.

JuMP in 2024, and by Gould again in 2026 with LANCELOT in quadruple precision, reported to the authors of [1] privately. On the upper-bound side Tornheim [9] proved $g_5 \le 4 + \frac{17}{18}$. Whether g_5 equals $4.1325\ldots$, and what kind of number that is, remained open.

1.2. **The polynomials of Chen, Edelman and Urschel.** Chen, Edelman and Urschel [1] give that constant an exact algebraic candidate. Combining a JuMP optimisation, a Gröbner basis and a discriminant, they produce an integer polynomial

$$P_5(x) \;=\; \sum_{i=0}^{61} p_i\, x^i \in \mathbb{Z}[x], \qquad p_{61} = 59049 = 3^{10},$$

their Equation (2.15), printed there in full; its coefficients run to 52 decimal digits and its constant term is

$$p_0 \;=\; -400768351683901408958416200638054732473753927680.$$

They conjecture that g_5 is the unique real root of P_5 in $[4, 5]$, and they print that root "to 200 decimal places" as their Equation (2.16). The same method applied at $n = 7$ produces the sextic

$$P_7(x) \;=\; 3x^6 - 86x^5 + 688x^4 - 1136x^3 - 5888x^2 + 12032x + 22528, \tag{1}$$

whose relevant root they print to 78 decimals; at $n = 6$ their construction collapses to the one-variable problem "maximize $|x^2 - 5|$ subject to $x \in [-1, 1]$", and at $n = 8$ it returns the value 8. They also improve Tornheim's bound at $n = 5$ to 4.84. Throughout, references to [1] are to `arXiv:2602.20390v2` of 15 March 2026, the current version as of 30 August 2026.

Write γ_5 for the unique real root of P_5 in $(4, 5)$ and γ_7 for the unique real root of P_7 in $(6, 7)$ (existence and uniqueness are Theorems 6 and 18 below). The conjecture of [1] is that $g_5 = \gamma_5$; the analogous expectation at $n = 7$ is $g_7 = \gamma_7$. Nothing in this paper bears on either conjecture: the reduction of the matrix problem to these polynomials is the work of [1] and is not revisited here. What we do is settle what is finite and checkable about the two polynomials themselves.

1.3. **What the source states, and what it leaves open.** Three things are asserted in [1] about the roots. Of P_5, in a single sentence: "we also applied Descartes' rule of signs to confirm rigorously that there was only one real root in $(4, 5)$ and no real roots in $(-5, 0)$". Of P_7: "This polynomial has only one real root with absolute value between 3 and 16". No *total* count of real roots is stated for either polynomial, and the software accompanying [1] contains no Sturm or root-counting call [2].

Nor is minimality of degree claimed. The source calls γ_7 "a root of a polynomial of degree 6", which it is by construction, and its subsection "The algebraic complexity of the growth factor" defines the degree of an algebraic number as "the minimum degree for which such a polynomial exists" and then proves only an upper bound: its Proposition 1.1 states that the largest growth factor of an $n \times n$ real matrix under complete pivoting is an algebraic number of degree at most $(2n)^{n^2-1}$, deduced from a general estimate of Jeronimo, Perrucci and Tsigaridas. The source evaluates that bound at $n = 5$, obtaining 10^{24}, and observes that this "appears strikingly pessimistic in comparison to our conjecture that the true maximum is the root of a 61^{st} degree polynomial"; at $n = 7$ the same formula reads $14^{48} \approx 10^{55}$. So

the minimal degree of these constants is a question the source raises and does not answer.

1.4. **Results.**

(i) *The real roots of P_5* (Section 3). P_5 has exactly seven real roots, one in each of $(-17,-16)$, $(-6,-5)$, $(1,2)$, $(2,3)$, $(4,5)$, $(7,8)$, $(17,18)$, and none elsewhere (Theorem 8). Two of the seven lie in $(1,3)$, so the isolation is not a partition artefact. The source's two statements — one root in $(4,5)$, none in $(-5,0)$ — come out as Theorems 6 and 7, the second strengthened to $P_5 < 0$ on the closed interval $[-5,0]$.

(ii) *The real roots of P_7* (Section 6). Exactly four, in $(-3,-2)$, $(-2,-1)$, $(6,7)$, $(16,17)$ (Theorem 17); exactly one of them has modulus in $(3,16)$, which is the source's own claim (Theorem 18). The fourth root, numerically $16.589\ldots$, misses that window by little.

