THE BONDED-MOLECULE DENSITY OF ONE-DIMENSIONAL RANDOM SEQUENTIAL ADSORPTION IS DECREASING IN k, AND AN EXACT REDUCTION FOR THE GAP DENSITIES

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Abstract. Fix $k \geq 2$ and bond uniformly chosen nearest-neighbour k-tuples in a row of n molecules until none is left. Two limiting densities describe the jammed configuration: the density m_k of bonded molecules and the density $g_{k;l}$ of maximal gaps of exactly l unbonded molecules,

$$m_k = k\int_0^1 \Phi_k, \qquad g_{k;l} = 2\int_0^1 (1-s)\, s^l\, \Phi_k, \qquad \Phi_k(s) = \exp\Big(2\sum\nolimits_{j=1}^{k-1} \tfrac{s^j-1}{j}\Big),$$

the second being a formula of Klaassen and Runnenburg. Pinsky (arXiv:2608.04730v1) records both and states two monotonicity assertions for which he has no proof: that $l\, g_{k;l}$ increases on $\{0,\dots,k-1\}$, and that m_k decreases in k — the latter attributed by him to his own 2014 book and asserted, without argument, in the physics literature. We settle the second one completely: $m_{k+1} < m_k$ for every $k \geq 1$.

The proof is an exact moment computation. Since $\Phi_k' = \Lambda_k \Phi_k$ with Λ_k a polynomial, integrating $s^{m+1}(1-s)\Phi_k$ by parts gives $(m+1)M_{k,m} - m\, M_{k,m+1} = 2M_{k,m+k}$ for the moments $M_{k,j} = \int_0^1 s^j\, \Phi_k$. Its instance $m = 0$ is Pinsky's own identity $\sum_l g_{k;l} = m_k/k$ read as an integral; its instances $m = 1$ and $m = k$ evaluate the elementary bound $e^x \leq 1 + x + x^2/2$ in closed form, with no loss, as

$$2k^2\big(m_k - m_{k+1}\big) \;\geq\; (k-1)J_k - k(k+1)N_k, \qquad J_k = \int_0^1 \Phi_k, \quad N_k = \int_0^1 (1-s)^2\Phi_k.$$

Three elementary estimates then reduce the entire tail $k \geq K$ to the single numerical inequality $2K(K+1)\Phi_K(0) < 1 - e^{-2(K-1)}$, which holds at $K = 5$; the three remaining values $k = 2, 3, 4$ are exact rational certificates. The same chain yields a rate, which the conjecture does not assert: $m_k - m_{k+1} \geq 1/(100k^2)$ for $k \geq 5$ and $\geq 1/(20k^2)$ for $k \geq 12$.

For the first assertion we prove the exact identity $(l+1)g_{k;l+1} - l\, g_{k;l} = g_{k;l} - 2g_{k;l+k}$, settle the steps $l = 0$ and $l = 1$ for every k, and close the assertion in full for every $k \leq 20$ by 153 exact rational certificates; a further 72 certificates locate, for each such k, the exact index at which the monotonicity finally fails, and two-sided certificates trap two entries of the source's tables inside explicit rational intervals that exclude the printed values. Every theorem below is machine-checked in Lean 4 over Mathlib, with no appeal to compiled evaluation.

1. Introduction

1.1. **The model and its two densities.**

Fix an integer $k \geq 2$ and, for $n \geq k$, a row of n molecules. From among the $n - k + 1$ nearest-neighbour k-tuples select one uniformly at random and bond its k molecules; from among the remaining unbonded nearest-neighbour k-tuples select another uniformly at random and bond it; continue until no unbonded nearest-neighbour k-tuple is left. This is the discrete random sequential adsorption (RSA) process, the lattice analogue of Rényi's car-parking problem [4]; $k = 2$ is Flory's model [5].

Two limiting densities describe the jammed configuration. Let $M_k^{(n)}$ be the expected number of bonded molecules and $G_{k;l}^{(n)}$ the expected number of *maximal* gaps of exactly l unbonded molecules between consecutive bonded k-tuples. Writing $m_k = \lim_n M_k^{(n)}/n$ and $g_{k;l} = \lim_n G_{k;l}^{(n)}/n$, Pinsky's recent paper [1] records, as its equations (mk) and ($lgaps$), the closed forms

$$m_k = k \int_0^1 \exp\Big(2 \sum_{j=1}^{k-1} \frac{s^j - 1}{j}\Big) ds, \tag{1}$$

$$g_{k;l} = 2 \int_0^1 (1-s)\, s^l \exp\Big(2 \sum_{j=1}^{k-1} \frac{s^j - 1}{j}\Big) ds, \qquad l = 0, 1, \dots, k-1. \tag{2}$$

Formula (1) goes back to Flory [5] for $k = 2$ and was proved in general by Pinsky [3], who records a fuller attribution history in [1], including earlier derivations at varying levels of rigour; formula (2) is Theorem `gapsthm` of [1], which credits it to Klaassen and Runnenburg [2]. The quantity m_k is the jamming coverage of the process, and it carries a second name in the physics literature: [7] writes $\theta_b = (b+1)\rho_b$ for the ratio of the jamming density ρ_b to the maximal possible density, and $\theta_b = m_{b+1}$ exactly. (Throughout, a numbered statement of another paper is cited together with its source, so that a bare number below is one of ours.)

1.2. **The two questions.** Immediately after its Table 1, [1] states, verbatim:

> "It follows from (2) that $g_{k;l}$ is decreasing for $l \in \{0, \cdots, k-1\}$. Numerical evidence suggests that $lg_{k;l}$ is increasing for $l \in \{0, \cdots, k-1\}$, but we don't have a proof."

(The displayed equation reference in the original is to its own equation ($lgaps$), which is (2) here.) So for a fixed k the assertion is a chain of $k-1$ strict inequalities,

$$l\, g_{k;l} < (l+1)\, g_{k;l+1}, \qquad l = 0, 1, \dots, k-2, \tag{3}$$

and it is a genuinely competitive assertion: the factor $g_{k;l}$ is strictly *decreasing* in l, so (3) says the linear factor wins. The source offers no method for it, and (2) is not an elementary function of l: the weight $\exp(2 \sum_{j<k} (s^j - 1)/j)$ takes transcendental values at every rational $s \in [0, 1)$ — its exponent is a nonzero rational there, so Lindemann's theorem applies — and no closed form for (2) is available except at the top index $l = k-1$.

Immediately after its Table 2, the same paper states a second assertion, verbatim:

> "It is clear from numerical evidence that m_k is decreasing in k; this was stated as a conjecture in [P14], and seems to be open still."

Here [P14] is the author's own 2014 book [3], whose text we were not able to obtain, so that nothing below depends on its wording. Its published review does corroborate the attribution, and sharpens it: the review displays the book's quantity $p_k = k \exp(-2 \sum_{j=1}^{k-1} \frac{1}{j}) \int_0^1 \exp(2 \sum_{j=1}^{k-1} \frac{s^j}{j})\, ds$, which is exactly m_k of (1), and records that "As an open problem the author asks about monotonicity of p_k and also its limit value $\lim_{k\to\infty} p_k$" (Zbl 1311.11002, reviewer M. Hassani). The book's open problem is therefore a *pair*, and only its first half is settled below; the limit is the Rényi parking constant and we prove nothing about it. The assertion is

$$m_{k+1} < m_k, \qquad k \geq 2. \tag{$*$}$$

As a *statement*, ($*$) is not new. It is asserted, in one sentence and with no argument, in the physics literature: [7] writes that θ_b "monotonically decays as b increases and it approaches $C = \theta_\infty$", and prints $\theta_1, \dots, \theta_5$ to fourteen decimals; those five numbers are $m_2, \dots, m_6$, and each printed string agrees with the value we compute to its last decimal.[1] What is open is a *proof*. None accompanies the sentence in [7], whose only proved bounds on the jamming density are $1/(2b+1) < \rho_b < 1/(b+1)$; none is given in [1], whose author wrote in August 2026 that the conjecture "seems to be open still"; and we found none in the searches recorded in §14. This paper supplies one.

[1]Four of the five are the truncations of our values and one, θ_2, is the rounding. [7] also prints a value for the limit, $\theta_\infty = 0.74759791502876\ldots$, whose digit string parts at the *eighth* decimal from the catalogued decimal expansion $0.7475979202\ldots$ of Rényi's parking constant (OEIS A050996) — which is also the value we obtain for $\int_0^\infty \exp\big(-2 \int_0^v \frac{1-e^{-u}}{u} du\big) dv$, the integral that [7] identifies with θ_∞, by high-precision quadrature outside the formal development. We make no claim about either printed value, and the limit is not a quantity this paper bounds.

1.3. **What we prove.** Sections 2–9 are those of the first version of this paper and settle (3) for every $k \le 20$; they are reproduced unchanged, so that every statement of that version keeps its number and its wording. The new material is therefore at the end, in §§10–13 and in Theorems 8.3–8.4, and the logical headline of this version is Theorem 13.4. In increasing order of dependence on computation:

(i) An exact reduction for the gap densities, uniform in k and l. Theorem 4.1 states that

$$(l+1)\, g_{k;l+1} - l\, g_{k;l} \;=\; g_{k;l} - 2\, g_{k;l+k}$$

for every $k \ge 1$ and every $l \ge 0$, where the right-hand side reads formula (2) at the index $l+k$, outside the range for which [1] states it. Hence (3) is *exactly* the statement $g_{k;l} > 2g_{k;l+k}$: an inequality for a sign-changing weighted integral becomes a comparison of two gap densities k places apart.

(ii) The moment relation behind it. Theorem 12.1: with $M_{k,j} = \int_0^1 s^j \Phi_k$ one has $(m+1)M_{k,m} - m\, M_{k,m+1} = 2M_{k,m+k}$ for every $m \ge 0$. This is one integration by parts, and it subsumes (i). Its instance $m = 0$ is Corollary 12.2, $\int_0^1 s^k \Phi_k = \frac{1}{2}\int_0^1 \Phi_k$; that identity is Pinsky's own, being the first line of his displayed equation (*dandg*) after substitution, and we claim no novelty for it. Its instances $m = 1$ and $m = k$ give Corollary 12.3, $4M_{k,2k} = J_k + kN_k$, which is what the rest needs and which no source we read states. The integration-by-parts technique is itself published for this very kernel, in equation (4.6) of the e-print of [2].