(iii) *The algebraic degree at $n=7$* (Theorem 20). P_7 is irreducible over $\mathbb{Q}$. Hence γ_7 is an algebraic number of degree exactly six and $P_7/3$ is its minimal polynomial; if $g_7 = \gamma_7$, as [1] supposes, the maximal 7×7 growth factor has algebraic degree exactly six. The proof is a two-prime certificate: the reductions of P_7 modulo 47 and modulo 97 factor into irreducibles of degree 3 and of degree 2 respectively, so every integer factor has degree divisible by 6.

(iv) *A certified enclosure and an erratum at $n=5$* (Section 4). We certify γ_5 to within 10^{-200} (Theorem 11) and find that the expansion printed in [1] agrees with γ_5 in its first 182 decimals and departs from it in the 183rd: the printed value exceeds γ_5 by between $2\cdot 10^{-183}$ and $3\cdot 10^{-183}$ (Theorem 12). The same instrument applied to the 78 decimals that [1] prints at $n=7$ reports every one of them correct (Theorem 19), which is what makes the $n=5$ report a statement about that expansion rather than about our arithmetic.

(v) *An off-by-one in the source's Descartes paragraph* (Theorem 16). Of the transformed polynomial $(t+1)^{61}P_5\big(\frac{5t+4}{t+1}\big)$ the source writes that the coefficients "are negative up to t^9 and positive from t^{10} onward". Computed exactly, they are negative for indices $0,\ldots,8$ and positive for indices $9,\ldots,61$. The number of sign changes is one either way, so the conclusion drawn from it — our Theorem 6 — is untouched.

(vi) *The $n=6$ cell* (Theorem 23). The one-variable problem to which [1] reduces $n=6$ has maximum 5, attained only at $x=0$.

(vii) *Why the same certificate does not reach $n=5$* (Section 7). The two-prime argument of (iii) needs each irreducible factor of a reduction to have degree at most three, since that is the range in which "no root in $\mathbb{F}_p$" certifies irreducibility. We report a scan showing that no useful prime of that kind exists for P_5 below 200, locate the smallest prime at which the reduction of P_5 is outright irreducible, and describe what a formal proof of irreducibility at degree 61 would require.

Of these, (iii) is the new mathematical statement; (i), (ii), (iv) and (vi) are facts that any computer algebra system settles in about a second, and what is offered for them here is a proof rather than an assertion. We are explicit about that division throughout.

1.5. **What rests on computation.** Every theorem below except Theorem 23, whose proof is the two-line estimate printed with it, is proved from finitely many exact integer facts, each checked by the kernel of the Lean 4 proof assistant on arbitrary-precision integers. The finite checks are of exactly four shapes: the sign pattern of a Möbius-transformed coefficient list; the sign of a polynomial at an integer or at a rational point, read off a single integer; a polynomial identity over $\mathbb{Z}$ closed by ring normalisation; and an exhaustive sweep over the elements of $\mathbb{F}_{47}$ or $\mathbb{F}_{97}$. No floating-point arithmetic appears in any proof, and no check is delegated to compiled code: in Lean's terms, every decision procedure used is `decide +kernel` and *no* use of `native_decide` occurs anywhere in the development, so neither a compiler nor a runtime is part of the trusted base. Search — floating-point root scanning, exact rational bisection, a computer algebra system — is used only to *find* the objects that the certificates then confirm: the windows, the split indices, the 200 decimal digits, the two primes. We say at each theorem which search was run.

2. Certificates for integer polynomials

The layer described in this section is independent of the growth-factor problem; it is a minimal apparatus for isolating the real roots of an integer polynomial given as a coefficient list, and for reading off the sign of that polynomial at a rational point.

2.1. **Coefficient lists.** A polynomial is carried as the finite list of its coefficients in ascending order: $p = (p_0, \dots, p_{N-1})$ denotes $p(x) = \sum_{i<N} p_i x^i$, and $|p| = N$ is the length of the list, one more than the degree when the leading entry is nonzero. Sums, products, negation $-p$, the reflection $p(-x)$ and the shift $p(c+s)$ are the evident list operations. Nothing here assumes the leading coefficient nonzero, so no normalisation step is needed anywhere.

2.2. **The Möbius transform.** Fix integers $a < b$. The map

$$t \;\longmapsto\; x = \frac{a+bt}{1+t}$$

is an increasing bijection $(0,\infty) \to (a,b)$, with inverse $x \mapsto t = (x-a)/(b-x)$. Multiplying $p(x)$ by $(1+t)^{|p|}$ clears denominators and leaves an integer coefficient list, written $M_{a,b}\,p$, with

$$\big(M_{a,b}\,p\big)(t) \;=\; (1+t)^{|p|}\,p\!\left(\frac{a+bt}{1+t}\right) \qquad (t>0), \tag{2}$$

a positive multiple of $p(x)$ at the corresponding point $x \in (a,b)$. Sign patterns of $M_{a,b}\,p$ therefore transfer directly to p on (a,b), and only two patterns are needed.

Definition 1. Let $q = M_{a,b}\,p$ and let $m \ge 1$ be an integer.

- p satisfies $\mathrm{Pos}(a,b)$ if every coefficient of q is ≥ 0 and at least one is > 0.
- p satisfies $\mathrm{One}(a,b;m)$ if the coefficients of q of index $< m$ are all ≤ 0, those of index $\ge m$ are all ≥ 0, and at least one coefficient of index $> m$ is > 0.
- p satisfies $\mathrm{Ray}(c)$ if every coefficient of the shifted list $p(c+s)$ is ≥ 0 and its constant term is > 0.

Lemma 2 (no sign change)**.** *If p satisfies $\mathrm{Pos}(a,b)$ then $p(x) > 0$ for every $x \in (a,b)$.*

Proof. A list with nonnegative coefficients, one of them positive, is positive at every $t > 0$; now apply (2) at $t = (x-a)/(b-x) > 0$, noting $(1+t)^{|p|} > 0$. □

Lemma 3 (one sign change)**.** *If p satisfies* One$(a, b; m)$ *then p has at most one zero in (a, b). If in addition $p(a) < 0 < p(b)$, or $p(a) > 0 > p(b)$, then p has exactly one zero in (a, b).*

Proof. Write $q = M_{a,b}\, p = L \,\|\, R$, where L collects the coefficients of index $< m$ (all ≤ 0) and R those of index $\ge m$ (all ≥ 0, with a positive one away from the constant term of R). For $0 < s < t$ one has $R(s) < R(t)$, because a list with nonnegative coefficients and a positive coefficient beyond the constant term is strictly increasing on $(0, \infty)$; and $L(s)\, t^m \le L(t)\, s^m$, because for a list A with nonnegative coefficients and $|A| \le m$ the quotient $A(x)/x^m$ is nonincreasing on $(0, \infty)$, applied to $A = -L$. Since $q(t) = L(t) + t^m R(t)$, adding the two estimates gives

$$q(s)\, t^m \;<\; q(t)\, s^m \qquad (0 < s < t),$$

so $q(t)/t^m$ is strictly increasing and q has at most one zero in $(0, \infty)$. By (2) the zeros of p in (a, b) correspond bijectively and monotonically to the zeros of q in $(0, \infty)$, which gives the first claim; the second follows from the first and the intermediate value theorem, p being continuous. □

This is the case $V \le 1$ of Descartes' rule of signs, which is all that root isolation needs. It is convenient to state the certificates for a polynomial that increases through the root of a window; a window on which p decreases is certified by applying Definition 1 to $-p$, and the conclusion is mirrored back. In the same way, Ray(c) applied to p, to $-p$, to $p(-x)$ or to $-p(-x)$ certifies a sign of p on $[c, \infty)$ or on $(-\infty, -c]$.

Lemma 4 (rays)**.** *If p satisfies* Ray(c) *then $p(x) > 0$ for every $x \ge c$.*

Proof. Immediate: $p(x) = \big(p(c + \cdot)\big)(x - c)$ and $x - c \ge 0$. □

2.3. **Signs at rational points.** For integers u and $v > 0$ the number

$$H(u, v; p) \;=\; \sum_{i < |p|} p_i\, u^i v^{|p|-i}$$

is a single integer with the same sign as $p(u/v)$, since $H(u, v; p) = v^{|p|}\, p(u/v)$. Evaluating it by a Horner recursion that carries the pair (value, power of v) avoids exponentiation entirely, which matters when u has 200 digits: for $p = P_5$ and $v = 10^{200}$ the intermediate integers reach about 12 400 decimal digits, and the whole evaluation is one pass over 62 coefficients.

2.4. **The two polynomials, and their transcription.** The coefficients of P_5 were transcribed mechanically from the LaTeX source of [1], and independently from a second, separately obtained copy of that source; the two extractions agree coefficient by coefficient, every exponent from 0 to 61 occurs exactly once, and the term-matching consumed the printed equation with no residue. The sextic (1) is short enough to read off directly. The following further checks are recorded as part of the development.