(iii) A uniform reduction of the second conjecture, and the conjecture itself. Theorem 13.1: $2k^2(m_k - m_{k+1}) \ge (k-1)J_k - k(k+1)N_k$, where the two identities of (ii) evaluate, in closed form and with no loss, the elementary bound $e^x \le 1 + x + x^2/2$ applied to Φ_{k+1}/Φ_k. Lemma 13.2 and Theorem 13.3 then reduce the whole tail $k \ge K$ to the single numerical inequality $2K(K+1)\Phi_K(0) < 1 - e^{-2(K-1)}$, both of whose sides move the right way in K; it holds at $K = 5$, and only $k = 2, 3, 4$ are left over. Theorem 13.4 assembles this into $(*)$ for every $k \ge 2$, and in fact for every $k \ge 1$: *Pinsky's second conjecture, with no window.* Corollary 13.5 is the chained form, $m_j < m_i$ for $1 \le i < j$.

(iv) A rate. Theorem 13.6: $m_k - m_{k+1} \ge 1/(100k^2)$ for every $k \ge 5$ and $m_k - m_{k+1} \ge 1/(20k^2)$ for every $k \ge 12$. Neither [1] nor [7] asserts any rate; both assert only that m_k decreases.

(v) The first two steps of (3)*, for every k.* Theorem 5.1: the steps at $l = 0$ and $l = 1$ hold for every $k \ge 2$. The case $l = 0$ is positivity; the case $l = 1$ is the one index at which the *unweighted* integrand has mean zero, so the weight may be recentred for free and a single sign change finishes the argument. Proposition 5.2 shows that this is exactly as far as such an argument reaches: already at $l = 2$ the constant weight makes the integral negative.

(vi) A certificate, and (3) *up to $k = 20$.* The kernel $\Phi_k(s) = \exp(2\sum_{j<k}(s^j - 1)/j)$ satisfies $\Phi_k' = \Lambda_k \Phi_k$ with $\Lambda_k(s) = 2\sum_{i=0}^{k-2} s^i$, and $\Phi_k(1) = 1$. Consequently $\int_0^1 (p' + \Lambda_k p)\Phi_k = p(1)$ *exactly*, for every polynomial p with $p(0) = 0$ (Lemma 6.1), so one integer polynomial with a coefficientwise-nonnegative residual proves one inequality about Φ_k with no quadrature, no interval arithmetic and no series tail bound (Theorems 6.2 and 10.1). We exhibit 153 such certificates for (3) (Theorem 6.3); with (v) they give Theorem 7.1, Pinsky's first assertion in full at each of $k = 2, \ldots, 20$. The same device with the polynomial χ_k of Theorem 11.1 gives 19 more and hence $(*)$ on the window $2 \le k \le 20$ (Theorem 11.2); three of those nineteen are the cells (iii) leaves behind.

(vii) Where the monotone phenomenon actually stops. Theorem 8.1 shows, with a negative scale, that the step *fails* at each of $(k,l) = (2,3), (3,4), (4,5), (5,6)$ and $(6,7)$, so the restriction on l cannot simply be dropped. Theorems 8.3 and 8.4 locate the turnover exactly for every $k \le 20$: writing $L(k) = k+1$ for $2 \le k \le 7$, $k+2$ for $8 \le k \le 13$ and $k+3$ for $14 \le k \le 20$, the step holds at every $l < L(k)$ and fails at $l = L(k)$. In particular it still holds at $l = k-1$ and at $l = k$, so Pinsky's range is not the exact range of monotonicity — it is the range in which $g_{k;l}$ is a gap density.

(viii) Two entries of the source's tables, certified. Two-sided certificates (Theorem 10.1) trap $g_{10;5}$ and m_5 inside explicit rational intervals,

$$0.00558816 \le g_{10;5} \le 0.00558906, \qquad 0.7922753 \le m_5 \le 0.7922765,$$

which exclude the values 0.0059 and 0.793 printed in Tables 1 and 2 of [1] and confirm the correctly rounded entries 0.0056 and 0.792 (Corollary 10.3). The source's own Table 3 prints 0.792 for the same quantity as its Table 2.

1.4. **What is not claimed.** Pinsky's *first* assertion (3) remains open for $k \geq 21$: nothing below proves it uniformly in k, and §9.2 says what a uniform proof would have to supply and records the numerical evidence that one should exist. The statement $(*)$ is, as §1.2 says, already in the literature; what is new here is its proof, together with the reduction of Theorem 13.1, the rate of Theorem 13.6 and the exact frontier of Theorems 8.3 and 8.4. We prove nothing about the limit objects of [1] — the Rényi parking constant C and the weak limit of the rescaled gap distribution — and in particular we do not determine $\lim_k k^2(m_k - m_{k+1})$, whose numerical value we do not claim; the constants in Theorem 13.6 are honest lower bounds and are not sharp.

1.5. **Changes from version 1.** Sections 2–9 are those of version 1. Every numbered statement of version 1 keeps its number and its wording verbatim: Definition 2.1, Propositions 3.1, 3.2 and 5.2, Theorems 4.1, 5.1, 6.2, 6.3, 7.1 and 8.1, Corollary 4.2, Lemma 6.1 and Remarks 8.2, 9.1 and 9.2, as do the numbered equations. What is new is: the abstract and this introduction; Definition 2.2; Theorems 8.3 and 8.4 and Remark 9.3; the sections §§10–13; and the prose of §9 and §14, rewritten for the enlarged development.

Three things version 1 recorded as computations outside the formal development have since become theorems, and nothing it stated has been withdrawn. Version 1's Remark 8.2 located the first failing index of (3) for each $k \leq 20$ by high-precision arithmetic and said that this was "a computation, not a theorem of this paper"; Theorems 8.3 and 8.4 now prove that table, and its last sentence should be read as superseded. Version 1's Remarks 9.1 and 9.2 computed $g_{10;5}$ and m_5 in high-precision arithmetic; Corollary 10.3 now brackets both by exact rational certificates, and Remark 9.3 says so at the place where the two remarks stand. Finally, version 1's "what is not claimed" said that the second conjecture of [1] was untouched and that we proved nothing about m_k; §§11 and 13 of this version are exactly that omission repaired.

2. The objects

Definition 2.1. For an integer $k \geq 1$ and $s \in \mathbb{R}$ put

$$\Phi_k(s) = \exp\Big(2\sum_{j=1}^{k-1} \frac{s^j - 1}{j}\Big), \qquad \Lambda_k(s) = 2\sum_{i=0}^{k-2} s^i.$$

For integers $k \geq 1$ and $l \geq 0$ put

$$A_{k,l} = \int_0^1 (1-s)\, s^l\, \Phi_k(s)\, ds, \qquad g_{k;l} = 2A_{k,l}, \qquad m_k = k\int_0^1 \Phi_k(s)\, ds.$$

Two conventions deserve emphasis. First, *the source names none of these weights*: [1] writes $\exp(2\sum_{j=1}^{k-1}(s^j-1)/j)$ inline inside (1) and (2) and attaches no symbol to it, so Φ_k, Λ_k and the integrand ψ_l of (6) below are our notation, introduced only to make the statements readable. (The e-print [2] carries the un-normalised variant $e_k(y) = \exp\{2[y + y^2/2 + \cdots + y^{k-1}/(k-1)]\}$, which is $e^{2H_{k-1}}\Phi_k$ with $H_{k-1} = \sum_{j=1}^{k-1} 1/j$.) Second, (2) is stated in [1] for $l \in \{0, \dots, k-1\}$ only, because outside that range there are no l-gaps to count; we nevertheless *define* $g_{k;l} = 2A_{k,l}$ for every $l \geq 0$ by the same integral, since the reduction of §4 evaluates it at $l + k$. For $l \leq k-1$ this agrees with the density of l-gaps; for $l \geq k$ it is a formula, not a density, and we use it only as such.

The two elementary facts that drive everything are

$$\Phi_k' = \Lambda_k\,\Phi_k, \qquad \Phi_k(1) = 1, \qquad \Phi_k > 0, \tag{4}$$

the middle one because the exponent $2\sum_j (s^j-1)/j$ vanishes at $s = 1$ — this is exactly what makes every certificate below rational — together with the geometric collapse

$$(1-s)\,\Lambda_k(s) = 2\,(1 - s^{k-1}). \tag{5}$$

Finally, write

$$\psi_l(s) = (1-s)\, s^l\,\big((l+1)s - l\big), \tag{6}$$

so that, directly from Definition 2.1,

$$(l+1)\, g_{k;l+1} - l\, g_{k;l} = 2\int_0^1 \psi_l(s)\,\Phi_k(s)\,ds. \tag{7}$$

Thus one step of (3) is the positivity of one weighted integral of ψ_l. The weight is positive, and (for $k \geq 2$) increasing, and for $l \geq 1$ the function ψ_l changes sign once on $(0,1)$, at $s = l/(l+1)$; the difficulty is that for $l \geq 2$ the negative part is the larger one, as Proposition 5.2 makes precise.

The second half of this paper works with the plain moments of the same kernel rather than with $A_{k,l}$, so we name them here as well.

Definition 2.2. For integers $k \geq 1$ and $j \geq 0$ put

$$M_{k,j} = \int_0^1 s^j\,\Phi_k(s)\,ds, \qquad N_k = \int_0^1 (1-s)^2\,\Phi_k(s)\,ds, \qquad J_k = M_{k,0} = \int_0^1 \Phi_k(s)\,ds.$$

All three integrands are continuous on $[0,1]$, so the integrals exist; and, directly from Definition 2.1,

$$m_k = k\,J_k, \qquad A_{k,l} = M_{k,l} - M_{k,l+1}, \qquad N_k = M_{k,0} - 2M_{k,1} + M_{k,2}.$$

3. Two facts from the source, re-proved

Both statements in this section are Pinsky's; only their proofs from Definition 2.1 are ours. They are recorded because they are the frame in which the new results sit: the first says the conjecture is a competition, the second is the one closed form available and it is what ties Definition 2.1 to the process.

Proposition 3.1. *For every k and every $l \geq 0$ one has $g_{k;l} > 0$, and $l \mapsto g_{k;l}$ is strictly decreasing — with no restriction on l.*

Proof. Positivity is immediate: the integrand $(1-s)s^l\Phi_k(s)$ of $A_{k,l}$ is continuous, nonnegative on $[0,1]$ and strictly positive on $(0,1)$. For the decrease, $A_{k,l} - A_{k,l+1} = \int_0^1 (1-s)^2 s^l \Phi_k(s)\,ds > 0$ by the same argument. ☐

[1] states the decrease for $l \in \{0,\dots,k-1\}$, the range in which $g_{k;l}$ is a density; dropping the range costs nothing.

Proposition 3.2. *For every $k \geq 1$,*

$$g_{k;k-1} = \exp\Bigl(-2\sum_{j=1}^{k-1}\frac{1}{j}\Bigr).$$

Proof. By (4) and (5), $\bigl((1-s)^2\Phi_k(s)\bigr)' = -2(1-s)s^{k-1}\Phi_k(s)$. Integrate over $[0,1]$: the value at $s=1$ vanishes because of the factor $(1-s)^2$, so only the left endpoint survives and $2A_{k,k-1} = \Phi_k(0) = e^{-2H_{k-1}}$. ☐

This is the Remark following Theorem `gapsthm` of [1], with the same antiderivative; we reproduce it because it is the check that Definition 2.1 is the source's object. At $k = 10$ it gives $0.0034897\ldots$, matching the entry 0.0035 printed in Table 1 of [1].

4. The exact reduction

Theorem 4.1. *For every integer $k \geq 1$ and every integer $l \geq 0$,*

$$(l+1)\,\bigl(A_{k,l} - A_{k,l+1}\bigr) = 2\,A_{k,l+k}, \tag{8}$$

equivalently

$$(l+1)\,g_{k;l+1} - l\,g_{k;l} \;=\; g_{k;l} - 2\,g_{k;l+k}. \tag{9}$$

Proof. Differentiate $F(s) = s^{l+1}(1-s)^2\Phi_k(s)$. By (4),

$$F'(s) = (l+1)s^l(1-s)^2\Phi_k(s) - 2s^{l+1}(1-s)\Phi_k(s) + s^{l+1}(1-s)^2\Lambda_k(s)\Phi_k(s),$$

and by (5) the last term equals $2(1-s)\big(s^{l+1} - s^{l+k}\big)\Phi_k(s)$, whose first half cancels the middle term exactly, leaving

$$F'(s) = (l+1)\, s^l(1-s)^2\Phi_k(s) - 2\,(1-s)\, s^{l+k}\,\Phi_k(s).$$

Both endpoint values of F vanish, at $s = 0$ because $l + 1 \geq 1$ and at $s = 1$ because of $(1-s)^2$. So $\int_0^1 F' = 0$, which is (8) once one notes $\int_0^1 s^l(1-s)^2\Phi_k = A_{k,l} - A_{k,l+1}$. For (9), multiply (8) by 2 and rearrange: $2(l+1)A_{k,l+1} - 2lA_{k,l} = 2A_{k,l} - 4A_{k,l+k}$. □

Corollary 4.2. *For every $k \geq 2$ and every l, the step* (3) *holds at l if and only if $g_{k;l} > 2\,g_{k;l+k}$. In particular Pinsky's conjecture at k is exactly the assertion*

$$A_{k,l} > 2\,A_{k,l+k} \qquad \text{for } l = 0, 1, \ldots, k-2.$$

The identity (8) deserves a word. Its ingredient — the antiderivative of $(1-s)^2\Phi_k$ — is printed in [1], but only in the special case $l = k-1$ handled in Proposition 3.2; the general-l identity, and with it the reformulation of Corollary 4.2, does not appear there, nor in [2]. It is what makes the rest of this paper possible: the quantity to be bounded is no longer a cancelling integral but a ratio $r_{k,l} = 2A_{k,l+k}/A_{k,l}$ of two members of one moment sequence, and $r_{k,l} < 1$ is the whole conjecture.

5. The two steps that hold for every k

Theorem 5.1. *For every integer $k \geq 2$,*

$$0 \cdot g_{k;0} < 1 \cdot g_{k;1} \qquad \text{and} \qquad 1 \cdot g_{k;1} < 2 \cdot g_{k;2}.$$

Proof. The first is $g_{k;1} > 0$, which is Proposition 3.1. For the second, by (7) it suffices that $\int_0^1 \psi_1\Phi_k > 0$ with $\psi_1(s) = (1-s)s(2s-1)$. Two facts about ψ_1: it is negative on $(0, \frac{1}{2})$ and positive on $(\frac{1}{2}, 1)$, and — this is the point —

$$\int_0^1 \psi_1(s)\,ds = \int_0^1 (1-s)s(2s-1)\,ds = 0.$$

Hence for any constant c we may replace Φ_k by $\Phi_k - c$ without changing the integral, and we take $c = \Phi_k(\frac{1}{2})$. The kernel Φ_k is strictly increasing on $[0,\infty)$ for $k \geq 2$, since $\Phi_k' = \Lambda_k\Phi_k$ with $\Lambda_k \geq 2$ there; so $\psi_1(s)\big(\Phi_k(s) - \Phi_k(\frac{1}{2})\big) \geq 0$ pointwise on $[0,1]$, with strict inequality on $(\frac{1}{2}, 1)$. Therefore $\int_0^1 \psi_1\Phi_k = \int_0^1 \psi_1 \cdot (\Phi_k - \Phi_k(\frac{1}{2})) > 0$. □

Nothing in that proof depends on k beyond "Φ_k is increasing". The next proposition says that this is a genuine ceiling, not a limitation of the presentation: no argument that uses only the positivity and the monotonicity of the weight can reach $l = 2$.

Proposition 5.2. $\displaystyle\int_0^1 (1-s)s^2(3s-2)\,ds = -\frac{1}{60} < 0$*; and for every $k \geq 2$ the kernel Φ_k is strictly increasing on* $[0,\infty)$.

Proof. The first is the quartic $\int_0^1(-3s^4 + 5s^3 - 2s^2)\,ds = -\frac{3}{5} + \frac{5}{4} - \frac{2}{3}$. The second is $\Phi_k' = \Lambda_k\Phi_k$ together with $\Lambda_k \geq 2$ and $\Phi_k > 0$ on $[0,\infty)$. □

Indeed an elementary Beta integral gives $\int_0^1 \psi_l(s)\,ds = (1-l)/\big((l+1)(l+2)(l+3)\big)$ for every $l \geq 0$, which is $\frac{1}{6}$ at $l = 0$, exactly 0 at $l = 1$, and negative for every $l \geq 2$; at $l = 2$ it is $-\frac{1}{60}$, the first assertion above, since $\psi_2(s) = (1-s)s^2(3s-2)$. So at $l = 1$ the recentring in the proof of Theorem 5.1 is free, and at no other index is it; and for $l \geq 2$ the constant weight 1 — itself a legitimate limit of positive increasing weights — already violates the desired inequality. The positivity of $\int_0^1 \psi_l\Phi_k$ for $l \geq 2$ must therefore use the specific Φ_k, and the second half of Proposition 5.2 records that the hypothesis used at $l = 1$ is at least not vacuous. This is why the next section exists.

6. An exact rational certificate

Lemma 6.1. *Let $k \geq 1$ and let p be a real polynomial with $p(0) = 0$. Then*

$$\int_0^1 \big(p'(s) + \Lambda_k(s)\, p(s)\big)\, \Phi_k(s)\, ds = p(1).$$

Proof. By (4) the integrand is $(p\,\Phi_k)'$, so the integral is $p(1)\Phi_k(1) - p(0)\Phi_k(0) = p(1)$. □

Lemma 6.1 is the whole engine. It says that against the measure $\Phi_k(s)\, ds$ an entire linear subspace of polynomials integrates to an *exactly rational* value, even though Φ_k takes transcendental values at every rational point of $[0, 1)$ and no closed form is available for the individual moments $\int_0^1 s^n \Phi_k(s)\, ds$. Combining it with positivity of Φ_k gives a sufficient condition for one step that involves no analysis at all.

Theorem 6.2. *Let $k \geq 1$ and $l \geq 0$, let Q be a nonzero integer, and let $p \in \mathbb{Z}[s]$ satisfy*

$$p(0) = 0, \qquad p(1) > 0, \qquad Q\,\psi_l - p' - \Lambda_k\, p \ \in\ \mathbb{Z}_{\geq 0}[s]$$

(the last condition being that every coefficient of the polynomial $Q\psi_l - p' - \Lambda_k p$ is a nonnegative integer). Then $Q \int_0^1 \psi_l \Phi_k > 0$. Consequently, if $Q > 0$ then $l\, g_{k;l} < (l+1)\, g_{k;l+1}$, and if $Q < 0$ then $(l+1)\, g_{k;l+1} < l\, g_{k;l}$.

Proof. Write $r = Q\psi_l - p' - \Lambda_k p$, so that $Q\psi_l = (p' + \Lambda_k p) + r$ identically. Integrating against Φ_k over $[0, 1]$ and applying Lemma 6.1,

$$Q \int_0^1 \psi_l \Phi_k = p(1) + \int_0^1 r\, \Phi_k.$$

On $[0, 1]$ a polynomial with nonnegative coefficients is nonnegative and $\Phi_k > 0$, so the last integral is ≥ 0; and $p(1) > 0$. The two consequences follow from (7) on dividing by Q. □

Two features are worth isolating. *The hypothesis is a finite statement about integer lists*: with p, Λ_k, ψ_l represented by their coefficient vectors, "$p(0) = 0$" is one entry being zero, "$p(1) > 0$" is a sum of integers, and the domination is one vector identity plus a sign test. No real number is approximated anywhere. *And the scheme is complete in principle*: the function $p^\star = \Phi_k^{-1} \int_0^s \psi_l \Phi_k$ solves $(p^\star)' + \Lambda_k p^\star = \psi_l$ with $p^\star(0) = 0$ and $p^\star(1) = \int_0^1 \psi_l \Phi_k$, and its Taylor coefficients are rational; a certificate is a polynomial truncation of $Q\, p^\star$ rounded so as to keep the residual nonnegative. Finding one is a search, which we did in exact rational arithmetic outside the formal development; checking one is a finite verification, which is what is verified.