Proposition 5. *The coefficient list of P_5 has 62 entries and leading entry $59049 = 3^{10}$, which is the one structural fact* [1] *states about it. Moreover the two certificate shapes discriminate: P_5 does* not *satisfy* Pos$(4, 5)$*, an interval that contains a*

root, and neither P_5 nor $-P_5$ satisfies $\mathrm{One}(1,3;m)$ *for any* $m \le 63$*, that interval containing two roots.*

The second half of Proposition 5 is what keeps the isolation theorems from being vacuous: it shows that the predicates of Definition 1 can fail, and fail exactly where they should. A far stronger check on the transcription arrives in Section 4 and is discussed there.

3. The real roots of P_5

Theorem 6. *P_5 has exactly one real root in the open interval $(4,5)$.*

Theorem 7. *$P_5(x) < 0$ for every x in the closed interval $[-5,0]$. In particular P_5 has no real root there.*

Theorem 6 is the statement [1] obtains by Descartes' rule, and Theorem 7 strengthens "no real roots in $(-5,0)$" to a sign on the closed interval. Both are instances of the following complete description.

Theorem 8. *P_5 has exactly seven real roots: there is exactly one in each of*

$$(-17,-16), \quad (-6,-5), \quad (1,2), \quad (2,3), \quad (4,5), \quad (7,8), \quad (17,18),$$

and P_5 has no other real root; the set $\{x \in \mathbb{R} : P_5(x) = 0\}$ has cardinality seven.

Proof of Theorems 6–8. The real line is covered by twelve intervals with integer endpoints together with two rays, and each piece carries one certificate of Section 2:

$(-\infty,-17]$	Ray, $P_5 < 0$	$(3,4)$	Pos on $-P_5$
$(-17,-16)$	$\mathrm{One}(\cdot\,;56)$ on P_5	$(4,5)$	$\mathrm{One}(\cdot\,;9)$ on P_5
$(-16,-6)$	Pos on P_5	$(5,7)$	Pos on P_5
$(-6,-5)$	$\mathrm{One}(\cdot\,;23)$ on $-P_5$	$(7,8)$	$\mathrm{One}(\cdot\,;61)$ on $-P_5$
$(-5,1)$	Pos on $-P_5$	$(8,17)$	Pos on $-P_5$
$(1,2)$	$\mathrm{One}(\cdot\,;50)$ on P_5	$(17,18)$	$\mathrm{One}(\cdot\,;10)$ on P_5
$(2,3)$	$\mathrm{One}(\cdot\,;10)$ on $-P_5$	$[18,\infty)$	Ray, $P_5 > 0$

Each of the seven One windows has a sign change of P_5 between its endpoints, so by Lemma 3 it holds exactly one root; each of the five Pos windows is root-free by Lemma 2; the two rays are root-free by Lemma 4; and P_5 is nonzero at each of the eleven interior integer endpoints, since its value there is a nonzero integer. Theorem 7 adds the value at $x = -5$ to the root-free window $(-5,1)$. The seven roots are separated by their windows, hence distinct, which gives the cardinality.

The computation this rests on is exhaustive and finite. Each Pos or One certificate is a sign check on the 63 integers of a Möbius transform, whose entries reach about 102 decimal digits; each ray certificate is the same check on a Taylor shift; and each endpoint sign is one integer. Fourteen such lists were formed and checked in total, together with the value of P_5 at each integer endpoint. The windows themselves were located by an ordinary floating-point root scan, and for each window the split index m was found by trying $m = 1, 2, \dots, 63$; only the resulting certificate enters the proof. That this search can fail — so that the certificates are not trivially satisfiable — is the content of Proposition 5. □

Two consequences are worth stating separately, because they are the readings of Theorems 6 and 7 that a hurried reader might adopt, and both are false.

Proposition 9. *P_5 has no real root in $(3,4)$ and none in $(-16,-6)$, so neither window of Theorem 8 may be widened by an integer. The sign of Theorem 7 does not persist on the negative axis: $P_5(-10) > 0$. And the 200-decimal value printed in* [1] *is not a root of P_5.*

Proposition 10. *P_5 has two distinct real roots inside $(1,3)$, so the splitting of that interval in Theorem 8 is forced. In particular P_5 has more than one real root, and so does P_7.*

4. The decimals of γ_5

4.1. **A certified enclosure.** Let

$$N = \begin{array}{l}\texttt{41325170786324728542233468532773712699527915377990}\\ \texttt{87694544180521967437834296817648258551799854830226}\\ \texttt{30152515659887825271609554209934703653072223897312}\\ \texttt{92506908826437876945891545075917154154916663756026 9}\end{array}$$

a 201-digit integer.