Theorem 6.3. *For every pair (k, l) with $4 \leq k \leq 20$ and $2 \leq l \leq k-2$ — that is, for all 153 such pairs — one has $l\, g_{k;l} < (l+1)\, g_{k;l+1}$.*

Proof sketch. Each of the 153 pairs is settled by one instance of Theorem 6.2 with an explicitly exhibited integer polynomial p and an explicitly exhibited nonnegative residual r; the scale is $Q = 10^6$ for 136 of the pairs and $Q = 10^9$ for the remaining 17. The certificates were produced by truncating the power series of $Q\, p^\star$ and rounding each coefficient down to denominator Q, greedily so that every residual coefficient stays nonnegative; the degree of p grows roughly like $4.5\, k$ (degree 16 at $k = 4$, 44 at $k = 10$, 91 at the largest cell $(k, l) = (20, 18)$), and no coefficient of any p or r exceeds 1 944 736 842 in absolute value. This is the one part of the paper that rests on computation, and it is worth saying exactly what kind: the computation is a *finite exact* one, an identity and a sign test on integer vectors of length at most 110, repeated 153 times. It involves no quadrature, no floating point, no interval arithmetic and no truncation error, because by Lemma 6.1 the transcendental weight never has to be evaluated. The search that produced the certificates is heuristic and is not part of the proof; only the exhibited data are. □

7. Pinsky's conjecture for $2 \le k \le 20$

Theorem 7.1. *Let $2 \le k \le 20$. Then for every l with $l < k-1$,*

$$l\, g_{k;l} < (l+1)\, g_{k;l+1};$$

that is, $l \mapsto l\, g_{k;l}$ is strictly increasing on $\{0, 1, \dots, k-1\}$.

Proof. Fix such a k and let $0 \le l \le k-2$. If $l = 0$ or $l = 1$, this is Theorem 5.1 (which needs only $k \ge 2$). Otherwise $2 \le l \le k-2$, which forces $k \ge 4$, and the pair (k, l) is one of the 153 pairs of Theorem 6.3. ☐

So the source's sentence is now a theorem at each of the nineteen values $k = 2, 3, \dots, 20$. Equivalently, by Corollary 4.2, for those k

$$A_{k,l} > 2\, A_{k,l+k} \qquad (0 \le l \le k-2),$$

so that in each of these gap sequences the l-th term exceeds twice the term k places later. We emphasise once more what is *not* proved: the range $k \le 20$ is the range in which certificates were built, and no assertion is made for $k \ge 21$.

8. Pinsky's range cannot simply be dropped

Theorem 8.1. *At each of the five pairs*

$$(k, l) = (2, 3),\ (3, 4),\ (4, 5),\ (5, 6),\ (6, 7)$$

the step fails: $(l+1)\, g_{k;l+1} < l\, g_{k;l}$.

Proof sketch. Five instances of Theorem 6.2 with $Q = -10^6$, each with an explicit integer polynomial p of degree at most 33 and an explicit nonnegative residual; the verification is the same finite exact check as in Theorem 6.3. The negative scale reverses the conclusion, so what is certified is $\int_0^1 \psi_l \Phi_k < 0$. ☐

Thus the restriction to $l \in \{0, \dots, k-1\}$ in the sentence quoted in §1 is not a convention forced by the combinatorial meaning of $g_{k;l}$ alone: even read as the pure formula of Definition 2.1, the monotonicity genuinely stops. It has to stop somewhere, because the Beta densities $(l+1)(l+2)(1-s)s^l$ concentrate at $s = 1$, so $(l+1)(l+2)A_{k,l} \to \Phi_k(1) = 1$ and hence $r_{k,l} = 2A_{k,l+k}/A_{k,l} \to 2$ as $l \to \infty$ with k fixed. The content of Theorem 8.1 is that for these five values of k it has already stopped at $l = k+1$, with only the two indices $l = k-1$ and $l = k$ in between; what happens at those two is not settled here. We stress what Theorem 8.1 does *not* say: it does not say that $l = k-2$ is the last index at which the step holds, and Remark 8.2 records that outside the formal development it is not.

Remark 8.2. Outside the formal development, and in high-precision arithmetic on the same moment series, we located, for every k with $2 \le k \le 20$, the first index l at which the step fails: it is $l = k+1$ for $2 \le k \le 7$, $l = k+2$ for $8 \le k \le 13$, and $l = k+3$ for $14 \le k \le 20$. In particular, for every one of these k the step still holds at $l = k-1$ and at $l = k$, so the range $\{0, \dots, k-1\}$ of Pinsky's sentence is not the exact range on which $l \mapsto l\, g_{k;l}$ increases; it is the range on which $g_{k;l}$ is a gap density. This is a computation, not a theorem of this paper; only the five refutations of Theorem 8.1 are proved.

Since version 1 that computation has become a theorem, so the last sentence of Remark 8.2 is superseded: the table it reports is Theorem 8.3 below, proved by 72 further certificates of exactly the kind used for Theorems 6.3 and 8.1. Write

$$L(k) = \begin{cases} k+1, & 2 \le k \le 7, \\ k+2, & 8 \le k \le 13, \\ k+3, & 14 \le k \le 20. \end{cases} \tag{10}$$

Theorem 8.3. *Let $2 \le k \le 20$. Then the step (3) holds at every $l < L(k)$ and fails at $l = L(k)$:*

$$l\, g_{k;l} < (l+1)\, g_{k;l+1} \quad \text{for all } l < L(k), \qquad \big(L(k)+1\big) g_{k;L(k)+1} < L(k)\, g_{k;L(k)}.$$

Proof sketch. For each k the forward half is Theorem 7.1 for $l \le k-2$ together with at most four further instances of Theorem 6.2, and the failure at $l = L(k)$ is one instance with a negative scale. In total 72 new certificates: 58 forward ones at $k-1 \le l \le L(k)-1$ and 14 reversed ones at $l = L(k)$ for $7 \le k \le 20$; the five reversed cells at $2 \le k \le 6$ are those of Theorem 8.1, reused. The scale is $Q = 10^6$ for 35 of the new cells and $Q = 10^9$ for the other 37, and the degree of p reaches 138. The two cells that decide the shape of (10) are the tightest in the paper: $r_{k,l} = 2A_{k,l+k}/A_{k,l}$ equals $0.99956\ldots$ at $(k, l) = (14, 16)$, where the step still holds, and $1.00083\ldots$ at $(k, l) = (20, 23)$, where it fails — margins of about $4 \cdot 10^{-4}$ and $8 \cdot 10^{-4}$, certified at $Q = 10^9$ with degrees 103 and 138. As before the verification is a finite exact check on integer vectors and the search that produced the data is not part of the proof. □

Theorem 8.4. *Theorem 8.3 holds in the three blocks of* (10) *as three statements quantified over* k. *Moreover Pinsky's range is not the exact range of monotonicity: for every* k *with* $2 \le k \le 20$ *the step* (3) *still holds at* $l = k-1$ *and at* $l = k$.

Proof. Both parts are assembled from the cells of Theorem 8.3 and from Theorems 5.1 and 6.3; the second part is the case $l \in \{k-1, k\}$ of the forward half, which is nonempty because $L(k) \ge k+1$ for every k in range. □

So the honest statement about the range is the one this section is titled with. The restriction $l \in \{0, \ldots, k-1\}$ in [1] is not a convention that could be dropped, because the sequence does turn over; but it is not sharp either, since the turnover happens at $l = L(k) \ge k+1$, at least two indices beyond where the source stops.

9. What is finite, and what is not

9.1. **The shape of the proof.** Theorems 4.1, 5.1, 6.2, 10.1, 11.1, 12.1, 13.1, 13.3 and 13.6, Propositions 3.1, 3.2, 5.2 and 10.4, Lemmas 6.1 and 13.2 and Corollaries 4.2, 12.2 and 12.3 are uniform in their parameters and involve no computation at all. (The forward references are to the sections written for this version; the accounting is easier to read in one place.)

Exactly six statements rest on exhaustive computation, and in every one the computation is finite and exact: Theorem 6.3 (153 certificates, one per pair (k, l) with $4 \le k \le 20$, $2 \le l \le k-2$), Theorem 8.1 (5), Theorem 8.3 (72), Proposition 10.2 (4), Theorem 11.2 (19) and part (iii) of Proposition 11.4 (1) — 254 certificates in all. Theorems 7.1 and 8.4 and Corollaries 10.3 and 11.3 inherit this and nothing more; and Theorem 13.4, with Corollary 13.5 and Proposition 13.7(iii), inherits exactly three of the 254 — the cells $k = 2, 3, 4$ of Theorem 11.2 — because §13 covers every $k \ge 5$ without any. The finite search actually run was, for each cell, a search over truncation degree and rounding denominator for an integer polynomial meeting the hypotheses of Theorem 6.2 or 10.1; degrees up to 138 and denominators $10^5, 10^6, 10^8, 10^9$ sufficed, and the search never had to be widened further.

9.2. **Why a proof uniform in k should exist.** The following observation is not formalised and is stated here as an observation, not as a claim of this paper. The measure $d\nu_k = (1-s)\Phi_k(s)\,ds$ on $[0, 1]$ is positive and finite, and $A_{k,l} = \int_0^1 s^l\,d\nu_k$; so $(A_{k,l})_{l\ge0}$ is a Hausdorff moment sequence (see [6], Ch. III, for the classical theory), and Cauchy–Schwarz gives log-convexity, $A_{k,l}A_{k,l+2} \ge A_{k,l+1}^2$. Log-convexity on the integers makes $\log A_{k,l+k} - \log A_{k,l}$ nondecreasing in l, hence $r_{k,l} = 2A_{k,l+k}/A_{k,l}$ is nondecreasing in l. With Corollary 4.2 this would say that for each k the whole of Pinsky's conjecture reduces to its single hardest case,

$$A_{k,k-2} > 2\,A_{k,2k-2},$$

and the hardest case has room that does not close up: computing $r_{k,k-2}$ in high-precision arithmetic gives

k	2	5	10	20	50	100	200	400	800
$r_{k,k-2}$	0.5443	0.8129	0.8858	0.9194	0.9387	0.9450	0.9482	0.9497	0.9505

increasing towards about $0.951 < 1$, i.e. about 5% of margin uniformly in k. A proof for all k therefore ought to exist, and would consist of a uniform upper bound on $r_{k,k-2}$; we do not have one, and finding it is the natural next question. (The log-convexity step is elementary but was not formalised here; the decimals in the table above are ours, computed outside the formal development.)

9.3. **Two entries of the source's tables.** The following two remarks concern printed numerical values and are computations outside the formal development; we state them neutrally, as computed values that differ from printed ones. Both were obtained twice, independently: once by summing the rational power series of $\exp(2\sum_{j<k} s^j/j)$ to 3000 terms in 60-significant-digit arithmetic against $\int_0^1 (1-s)s^{l+n} ds = 1/\big((l+n+1)(l+n+2)\big)$, and once by composite Simpson quadrature on the integral itself; the two agree to every digit quoted.

Remark 9.1. Table 1 of [1] prints, for $k = 10$, the value $g_{10;5} = 0.0059$. We compute $g_{10;5} = 0.0055886\ldots$, which rounds to 0.0056. The other nine entries of that column reproduce the printed values exactly. Two internal consistency checks bear on the difference. The second column of the same table prints $5\,g_{10;5} = 0.0280$, which is incompatible with $g_{10;5} = 0.0059$ (that would give 0.0295) and agrees to one unit in the last place with $5 \times 0.0055886\ldots = 0.0279431\ldots$; and the printed first column sums to 0.0773, whereas its caption gives $\sum_{l=0}^{9} g_{10;l} = m_{10}/10 \approx 0.0770$, a discrepancy that disappears exactly when 0.0059 is replaced by 0.0056. We compute $m_{10}/10 = 0.0769577\ldots$ and $1 - m_{10} = 0.2304225\ldots$, both agreeing with that caption. [1] notes, of that table, "(Due to roundoff error, there is a discrepancy in the fourth decimal place between the sum of the values in each of the two columns and the corresponding values appearing in the caption to the table.)"

Remark 9.2. Table 2 of [1] prints $m_5 = 0.793$. We compute $m_5 = 0.7922759\ldots$, which rounds to 0.792. The other four entries of that column, at $k_1 = 2, 3, 10, 100$, reproduce the printed 0.865, 0.824, 0.770, 0.750 exactly, and the entry at $k_1 = 2$ agrees with the exact value $m_2 = 1 - e^{-2} = 0.8646647\ldots$ quoted in [1] itself. Table 3 of [1] carries the same column m_{k_1} over the same five values of k_1 and prints 0.792 at $k_1 = 5$; so the two tables of the source disagree at exactly this entry, and Table 3 is the one that agrees with our computation.

Remark 9.3. Both of the preceding remarks have since become theorems. Corollary 10.3 brackets $g_{10;5}$ and m_5 between exact rationals produced by two-sided certificates, and those brackets exclude the printed values 0.0059 and 0.793 while confirming the rounded values 0.0056 and 0.792. The high-precision computations reported above remain what they were, independent arithmetic agreeing with the certified intervals; nothing in Remarks 9.1 and 9.2 is withdrawn.

9.4. **Cost.** One clean pass over the formal development takes about ten and a half minutes of wall time in total and peaks at about 8.25 GB of resident memory, of which about 6.6 GB is the library import; the three modules carrying the largest certificate blocks account for 158 s (Theorems 6.3 and 8.1), 65 s (Theorem 8.3) and 21 s (Theorem 11.2) of it, and the whole of §13, which carries no certificate, for 26 s. The certificate searches, in exact rational arithmetic, took under ten minutes for the 158 cells of version 1 and 40 s for the 96 added here. Extending Theorem 11.2 to $k \le 60$ or Theorem 6.3 to $k \le 30$ is a matter of minutes in each case, so the ceilings are choices and not computational limits; after Theorem 13.4 a larger m_k window has no value at all, and by §9.2 a larger $g_{k;l}$ table is worth much less than a uniform bound.

10. Two-sided certificates, and two entries of the source's tables

Theorem 6.2 certifies the *sign* of one weighted integral. Nothing in its proof used the shape of ψ_l, and nothing used the sign of Q until the last line; keeping both free turns Lemma 6.1 into a device that traps a transcendental integral inside an explicit rational interval.

Theorem 10.1. *Let $k \ge 1$, let $c \in \mathbb{Z}[s]$, let Q be a nonzero integer and let $p \in \mathbb{Z}[s]$ satisfy*

$$p(0) = 0, \qquad Q\,c - p' - \Lambda_k\, p \;\in\; \mathbb{Z}_{\ge 0}[s].$$

Then $p(1) \le Q \int_0^1 c(s)\Phi_k(s)\,ds$. In particular $\int_0^1 c\,\Phi_k \ge p(1)/Q$ if $Q > 0$, and $\int_0^1 c\,\Phi_k \le p(1)/Q$ if $Q < 0$.

Proof. Verbatim the proof of Theorem 6.2: with $r = Qc - p' - \Lambda_k p$ one has $Q\int_0^1 c\,\Phi_k = p(1) + \int_0^1 r\,\Phi_k$ by Lemma 6.1, and $\int_0^1 r\,\Phi_k \geq 0$ because r has nonnegative coefficients and $\Phi_k > 0$ on $[0,1]$. Dividing by Q reverses the inequality exactly when $Q < 0$. □

A pair of certificates, one at $Q > 0$ and one at $Q < 0$, therefore brackets $\int_0^1 c\,\Phi_k$ between two explicit rationals; Theorem 6.2 is the case $c = \psi_l$, where the extra hypothesis $p(1) > 0$ turns the conclusion into a strict sign. We use the two-sided form on the two integrals behind Remarks 9.1 and 9.2, namely $c(s) = (1-s)s^5$ at $k = 10$ and $c = 1$ at $k = 5$.

Proposition 10.2.

$$0.00279408 \leq A_{10,5} \leq 0.00279453, \qquad 0.15845506 \leq \int_0^1 \Phi_5(s)\,ds \leq 0.15845530.$$

Proof sketch. Four instances of Theorem 10.1, at $Q = 10^8$ and $Q = -10^8$; the certificate degrees are 65 and 72 for the first pair and 36 and 39 for the second. The four bounds are the exact rationals $p(1)/Q$ themselves, and the check is again a finite identity and sign test on integer coefficient lists: no value of Φ_k is evaluated anywhere. □

Corollary 10.3. $0.00558816 \leq g_{10;5} \leq 0.00558906$ *and* $0.7922753 \leq m_5 \leq 0.7922765$*. Consequently*

$$\bigl|g_{10;5} - 0.0059\bigr| > \tfrac{1}{2}\cdot 10^{-4}, \qquad \bigl|m_5 - 0.793\bigr| > \tfrac{1}{2}\cdot 10^{-3},$$

while $|g_{10;5} - 0.0056| < \frac{1}{2}\cdot 10^{-4}$ *and* $|m_5 - 0.792| < \frac{1}{2}\cdot 10^{-3}$*. So the entries* 0.0059 *and* 0.793 *printed in Tables 1 and 2 of* [1] *are not roundings of the quantities they name, and the correctly rounded entries are* 0.0056 *and* 0.792*.*

Proof. $g_{10;5} = 2A_{10,5}$ and $m_5 = 5\int_0^1 \Phi_5$ by Definition 2.1, so both brackets are Proposition 10.2 multiplied by a positive integer, and the four displayed inequalities follow by arithmetic on rationals. □

Proposition 10.4. $m_2 = 1 - e^{-2}$*.*

Proof. $\Phi_2(s) = e^{2(s-1)}$, so $m_2 = 2\int_0^1 e^{2(s-1)}ds = \bigl[e^{2(s-1)}\bigr]_0^1 = 1 - e^{-2}$. □

Proposition 10.4 does for m_k what Proposition 3.2 does for $g_{k;l}$: it checks Definition 2.1 against the source at the one index where the source gives a closed form, $m_2 = 1 - e^{-2} = 0.8646647\ldots$, so that Corollary 10.3 is a statement about Pinsky's m_5 and not merely about an integral of ours. Three computations independent of ours agree with the certified brackets: the two high-precision evaluations of Remarks 9.1 and 9.2, the source's own Table 3, which prints 0.792 where its Table 2 prints 0.793, and the value $\theta_4 = 0.79227591371305$ printed in [7], which is m_5.

11. The bonded-molecule density: a reduction, and the window $k \leq 20$

We turn to the second assertion $(*)$. The first thing to notice is that the difference of two consecutive m_k is a single integral against the *unchanged* kernel, so that the certificate layer of §6 applies to it with no modification at all.

Theorem 11.1. *Let* $k \geq 1$*. Then*

(i) $\Phi_{k+1}(s) = \Phi_k(s)\exp\bigl(2(s^k-1)/k\bigr)$ *for every* s*;*

(ii) $m_k - m_{k+1} = \displaystyle\int_0^1 \Bigl(k - (k+1)\exp\bigl(2(s^k-1)/k\bigr)\Bigr)\Phi_k(s)\,ds$*;*

(iii) *the integer polynomial*

$$\chi_k(s) = 4k^5 - (k+1)\Bigl(2k^2 - 2k\bigl(1-s^k\bigr) + \bigl(1-s^k\bigr)^2\Bigr)^2$$

of degree $4k$ *satisfies* $\chi_k(s) \leq 4k^4\bigl(k - (k+1)\exp(2(s^k-1)/k)\bigr)$ *for every* $s \in [0,1]$*.*

Proof. (i) The exponent of Φ_{k+1} is that of Φ_k plus the single term $2(s^k-1)/k$. (ii) Substitute (i) into $m_{k+1}=(k+1)\int_0^1\Phi_{k+1}$ and subtract from $m_k=k\int_0^1\Phi_k$. (iii) For $x\le 0$ one has $e^{x/2}\le 1+x/2+x^2/8$, both sides being nonnegative there, so $e^x=(e^{x/2})^2\le(1+x/2+x^2/8)^2$. Apply this at $x=-2(1-s^k)/k\in[-2/k,0]$ and clear the denominator $4k^4$. □

The squaring in (iii) is not decoration. The unsquared bound $e^x\le 1+x+x^2/2$ produces, at $k=2$, the polynomial $2-6s^4$, which is a genuine pointwise lower bound on $k^2(k-(k+1)e^{2(s^k-1)/k})$ but whose integral against Φ_2 is *negative*; part (iii) of Proposition 11.4 records this, and it is why the certificates of Theorem 11.2 use χ_k and not the quadratic. §13 rescues the unsquared bound from a different direction: there it is not integrated against a certificate but evaluated in closed form, and the loss disappears.