Theorem 11. $\dfrac{N}{10^{200}} < \gamma_5 < \dfrac{N+1}{10^{200}}$. *Equivalently, $N = \lfloor 10^{200}\gamma_5 \rfloor$: the first 200 decimals of γ_5 are the digits of N after its leading 4.*

Proof. By Theorem 6, P_5 has a single root in $(4,5)$, and by the sign of P_5 at the endpoints it is negative to the left of γ_5 and positive to the right of it inside that interval. Both $N/10^{200}$ and $(N+1)/10^{200}$ lie in $(4,5)$, which is an integer comparison; $H(N, 10^{200}; P_5) < 0$ and $H(N+1, 10^{200}; P_5) > 0$ are two integer evaluations in the sense of Section 2. The enclosure was *located* by an ordinary bisection in exact rational arithmetic, performed outside the proof; the proof itself consumes only the two sign evaluations, each a single integer of some 12 400 digits. □

4.2. **The printed expansion.** Let D be the 201-digit integer for which $D/10^{200}$ is the value printed as Equation (2.16) of [1] — the root of its Equation (2.15), that is of P_5 — and "listed to 200 decimal places".

Theorem 12. *$\gamma_5 < D/10^{200}$, and*

$$\frac{2}{10^{183}} < \frac{D}{10^{200}} - \gamma_5 < \frac{3}{10^{183}}.$$

Moreover $\lfloor D/10^{18} \rfloor = \lfloor N/10^{18} \rfloor$: the printed and the true expansion agree in their first 182 decimals, and they first differ in the 183rd.

Proof. The integer identity

$$D = N + 252529878247276006$$

holds, and the difference has 18 digits; the floor identity and the two-sided estimate follow after division by 10^{200} and comparison with Theorem 11. That $D/10^{200}$ lies strictly to the right of γ_5 follows from $H(D, 10^{200}; P_5) > 0$ together with the sign behaviour of P_5 on $(4,5)$ used in the previous proof. □

Remark 13 (the form of the discrepancy). The two expansions first differ in the 183rd decimal place, where N has the digit 5 and D the digit 7. As the printed value lies to the right of the root, the printed expansion is an overestimate.

Remark 14 (why this is not a transcription error of ours). Changing the coefficient p_i of P_5 by 1 moves the root in $(4,5)$ by about $\gamma_5^i/|P_5'(\gamma_5)|$ to first order, and $|P_5'(\gamma_5)| < 10^{29}$, so the shift is at least of order 10^{-29} for every i, and far larger for the high-index coefficients. The discrepancy reported above is of order 10^{-183}, smaller by more than 150 orders of magnitude, so it cannot be the effect of a mistyped coefficient. The agreement of our exact enclosure with the independently printed expansion to 182 places is therefore itself the strongest available check that the coefficient list used here is the one whose root [1] printed. The complementary check runs the same instrument on the same paper's other printed constant and reports a clean pass; that is Theorem 19 below, and without it the present theorem would carry much less weight.

Remark 15 (scope and courtesy). Theorem 12 concerns one printed decimal expansion in `arXiv:2602.20390v2` of 15 March 2026, the current version as of 30 August 2026, and nothing else; the reading reported here was made on that date. No conclusion, conjecture, bound or table of [1] depends on digits beyond the first few of that expansion: the source reports that a quadruple-precision LANCELOT computation of Gould, communicated to its authors privately, "agrees with our exact number to 30 digits", and those 30 digits are confirmed by the enclosure of Theorem 11. Beyond those first digits the expansion appears never to have been reproduced anywhere; we have not been in contact with the authors, and we report the discrepancy as an observation about a printed expansion, with no inference about its cause.

5. The source's Descartes sign pattern

The argument by which [1] obtains one root in $(4,5)$ is the substitution $x = (5t+4)/(t+1)$, which carries $(0,\infty)$ onto $(4,5)$, followed by the count of sign changes; the source describes the resulting integer polynomial $(t+1)^{61}P_5\big(\frac{5t+4}{t+1}\big)$ by writing that its coefficients "are negative up to t^9 and positive from t^{10} onward".

Theorem 16. *The polynomial $(t+1)^{61}P_5\big(\frac{5t+4}{t+1}\big)$ has degree 61; its coefficients of index $0,\dots,8$ are strictly negative and its coefficients of index $9,\dots,61$ are strictly positive. In particular the coefficient of t^9 is positive.*

Proof. The degree-preserving Möbius transform is formed exactly over $\mathbb{Z}$ — it differs from the transform $M_{4,5}$ of Section 2 by one factor of $(1+t)$ — and the 62 resulting integers are tested for sign. This is a single finite check. □

The block therefore turns over one index earlier than stated. The number of sign changes is one either way, so the conclusion the source draws — exactly one root of P_5 in $(4,5)$, our Theorem 6 — is correct as it stands, and only the index is wrong. We verified the coefficient signs independently in exact rational arithmetic and in PARI/GP [8] before formalising them.

6. The case $n = 7$

6.1. Real roots and printed decimals.

Theorem 17. *P_7 has exactly four real roots, one in each of $(-3,-2)$, $(-2,-1)$, $(6,7)$ and $(16,17)$, and no others.*

Proof. As in Theorem 8, with seven windows and two rays: One certificates with split indices 6, 5, 1, 5 on $-P_7$, P_7, $-P_7$, P_7 for the four windows above; Pos certificates on $(-1,3)$, $(3,6)$ (for P_7) and $(7,16)$ (for $-P_7$); and rays giving $P_7 > 0$ on $(-\infty,-3]$ and on $[17,\infty)$. The lists here are seven and eight integers long. □

Theorem 18. *P_7 has exactly one real root in $(6,7)$, and exactly one real root x with $3 < |x| < 16$.*

Proof. The first assertion is one window of Theorem 17. For the second, the four roots have moduli in $(2,3)$, $(1,2)$, $(6,7)$ and $(16,17)$ respectively, so exactly one of them has modulus between 3 and 16. The margin is not large: the root in $(16,17)$ is numerically $16.589\ldots$, and it is only this that keeps the source's window from containing two roots. □

The second assertion of Theorem 18 is the claim made in [1]; the first identifies the root in question. Next, the control promised in Remark 14. Let Q be the 79-digit integer such that $Q/10^{78}$ is the value printed in [1] for the $n = 7$ root.

Theorem 19. $\dfrac{Q}{10^{78}} < \gamma_7 < \dfrac{Q+1}{10^{78}}$*: every one of the 78 decimals printed in* [1] *for the $n = 7$ root is a decimal of γ_7.*

Proof. Two rational sign evaluations, as in Theorem 11, together with the uniqueness of the root in $(6,7)$ and the fact that P_7 decreases through it. □

6.2. **The algebraic degree.**

Theorem 20. *P_7 is irreducible over $\mathbb{Q}$. Consequently γ_7 is an algebraic number of degree exactly six, and its minimal polynomial over $\mathbb{Q}$ is $P_7/3$.*

Proof. P_7 is primitive: 3 and -86 are its coefficients in degrees 6 and 5, and $3 \cdot 29 - 86 = 1$, so no prime divides all coefficients. By Gauss's lemma it therefore suffices to prove irreducibility in $\mathbb{Z}[x]$.

Suppose $P_7 = GH$ in $\mathbb{Z}[x]$ with neither factor a unit. Primitivity forces both factors to have positive degree, so $1 \le \deg G \le 5$. Since $\mathrm{lc}(G)\,\mathrm{lc}(H) = 3$ and neither 47 nor 97 divides 3, the degree of G is preserved by reduction modulo either prime.

Modulo 47 we have the identity

$$3\,(x^3 + 9x^2 + 6x + 21)(x^3 + 25x^2 + 14x + 45) \\ = P_7 + 47\,(4x^5 + x^4 + 46x^3 + 190x^2 - 220x - 419),$$

an identity in $\mathbb{Z}[x]$, so that the reduction of P_7 modulo 47 is 3 times the product of those two cubics. Each cubic has no root in $\mathbb{F}_{47}$, and a polynomial of degree at most three over a field with no root is irreducible; hence every irreducible factor of the reduction has degree 3, and so, by unique factorisation, every divisor of it has degree divisible by 3. In particular $3 \mid \deg G$.

Modulo 97, in the same way,

$$3\,(x^2 + 15x + 37)(x^2 + 50x + 28)(x^2 + 68x + 6) \\ = P_7 + 97\,(5x^5 + 155x^4 + 1808x^3 + 5018x^2 + 2476x - 40),$$

each quadratic is root-free over $\mathbb{F}_{97}$ and hence irreducible, and $2 \mid \deg G$.

So $6 \mid \deg G$ with $1 \le \deg G \le 5$, a contradiction. Therefore P_7 is irreducible over $\mathbb{Z}$, hence over $\mathbb{Q}$. It annihilates γ_7 and is irreducible, so its monic scalar multiple $P_7/3$ is the minimal polynomial of γ_7, of degree six. □

Three remarks on that proof. First, no polynomial factorisation is trusted: the two displayed identities are identities in $\mathbb{Z}[x]$, verified by expanding both sides, so that the only object reduced modulo a prime is the numeral 47 or 97 itself. Second, the exhaustive part of the argument is small and explicit: it is the $2\cdot 47+3\cdot 97 = 385$ field-element evaluations behind the five root-free claims. Third, the two primes were found by scanning the small primes not dividing the leading coefficient for a reduction all of whose irreducible factors have degree at most three, that being the range in which the root-free criterion applies; 47 gives the pattern $(3, 3)$ and 97 the pattern $(2, 2, 2)$, whose sets of achievable divisor degrees, $\{0, 3, 6\}$ and $\{0, 2, 4, 6\}$, meet in $\{0, 6\}$.