Theorem 11.2. *For every k with $2\le k\le 20$ one has $m_{k+1}<m_k$.*

Proof sketch. By Theorem 11.1 it suffices that $\int_0^1\chi_k\Phi_k>0$, and that is one instance of Theorem 10.1 with $c=\chi_k$ and $Q>0$: nineteen certificates, one per k, at scale $Q=10^5$, with $\deg p=4k+3$ (degree 11 at $k=2$, 83 at $k=20$). As in Theorem 6.3 the check is a finite exact identity and sign test on integer vectors, the transcendental weight is never evaluated, and the search that produced the polynomials is heuristic and is not part of the proof. This is a window and not a proof of $(*)$: the certificate degree grows linearly in k, so no finite family of them can reach every k. §13 removes the window. □

Corollary 11.3. *$m_j<m_i$ whenever $2\le i<j\le 21$.*

Proof. Induction on j over the nineteen steps of Theorem 11.2. □

Proposition 11.4. (i) *The reverse inequality fails at the first index: $m_2\le m_3$ is false.* (ii) $\chi_2(1)=-64<0$*, so the integrand of Theorem 11.2 changes sign and its positivity is a genuine integral fact.* (iii) *On* $[0,1]$ *one has* $2-6s^4\le 4\big(2-3e^{s^2-1}\big)$*, the case $k=2$ of the unsquared quadratic bound; and nevertheless* $\int_0^1(2-6s^4)\,\Phi_2(s)\,ds\le -0.026323<0$.

Proof. (i) is Theorem 11.2 at $k=2$. (ii) is arithmetic: $1-s^k$ vanishes at $s=1$, so $\chi_2(1)=4\cdot 2^5-3\cdot(2\cdot 4)^2=128-192=-64$. (iii) The first half is $e^x\le 1+x+x^2/2$ at $x=s^2-1\le 0$, cleared by $k^2=4$; the second half is one instance of Theorem 10.1 with $c=2-6s^4$ and $Q=-10^6$, an upper bound of degree 14. □

Part (iii) is the counterpart, for m_k, of Proposition 5.2: it delimits the method rather than illustrating it. A pointwise lower bound on the bracket of Theorem 11.1(ii) may be perfectly valid and still integrate to a negative number, in which case no certificate of the kind used here can exist for it.

12. The moment relation

Everything so far about m_k has been finite. The uniform argument rests instead on one integration by parts, one degree below the one behind Theorem 4.1.

Theorem 12.1. *For every integer $k\ge 1$ and every integer $m\ge 0$,*

$$(m+1)\,M_{k,m}-m\,M_{k,m+1}=2\,M_{k,m+k}. \tag{11}$$

Proof. Differentiate $f(s)=s^{m+1}(1-s)\Phi_k(s)$. By (4),

$$f'(s)=\big[(m+1)s^m-(m+2)s^{m+1}\big]\Phi_k(s)+s^{m+1}(1-s)\Lambda_k(s)\Phi_k(s),$$

and by (5) the second term is $2s^{m+1}(1-s^{k-1})\Phi_k(s)=2\big(s^{m+1}-s^{m+k}\big)\Phi_k(s)$, the last step using $k\ge 1$. Hence

$$f'(s)=\big[(m+1)s^m-m\,s^{m+1}-2s^{m+k}\big]\Phi_k(s).$$

Both endpoint values of f vanish, at $s=0$ because $m+1\ge 1$ and at $s=1$ because of the factor $1-s$, so $\int_0^1 f'=0$, which is (11). □

Two remarks on provenance. First, the *technique* — find a polynomial whose total derivative along Φ_k is the target, then read off the endpoints — is published for this very kernel: the display numbered (4.6) in the e-print of [2] integrates $(ky-(k-1)y^2)e_k(y)$, with $e_k = e^{2H_{k-1}}\Phi_k$ as in §2, in exactly this way, to prove $\sum_j j\, g_{k;j} = 1-m_k$. Of the asymptotic mean it then reads off, [2] writes that it "is in line with (A28) and (A25) of Mackenzie (1962)" [8]; that sentence, and not a fuller attribution, is what stands there. (The number (4.6) is the e-print's; the published version was not obtained here, and the e-print carries no bibliography entry for Mackenzie.) Second, Theorem 4.1 follows from (11): subtracting the instance $m = l+1$ from the instance $m = l$ gives $(l+1)(M_{k,l}-2M_{k,l+1}+M_{k,l+2}) = 2(M_{k,l+k}-M_{k,l+k+1})$, which is (8) because $A_{k,l} = M_{k,l} - M_{k,l+1}$. We therefore claim no novelty for (11) itself; what we did not find in any source read is the relation for a general m, nor Corollary 12.3 below; those two are what the uniform argument needs.

Corollary 12.2. *For every $k \geq 1$,*

$$\int_0^1 s^k\, \Phi_k(s)\, ds = \frac{1}{2}\int_0^1 \Phi_k(s)\, ds, \qquad \textit{that is,} \qquad M_{k,k} = \tfrac{1}{2}J_k.$$

Equivalently, under the probability measure $\Phi_k(s)\,ds/J_k$ on $[0,1]$ one has $\mathbb{E}[s^k] = \frac{1}{2}$, exactly.

Proof. The instance $m = 0$ of (11). □

This identity is Pinsky's, and we claim no novelty for it. His displayed equation (*dandg*) has as its first line $\sum_{l=0}^{k-1} g_{k;l} = m_k/k$; substituting (2), (1) and the geometric sum $\sum_{l<k} s^l = (1-s^k)/(1-s)$ turns that line into $2\int_0^1 (1-s^k)\Phi_k = \int_0^1 \Phi_k$, which is exactly the display above. He derives (*dandg*) combinatorially, from $\sum_l G^{(n)}_{k;l} = M^{(n)}_k/k+1$, and never writes the integral form; the proof given here is the one that generalises to (11). The second line of (*dandg*) is a different identity and is the one proved by (4.6) of [2]; it does not give the next corollary.

Corollary 12.3. *For every $k \geq 1$,* $\quad 4M_{k,2k} = J_k + k\, N_k$.

Proof. The instance $m = 1$ of (11) reads $2M_{k,1} - M_{k,2} = 2M_{k,k+1}$, and $N_k = M_{k,0} - 2M_{k,1} + M_{k,2}$ by Definition 2.2, so $2M_{k,k+1} = J_k - N_k$. The instance $m = k$ reads $(k+1)M_{k,k} - k\, M_{k,k+1} = 2M_{k,2k}$; doubling it and substituting $2M_{k,k} = J_k$ from Corollary 12.2 and $2M_{k,k+1} = J_k - N_k$ gives $4M_{k,2k} = (k+1)J_k - k(J_k - N_k) = J_k + kN_k$. □

Corollary 12.2 also gives Pinsky's first assertion a probabilistic form, worth recording although we make no use of it. Write $d\mu_k = \Phi_k(s)\,ds/J_k$ and $d\nu_{k,l} = (1-s)s^l\Phi_k(s)\,ds/A_{k,l}$, both probability measures on $[0,1]$. Then $A_{k,l+k} = A_{k,l}\,\mathbb{E}_{\nu_{k,l}}[s^k]$, so by Corollary 4.2 the step (3) at l says precisely that

$$\mathbb{E}_{\nu_{k,l}}\big[s^k\big] < \tfrac{1}{2} = \mathbb{E}_{\mu_k}\big[s^k\big],$$

the equality being Corollary 12.2. Tilting μ_k by the density proportional to $(1-s)s^l$ must therefore lower the k-th moment — and, by Theorem 8.3, it does so only for l below $L(k)$, so any proof of (3) uniform in k has to be quantitative in l.

13. The uniform reduction, and Pinsky's second conjecture

The certificates of Theorem 11.2 came from the *squared* bound of Theorem 11.1(iii), because the unsquared one integrates to a negative number at $k = 2$. The point of this section is that the unsquared bound is not weak at all: it is only being integrated badly. Corollaries 12.2 and 12.3 evaluate its integral in closed form, and what comes out is positive for every $k \geq 5$.

Theorem 13.1. *For every $k \geq 1$,*

$$2k^2\big(m_k - m_{k+1}\big) \;\geq\; (k-1)\,J_k - k(k+1)\,N_k. \tag{12}$$

Between Theorem 11.1(ii) and (12) the only inequality used is $e^x \leq 1 + x + x^2/2$ for $x \leq 0$; after it, every step is an identity.

Proof. Put $x = 2(s^k - 1)/k \le 0$ and $u = s^k$. From $e^x \le 1 + x + x^2/2$,

$$k^2\big(k - (k+1)e^x\big) \;\ge\; k^2\Big(k - (k+1)\big(1 + x + \tfrac{x^2}{2}\big)\Big) = (k^2-2) - (k+1)(2k-4)\,u - 2(k+1)\,u^2,$$

the last equality being an algebraic identity in u, since $kx = 2(u-1)$. Integrating against Φ_k and using Theorem 11.1(ii),

$$k^2\big(m_k - m_{k+1}\big) \;\ge\; (k^2-2)J_k - (k+1)(2k-4)M_{k,k} - 2(k+1)M_{k,2k}.$$

By Corollary 12.2, $M_{k,k} = \frac{1}{2}J_k$, and $(k^2-2) - (k+1)(k-2) = k$, so the right-hand side is $k\,J_k - 2(k+1)M_{k,2k}$; by Corollary 12.3, $2(k+1)M_{k,2k} = \frac{1}{2}(k+1)(J_k + kN_k)$, and $k - \frac{1}{2}(k+1) = \frac{1}{2}(k-1)$. Multiplying by 2 gives (12). □

So the whole question is whether $(k-1)J_k$ beats $k(k+1)N_k$. Both sides are transcendental integrals, but each needs only a one-sided elementary estimate.

Lemma 13.2. (i) *$N_k \le \Phi_k(0)$ for every $k \ge 1$.* (ii) *$J_k \ge \big(1 - e^{-2(k-1)}\big)/\big(2(k-1)\big)$ for every $k \ge 2$.* (iii) *$k^2\,\Phi_k(0) \le K^2\,\Phi_K(0)$ whenever $1 \le K \le k$.*

Proof. (i) By (4) and (5) the function $G(s) = (1-s)^2\Phi_k(s)$ has $G'(s) = -2(1-s)s^{k-1}\Phi_k(s) \le 0$ on $[0,1]$, so for $0 \le s \le 1$

$$\Phi_k(0) - (1-s)^2\Phi_k(s) = \int_0^s 2(1-t)t^{k-1}\Phi_k(t)\,dt \;\ge\; 0,$$

and integrating $(1-s)^2\Phi_k(s) \le \Phi_k(0)$ over $[0,1]$ gives (i). (ii) Bernoulli's inequality $1 + j(s-1) \le s^j$ for $s \ge 0$, $j \ge 1$ gives $(s^j - 1)/j \ge s - 1$, so the exponent of Φ_k is at least $2(k-1)(s-1)$ and $\Phi_k(s) \ge e^{-2(k-1)(1-s)}$; integrating the right-hand side over $[0,1]$ gives (ii). (iii) By Theorem 11.1(i) at $s = 0$, $\Phi_{k+1}(0) = e^{-2/k}\Phi_k(0)$, and $e^{1/k} \ge 1 + 1/k$ gives $(k+1)^2e^{-2/k} \le k^2$; so $k \mapsto k^2\Phi_k(0)$ is nonincreasing, and (iii) follows by induction. □

Theorem 13.3. *Let $K \ge 2$ satisfy*

$$2K(K+1)\,\Phi_K(0) \;<\; 1 - e^{-2(K-1)}, \qquad \textit{where} \quad \Phi_K(0) = \exp\Big(-2\sum_{j=1}^{K-1}\tfrac{1}{j}\Big). \tag{13}$$

Then $m_{k+1} < m_k$ for every $k \ge K$. The hypothesis (13) *holds at $K = 12$ and at $K = 5$.*

Proof. Let $k \ge K$. Combining (12) with Lemma 13.2(ii), (i) and (iii),

$$\begin{aligned} 2k^2\big(m_k - m_{k+1}\big) \;&\ge\; \frac{1 - e^{-2(k-1)}}{2} - \Big(1 + \frac{1}{k}\Big)k^2\Phi_k(0) \\ &\ge\; \frac{1 - e^{-2(k-1)}}{2} - \Big(1 + \frac{1}{k}\Big)K^2\Phi_K(0) \\ &\ge\; \frac{1 - e^{-2(k-1)}}{2} - \Big(1 + \frac{1}{K}\Big)K^2\Phi_K(0), \end{aligned}$$

using $k(k+1) = (1+1/k)k^2$, then Lemma 13.2(iii), then $1 + 1/k \le 1 + 1/K$. Since $e^{-2(k-1)} \le e^{-2(K-1)}$ and $(1+1/K)K^2 = K(K+1)$, the right-hand side is at least $\frac{1}{2}\big[(1 - e^{-2(K-1)}) - 2K(K+1)\Phi_K(0)\big]$, which is positive by (13).

For $K = 12$: $\Phi_{12}(0) = e^{-2H_{11}} = e^{-83711/13860}$ and $83711/13860 > 6$, so $\Phi_{12}(0) < e^{-6} < 1/403$ because $e^6 > 403$; the same bound gives $e^{-22} < 1/403$, so $2\cdot 12\cdot 13\,\Phi_{12}(0) < 312/403 < 402/403 < 1 - e^{-22}$. For $K = 5$: $\Phi_5(0) = e^{-2H_4} = e^{-25/6}$ and $e^{25/6} = e^4e^{1/6} > 54\cdot\frac{7}{6} = 63$, using $e^4 > 54$ and $e^{1/6} \ge 1 + \frac{1}{6}$, so $2\cdot 5\cdot 6\,\Phi_5(0) < 60/63 = \frac{20}{21}$; and $e^{-8} \le e^{-4} < \frac{1}{54}$ gives $1 - e^{-8} > \frac{53}{54}$, while $\frac{20}{21} < \frac{53}{54}$. The only analytic inputs are $e^6 > 403$ and $e^4 > 54$, both from a rational lower bound for e, and $e^{1/6} \ge \frac{7}{6}$. □

Theorem 13.4. *For every integer $k \ge 2$ one has $m_{k+1} < m_k$; and the same holds at $k = 1$, so $m_{k+1} < m_k$ for every $k \ge 1$.*

Proof. For $k \geq 5$ this is Theorem 13.3 at $K = 5$. For $k = 2, 3, 4$ it is three of the nineteen certificates of Theorem 11.2. For $k = 1$, formula (1) reads $\Phi_1 \equiv 1$ and $m_1 = 1$, while $m_2 = 1 - e^{-2} < 1$ by Proposition 10.4. □

Corollary 13.5. *$k \mapsto m_k$ is strictly decreasing on $\{1, 2, 3, \dots\}$: $m_j < m_i$ whenever $1 \leq i < j$.*

Proof. Induction on j over Theorem 13.4. □

This is Pinsky's second assertion, in the form in which he states it, for every k and with no window. Of the 254 certificates in this paper exactly three enter its proof, and they enter only at $k = 2, 3, 4$; the tail $k \geq 5$ is Theorem 13.3, which contains no computation beyond two rational comparisons. That the certificate window $2 \leq k \leq 20$ of Theorem 11.2 and the analytic tail $k \geq 5$ overlap is a convenience, not a necessity: what forces the three exceptional cells is Proposition 13.7(i) below, which shows that the right-hand side of (12) is negative at $k = 2$.

Theorem 13.6. *For every $k \geq 5$, $m_k - m_{k+1} \geq \dfrac{1}{100\,k^2}$; and for every $k \geq 12$, $m_k - m_{k+1} \geq \dfrac{1}{20\,k^2}$.*

Proof. The last bound in the display in the proof of Theorem 13.3 depends on k only through $e^{-2(k-1)}$, so it is bounded below by a constant on each tail. At $K = 12$, using $\Phi_{12}(0) < 1/403$ and $e^{-2(k-1)} \leq e^{-22} < 1/403$, that constant is $\frac{1}{2}\bigl(1 - \frac{1}{403}\bigr) - \frac{13}{12} \cdot \frac{144}{403} = \frac{45}{403} > \frac{1}{10}$, whence $m_k - m_{k+1} \geq \frac{1}{20k^2}$ for $k \geq 12$. At $K = 5$, using $\Phi_5(0) < \frac{1}{63}$ and $e^{-2(k-1)} \leq e^{-8} < \frac{1}{403}$, it is $\frac{201}{403} - \frac{10}{21} > \frac{1}{50}$, whence $m_k - m_{k+1} \geq \frac{1}{100k^2}$ for $k \geq 5$. □

Neither [1] nor [7] states any rate; both assert only that m_k decreases. The constants above are not sharp, and we do not claim a limit for $k^2(m_k - m_{k+1})$; the only loss in the chain worth naming is Lemma 13.2(ii), which replaces J_k by a Bernoulli bound and costs an asymptotic factor of about 1.5.

Proposition 13.7. (i) *$J_2 = \frac{1}{2}\bigl(1 - e^{-2}\bigr)$ and $N_2 = \frac{1}{4} - \frac{5}{4}e^{-2}$, so the right-hand side of* (12) *at $k = 2$ equals $7e^{-2} - 1 = -0.0526530\ldots < 0$.* (ii) *$m_5 \leq m_6$ is false, and $m_{12} \leq m_{13}$ is false.* (iii) *$m_j \neq m_i$ for $1 \leq i < j$.* (iv) *$M_{k,m} > 0$ for all k, m, and $M_{k,k} < M_{k,0}$ for $k \geq 1$.*

Proof. (i) $\Phi_2(s) = e^{2(s-1)}$, so both integrals are elementary; the stated value follows by arithmetic. (ii) is Theorem 13.4 at $k = 5$ and $k = 12$, and (iii) is Corollary 13.5. (iv) The integrand of $M_{k,m}$ is continuous, nonnegative on $[0, 1]$ and positive on $(0, 1)$; the strict inequality is then Corollary 12.2, $M_{k,k} = \frac{1}{2}M_{k,0}$. □

Part (i) is the boundary of the method, and it is exact: the uniform argument *cannot* start at $k = 2$, so the three certificate cells of Theorem 13.4 are forced and not a matter of taste. It also agrees with the independent route of Proposition 11.4(iii): halving $7e^{-2} - 1$ gives $\int_0^1 (2 - 6s^4)\Phi_2 = -0.0263265\ldots$, and the certificate there proves the bound ≤ -0.026323. Part (iv) says identity (11) is not vacuous at $m = 0$: it relates two genuinely different integrals.

Remark 13.8. Outside the formal development we measured how much of the true difference survives the chain. Write

$$B_k = \tfrac{1}{2}\bigl(1 - e^{-2(k-1)}\bigr) - \bigl(1 + \tfrac{1}{k}\bigr)K^2\Phi_K(0)$$

for the lower bound on $2k^2(m_k - m_{k+1})$ that (12) and Lemma 13.2 give at the base K — the bound reached in the display in the proof of Theorem 13.3 after Lemma 13.2(iii) has been applied and before $1 + \frac{1}{k}$ is weakened to $1 + \frac{1}{K}$. Then high-precision arithmetic gives, at the base $K = 12$, $B_k/\bigl(2k^2(m_k - m_{k+1})\bigr) = 31.33\%$ at $k = 12$, 33.41% at $k = 20$ and 33.91% at $k = 24$; at the base $K = 5$ the same ratio is 9.08% at $k = 5$, 19.55% at $k = 12$ and 22.88% at $k = 24$. The retained fraction increases with k in both cases, which is why one numerical check at the base covers the whole tail. These percentages are computations, not theorems. In the same arithmetic, (13) *fails* at $K = 2, 3, 4$, the ratio of its two sides being about 1.878, 1.217 and 1.025; so $K = 5$ is the sharpest base this chain admits, and again the three exceptional cells of Theorem 13.4 are forced.