In particular, if the maximal 7×7 complete-pivoting growth factor is the value γ_7 that the method of [1] produces at $n = 7$, then it is an algebraic number of degree exactly six.

We state the fact soberly. That P_7 is irreducible is settled in under a second by any computer algebra system, and we do not present the assertion itself as difficult. What is offered here is, first, that the question is one the source raises — its own bound for the algebraic degree of the $n \times n$ growth factor reads 14^{48} at $n = 7$, against the six proved here for the value its method produces — and appears never to have been answered in print; and second, that the answer now rests on a certificate a kernel checks, in which no factorisation algorithm, no floating point and no compiled code participates.

7. Why the same certificate does not reach $n = 5$

The natural next question is whether P_5 is irreducible, that is, whether γ_5 has algebraic degree 61. We have not settled it, and this section records precisely what blocks the method of Theorem 20 and what a proof would instead require. The computations reported in this section were carried out in PARI/GP [8]; they are not part of the formal development, and no statement of this section is used anywhere else in this paper.

The certificate of Theorem 20 has one hard requirement: every irreducible factor of the reduction must have degree at most three, since "no root in $\mathbb{F}_p$" is the only irreducibility test used, and beyond degree three it is not a test at all. For P_5 that requirement fails comprehensively.

Remark 21 (scan, outside the formal development). Among the 45 primes $p \le 200$ with $p \ne 3$, the only p for which every irreducible factor of $P_5 \bmod p$ has degree at most three is $p = 2$, where the reduction is x^{61} — all coefficients but the leading one are even — and the degrees of its divisors are unconstrained. For every other such prime the reduction has an irreducible factor of degree at least four; the two patterns that come closest to being usable are $p = 7$, where the degrees with multiplicity are $(2, 2, 6, 51)$, and $p = 37$, where they are $(3, 58)$. We scanned no further than 200 and make no claim about larger primes.

The degree arithmetic itself is not the obstruction. The pair $p = 7$, $p = 37$ would prove irreducibility if one could certify those two factorisations: the achievable divisor degrees are $\{0, 2, 4, 6, 8, 10, 51, 53, 55, 57, 59, 61\}$ at 7 and $\{0, 3, 58, 61\}$ at 37, meeting exactly in $\{0, 61\}$, so no proper factor could exist. (Many other pairs of primes below 200 have this property; the difficulty is uniform across them.) What is missing is a cheap certificate of irreducibility for a factor of degree 51 over $\mathbb{F}_7$ or

of degree 58 over $\mathbb{F}_{37}$. Trial division would need on the order of $7^{25} \approx 1.3 \cdot 10^{21}$, respectively $37^{29} \approx 3 \cdot 10^{45}$, divisions, which is out of reach.

A single prime does suffice, if one is willing to build more machinery.

Remark 22 (outside the formal development). P_5 mod 373 is irreducible in $\mathbb{F}_{373}[x]$, and among the primes below 659 exactly three have this property: 373, 563 and 631. In particular 373 is the smallest.

Because 61 is prime, a formal proof of irreducibility modulo 373 needs only two ingredients. The first is the divisibility $\overline{P_5} \mid x^{373^{61}} - x$: by the standard criterion that an irreducible factor q of $x^{|K|^n} - x$ over a finite field K satisfies $\deg q \mid n$, it forces every irreducible factor of $\overline{P_5}$ to have degree 1 or 61. The second is that $\overline{P_5}$ has no root in $\mathbb{F}_{373}$, a 373-case check, which excludes degree 1. The second is cheap; the first is the difficulty. Here $373^{61} \approx 7.5 \cdot 10^{156}$, far too large for the exponent to be evaluated as a numeral, so the divisibility has to come instead from an iterated Frobenius computation in a computable model of $\mathbb{F}_{373}[x]/(\overline{P_5})$, that is, 61 coordinates in $\mathbb{F}_{373}$ with a proved ring isomorphism to the quotient. That is a substantial piece of formalisation — we estimate several hundred to a thousand lines — and it is the reason P_5 is left open here. The obstruction is one of formalisation cost, not of computation: the computation takes a second.

Finally, a cheaper classical route also fails. Since $P_5 \equiv x^{61} \pmod 2$ one might hope for an Eisenstein-type or Newton-polygon criterion at $p = 2$. The 2-adic valuations of the constant and linear coefficients are 87 and 85, and $\gcd(87, 61) = 1$, which is the pattern that would make the polygon argument work; but $61 \cdot 85 = 5185 < 5220 = 87 \cdot 60$, so the lower convex hull of the Newton polygon is not a single segment and the criterion does not apply. A shorter route to a weaker statement remains available and is not carried out here: P_5 mod 7 has no root, while a rational root of P_5 in lowest terms would have denominator dividing 3^{10}, hence prime to 7, and so would reduce to a root in $\mathbb{F}_7$. So P_5 has no rational root and γ_5 is irrational, that is, of algebraic degree at least two.

8. The case $n = 6$

At $n = 6$ the construction of [1] produces a matrix with only six entries different from ± 1, so that no Gröbner basis is needed and the problem reduces, in the source's words, to "maximize $|x^2 - 5|$ subject to $x \in [-1, 1]$", with reported value 5. The arithmetic half of that is immediate and we record it for completeness; the reduction of the 6×6 growth problem to it is the source's work and is not revisited here.

Theorem 23. $\max\{|x^2 - 5| : x \in [-1, 1]\} = 5$*, and the maximum is attained only at* $x = 0$.

Proof. For $x \in [-1, 1]$ one has $0 \le x^2 \le 1$, so $x^2 - 5 \le -4 < 0$ and $|x^2 - 5| = 5 - x^2 \le 5$, with equality exactly when $x^2 = 0$. □

9. Verification

Every theorem and proposition of this paper is formalised in Lean 4 [6] against Mathlib [7] and checked by the Lean kernel; the exceptions are the whole of Section 7, which reports computations made in PARI/GP [8] and is marked as such, and the decimal approximations quoted in passing, such as the numerical position

of a root. The toolchain is `leanprover/lean4:v4.32.0`, and Mathlib is taken at revision `81a5d257c8e410db227a6665ed08f64fea08e997`. The finite data — the 62 coefficients of P_5, the seven of P_7, the 201-digit integers N and D, the 79-digit integer Q — and the 87 finite checks over them live in a module that imports nothing, so that every Möbius transform, sign sweep, Taylor shift and rational evaluation is decided by the kernel on core integer and list operations; the real-analytic content of Section 2 and the algebraic content of Theorem 20 are supplied separately over Mathlib. *No use of* ***`native_decide`*** *occurs anywhere*, so no compiled code, and no floating-point arithmetic, is part of the trusted base of any theorem above; there is no `sorry` and no added axiom. The two facts of Proposition 5 about certificate failure depend on no axioms at all, the integer statements of Theorem 16 and of Proposition 5 depend only on `propext`, and every remaining theorem depends only on `propext`, `Classical.choice` and `Quot.sound`, the three axioms of Lean's standard foundation. The whole development is about 2 100 lines and compiles in 90 seconds of wall time; the import-free data module accounts for 35 of those seconds at a peak of 1.5 GB, and each Mathlib-importing module costs at most 0.45 GB above the `import Mathlib` baseline measured in the same session. Every number reported in this paper was also computed independently in exact arithmetic in two unrelated engines before it was formalised, and the two agreed. The repository is private at the time of writing; repository reference to be added at submission.

Acknowledgement of provenance

The polynomials P_5 and P_7, the conjecture that $g_5 = \gamma_5$, the Descartes argument for the window $(4, 5)$, the reduction of the $n = 6$ case and the printed decimal expansions are due to Chen, Edelman and Urschel [1], as is the numerical work behind them. What is new here is the complete real-root structure of the two polynomials, the irreducibility of P_7 and the resulting algebraic degree at $n = 7$, the certified 10^{-200} enclosure of γ_5, the two errata of Sections 4 and 5, and the analysis of Section 7.

References

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[8] The PARI Group, *PARI/GP*, version 2.13.3, Univ. Bordeaux, 2021, `https://pari.math.u-bordeaux.fr/`; the version used for the computations of Section 7 and for the independent checks reported elsewhere above.

[9] L. Tornheim, *A bound for the fifth pivot in Gaussian elimination*, Tech. Report, Chevron Research Co., Richmond, CA, 1969. (An unpublished report, not consulted here; the title, issuing laboratory and year are those given for it in the bibliographies of Chen–Edelman–Urschel and of Edelman–Urschel above, which are also where the bound $4 + \frac{17}{18}$ is attributed to it. Tornheim wrote several such reports; this is the one on the fifth pivot.)