14. Verification

Every statement asserted above as a definition, lemma, proposition, theorem or corollary is checked by the kernel of the Lean 4 proof assistant [9] over the Mathlib library [10], in the form stated except for Lemma 6.1 and Corollary 4.2, whose standing is described below. The development is thirteen modules and about 5 700 lines carrying 420 theorems and 18 definitions: the kernel Φ_k, its derivative, Λ_k, $A_{k,l}$, $M_{k,j}$, N_k, $g_{k;l}$ and m_k; a small integer-polynomial layer (coefficient lists in Horner form with addition, scaling, multiplication, formal derivative and their evaluation homomorphisms); the two integrations by parts of Theorems 4.1 and 12.1 and the closed forms of Propositions 3.2 and 10.4; the sign-change argument of Theorem 5.1 and the controls; the certificate lemma in both its forms, Theorems 6.2 and 10.1; the 254 certified cells; the telescoping and the polynomial bound of Theorem 11.1; and the whole of §13, which is one further module and contains no certificate at all. It has no `sorry` and declares no axiom of its own. Three separate audits printed the axiom set of every headline declaration — 19 of them for the material of §§2–9, 39 for §§10–11 and the frontier theorems, 33 for §§12–13 — and every one returned `propext`, `Classical.choice` and `Quot.sound` and nothing else: in particular no `sorryAx` and no compiled-evaluation axiom. There is no use of `native_decide`, no external solver, no floating point and no interval arithmetic anywhere in the development. This matters here more than it usually would, because the objects are transcendental integrals: what is verified is Definitions 2.1 and 2.2 themselves, not a numerical surrogate for them, and the hypotheses of Theorems 6.2 and 10.1 that carry the certificates are decided as identities and sign tests on lists of integers. Total elaboration for one clean pass is about 0.18 core-hours; the whole development, including failed and iterated runs, about 2.2 core-hours.

Three points of form. Lemma 6.1 and Corollary 4.2 are not separately named declarations: the first occurs inside the verified proofs of Theorems 6.2 and 10.1, where it is the evaluation of $\int_0^1 (p\,\Phi_k)'$ for p with integer coefficients, which is the only case any statement of this paper uses; the second is Theorem 4.1 rearranged by one line of arithmetic. Next, the certified cells reach (3) through (7) directly rather than through the reduction, so §4 and §6 are logically independent routes to the same inequality; likewise §13 is independent of §11 except at $k = 2, 3, 4$. Finally, the probabilistic phrasings — $\mathbb{E}_{\mu_k}[s^k] = \frac{1}{2}$ in Corollary 12.2, and the reformulation of (3) as $\mathbb{E}_{\nu_{k,l}}[s^k] < \frac{1}{2}$ after Corollary 12.3 — are restatements for the reader; what is verified is the identity between integrals in each case.

Five things are deliberately outside the formal development and are labelled where they appear: the general Beta integral $\int_0^1 \psi_l = (1-l)/((l+1)(l+2)(l+3))$ quoted in §5 (only its two instances $l = 1$ and $l = 2$, the ones the proofs use, are verified); §9.2 in its entirety (the log-convexity reduction and the table of $r_{k,k-2}$, which would need Cauchy–Schwarz for interval integrals); Remarks 9.1 and 9.2 together with the decimal values quoted after Proposition 3.2 (these are high-precision computations, and Corollary 10.3 is the certified statement that replaces them); Remark 13.8 (the retained-signal percentages and the failure of (13) at $K = 2, 3, 4$); and the identification of the values printed in [7] with $m_2, \ldots, m_6$, together with the two decimal comparisons in the footnote to §1.2. As independent checks made before any claim above, the printed values of Table 1 and of the first column of Table 2 of [1] were recomputed from (1) and (2) by the two methods described in §9.3 — they agree with the source except at the two entries of Remarks 9.1 and 9.2 — and the five values $\theta_1, \ldots, \theta_5$ printed in [7] were recomputed the same way and agree with our values to the last decimal each of them prints. Every certificate was additionally re-checked outside the formal development by a second, independent implementation of the polynomial operations before being written into the development; the kernel remains the authority.

On the standing of $(*)$ we record what was searched rather than what was concluded. We found no proof of it in: the works citing [7] recorded by OpenAlex under its DOI; the forward citations of [2] and of [1], the latter having none; Semantic Scholar; five arXiv API queries pairing random sequential adsorption, jamming or parking with monotonicity; a local mirror of arXiv full texts (86 GB in 373 shards), of which 83 mention random sequential adsorption and in which the only sentence pairing a jamming density with a monotonicity claim is the one quoted in §1.2; the OEIS; and a web search. "Not found" is not "new": what is recorded is the search, together with the fact that the author of [1] wrote in August 2026 that the statement "seems to be open still". For the first assertion (3) we add that [2] contains none of the words *increasing*, *decreasing*, *monotone*, *conjecture* or *open problem*, and that [7] does not mention gaps at all, so neither bears on §§4, 5 or 6.

The source of the development is currently held in a private repository: repository reference to be added at submission.

On the reference list. [1] was read here in full from its arXiv LaTeX e-print, and every quotation from it above is from that source; its abs page was re-fetched on 3 September 2026 (still v1, submitted 5 August 2026, primary category math.PR, cross-list math-ph, MSC 60C05/60F05/60F99/82B20). The e-print of [2] was fetched and read, and equation (4.6) is quoted from it, the published version not having been obtained; [7] was read in its published form, and the sentence on θ_b and the values $\theta_1, \ldots, \theta_5$ are quoted from that. [3], [4], [5], [6] and [8] are cited for attribution and context only and were not read here; the bibliographic data of all five were verified at Crossref, zbMATH and dblp rather than taken from the reference list of [1], and one discrepancy is noted with [4] below.

References

[1] R. G. Pinsky, *A new look at some aspects of one-dimensional random sequential adsorption and its continuum limit*, arXiv:2608.04730v1 [math.PR], 5 August 2026. Read here in full from the arXiv LaTeX e-print of v1; all quotations above are from that source, and its equation labels (mk), ($lgaps$), its Theorem `gapsthm` and its Tables 1, 2 and 3 are those of that e-print as it compiles, quoted version-pinned to v1. Re-fetched from the arXiv API on 3 September 2026: still v1, submitted 5 August 2026 11:53:10 UTC, sole author Ross G. Pinsky, primary category math.PR, cross-list math-ph, no journal reference and no DOI beyond the arXiv one.

[2] C. A. J. Klaassen and J. T. Runnenburg, *Discrete spacings*, Statistica Neerlandica **57** (2003), no. 4, 470–483, doi:10.1111/1467-9574.00240; e-print arXiv:math/0112056v1 [math.PR], 6 December 2001 (the e-print carries no journal reference). The e-print was fetched and read here, and is the version whose notation is compared with ours in §2 and whose equation numbering and Mackenzie reference are quoted in §12. Journal data verified at Crossref and zbMATH (Zbl 1090.60505).

[3] R. G. Pinsky, *Problems from the Discrete to the Continuous: Probability, Number Theory, Graph Theory, and Combinatorics*, Universitext, Springer, Cham, 2014, doi:10.1007/978-3-319-07965-3. (The words after the colon are the catalogued subtitle; [1] prints the main title only.) Cited for attribution only; the text was not obtained here, and no statement of this paper depends on it. The sentence quoted from its published review in §1.2 is from the zbMATH record Zbl 1311.11002 (reviewer M. Hassani), which also displays the book's p_k.

[4] A. Rényi, *On a one-dimensional problem concerning random space filling*, Publ. Math. Inst. Hungar. Acad. Sci. **3** (1958), 109–127 (in Hungarian; catalogued also as Magyar Tud. Akad. Mat. Kutató Int. Közl. **3** (1958), no. 1–2, and translated as Sel. Transl. Math. Stat. Probab. **4** (1963), 203–218). Cited for context only. The title is as catalogued at zbMATH (Zbl 0105.11903); the reference list of [1] omits the word "random".

[5] P. J. Flory, *Intramolecular reaction between neighboring substituents of vinyl polymers*, J. Amer. Chem. Soc. **61** (1939), no. 6, 1518–1521, doi:10.1021/ja01875a053. Cited for attribution only.

[6] D. V. Widder, *The Laplace Transform*, Princeton Mathematical Series **6**, Princeton University Press, Princeton, 1941, x+406 pp. Cited only for the classical theory of the Hausdorff moment problem, which is its Chapter III (§1, p. 100, and §4, p. 108); the log-convexity used in §9.2 is not taken from there but is the one-line Cauchy–Schwarz argument given in the text.

[7] P. L. Krapivsky, *Large deviations in one-dimensional random sequential adsorption*, Phys. Rev. E **102** (2020), no. 6, 062108, 10 pp., doi:10.1103/PhysRevE.102.062108; e-print arXiv:2009.08609 [cond-mat.stat-mech]. The published article was obtained and read here (received 24 September 2020, accepted 13 November 2020, published 3 December 2020), and everything quoted from it above is on its page 062108-7: the sentence on θ_b and the values $\theta_1, \ldots, \theta_5$, the bounds $1/(2b+1) < \rho_b < 1/(b+1)$ of its equation (75), and its equation (77), $\rho_b = \int_0^1 \exp\bigl[-2\sum_{j=1}^{b}(1-y^j)/j\bigr]\,dy$, from which $\theta_b = (b+1)\rho_b = m_{b+1}$ is read. The e-print carries the same text.

[8] J. K. Mackenzie, *Sequential filling of a line by intervals placed at random and its application to linear adsorption*, J. Chem. Phys. **37** (1962), no. 4, 723–728, doi:10.1063/1.1733154. Author, title, volume, issue, year and pages verified at Crossref, OpenAlex and Semantic Scholar under that DOI; the same journal, volume, year and page range are printed in the reference list of [1] (its entry [M]). Cited only for the two equation numbers (A25) and (A28) that [2] names, as quoted in §12; those two numbers are [2]'s and the article itself was not obtained here. The e-print of [2] cites it as "Mackenzie (1962)" and carries no bibliography entry for it.

[9] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, in: Automated Deduction — CADE 28, Lecture Notes in Computer Science **12699**, Springer, 2021, 625–635, doi:10.1007/978-3-030-79876-5_37.

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