MINIMUM WINDING LENGTHS FOR k-OUT-OF-n PICTURE HANGING, AND THE OPEN CASE 3-OUT-OF-5

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ABSTRACT. A picture hangs from a loop of string wound around n nails. A winding is an element of the free group F_n on the nails, removing a set of nails sets the corresponding generators to 1, and the picture falls exactly when the winding becomes trivial. The *k-out-of-n puzzle* asks for a winding that falls as soon as any k nails are removed and survives every removal of fewer than k; $M(k,n)$ is the least length of such a winding. Verhoeff's recent paper determines $M(2,4) = 16$ by exhaustive search and asks for the exact minima of the next cases, naming 3-out-of-5 as the smallest open one. We prove $M(3,5) = 22$ and exhibit an attaining winding. The engine is a projection principle: restricting a solution to a subset of its nails is again a solution of a smaller puzzle, so every four-nail projection of a 3-out-of-5 solution is a 2-out-of-4 solution. With parity this excludes every length below 20 outright and leaves a single occurrence vector at length 20, which one enumeration of 667 979 nodes rules out. Summing the same principle over all co-singletons gives a density bound $M(k,n)/n \geq M(k-1,n-1)/(n-1)$, hence lower bounds for whole diagonal families: $5\,M(n-2,n) \geq 22n$ for $n \geq 5$, $5\,M(n-3,n) \geq 24n$ for $n \geq 5$, and $M(3,6) \geq 29$. For the next cell we prove $24 \leq M(2,5) \leq 30$; the upper bound comes from a product of two commutator blocks and improves the best published construction, which has length 54. All statements are machine-checked in Lean 4.

1. Introduction

Hang a picture from a loop of string wound around two nails in the wall so that it falls when *either* nail is pulled out. The classical answer is the commutator $x_1x_2x_1^{-1}x_2^{-1}$. Formalising the puzzle is what makes it a mathematical question: a winding is an element w of the free group F_n on generators $x_1, \ldots, x_n$, one per nail; pulling out the nails in a set S replaces each x_i with $i \in S$ by 1; and the picture falls precisely when the resulting element of F_n is trivial. Demaine, Demaine, Minsky, Mitchell, Rivest and Pătraşcu [1] made this the standard setting and gave constructions for many variants. The variant we study is the *threshold* one: for $1 \leq k \leq n$, a winding *solves the k-out-of-n puzzle* when it becomes trivial on removing any k nails and stays nontrivial on removing any $k-1$. Such windings exist for all $1 \leq k \leq n$ [1], and we write

$$M(k,n) \;=\; \min\{\,|w| : w \in F_n \text{ solves } k\text{-out-of-}n\,\},$$

the minimum being over reduced lengths. Constructions are plentiful and exact minima are scarce. Demaine et al. are explicit about the gap: their appendix records that "the solutions are not unique; shorter solutions may well exist".

Verhoeff's paper [4] is the source of the questions answered here. Two of the values are settled there by argument, at every n at once: $M(n,n) = n$ and $M(n-1,n) = 2n$ ([4, Propositions 1 and 2]). Beyond the two diagonals the only route known is exhaustive search, and [4, §6] reports what a symmetry-pruned search in Python reaches: $M(2,3) = 6$, $M(1,3) = 10$, $M(1,4) = 16$ and — its headline — $M(2,4) = 16$, attained by exactly two windings up to symmetry, the elimination of length 14 alone taking "just over two hours". The values $M(1,n)$ are in fact older and are known in closed form: Gartside and Greenwood [2] prove that a minimal Brunnian n-word, which is exactly a minimal 1-out-of-n winding, has length $2^m(3k + 2^m)$ when $n = 2^m + k$ with $0 \leq k < 2^m$, giving the sequence $1, 4, 10, 16, 28, 40, 52, 64, 88, \ldots$ [3]; Wästlund [6] gives further 1-out-of-n constructions and describes their minimality as open, which [2] had in fact already settled. Section 8 of [4] then asks, verbatim, "what are the exact minima for the smallest open cases beyond 2-out-of-4?", and names the next one: “the next case, 3-out-of-5 (equivalently, $(n-2)$-out-of-n at $n = 5$, the first non-trivial case in the co-rank family for odd n), should be within reach with a more efficient implementation.”

This paper answers that question and pins the cell after it.

Theorem 1.1. *$M(3,5) = 22$, attained by the winding*

$$w_{3,5} \;=\; x_1x_2x_1^{-1}x_3x_2^{-1}x_4x_5x_2x_1x_5^{-1}x_3^{-1}x_1^{-1}x_4^{-1}x_1x_5x_4x_2^{-1}x_3x_4^{-1}x_1^{-1}x_3^{-1}x_5^{-1}.$$

Theorem 1.2. *$24 \le M(2,5) \le 30$.*

Neither value nor bound was previously recorded: no source we are aware of gives any value or upper bound for 3-out-of-5, and the only lower bound on record is the one implicit in the abandoned run of [5], which eliminates lengths up to 16 and stops there. For 2-out-of-5 [4] itself names the quantity “the unknown true minimum” and offers a length-54 construction.

The tool behind both is elementary and is the reason the searches are small enough to be kernel-checked. Removing a set of nails and keeping its complement are the same operation, and removals compose; so restricting a k-out-of-n solution to t of its nails is a $(k-(n-t))$-out-of-t solution, of length at most the number of letters it keeps (Proposition 3.1). Verhoeff’s sixth open question states the underlying substitution and calls it “the easy move in the wrong direction” — wrong for constructions, which is what that question is about, and exactly right for lower bounds. Two things follow. Summed over all n co-singletons it gives a *density bound* $M(k,n)/n \ge M(k-1,n-1)/(n-1)$ (Theorem 3.3), which propagates along a diagonal and yields lower bounds for the co-rank families at every n (Corollaries 3.5, 3.6 and 3.7). Applied to a single nail it bounds that nail’s multiplicity, and this is what collapses the search: for 3-out-of-5 at length L, every four-nail projection is a 2-out-of-4 solution, so $16 + m_h \le L$ for every nail h, and together with parity this leaves *no* occurrence vector at all below length 20 and exactly one, $(4,4,4,4,4)$, at length 20.

Section 2 fixes the definitions, Section 3 proves the projection and density bounds and their diagonal consequences, Section 4 records the values that were already known and the windings that witness the upper bounds, Section 5 describes the enumeration and the two facts about it that have to be theorems rather than assumptions, Section 6 proves Theorem 1.1, Section 7 proves Theorem 1.2, Section 8 records the controls that pin the statements from both sides, and Section 9 describes the formal verification. The state of the small cells afterwards is:

(k,n)	$M(k,n)$	source
(n,n)	n	[4, Prop. 1]
$(n-1,n)$	$2n$	[4, Prop. 2]
$(1,3)$	10	[2, Thm. 7]; [4, §6]
$(1,4)$	16	[2, Thm. 7]; [4, §6]
$(2,4)$	16	[4, §6]
$(3,5)$	22	Theorem 1.1
$(2,5)$	$24 \le \cdot \le 30$	Theorem 1.2

2. Preliminaries

Throughout, $n \ge 1$, $[n] = \{1,\dots,n\}$, and F_n is the free group on $x_1,\dots,x_n$. Every element of F_n has a unique reduced word, $|w|$ denotes its length, and $m_g(w)$ denotes the number of occurrences of $x_g^{\pm 1}$ in it. Thus $\sum_{g\in[n]} m_g(w) = |w|$.

Definition 2.1. For $S \subseteq [n]$ let $\rho_S\colon F_n \to F_n$ be the endomorphism with $\rho_S(x_i) = 1$ for $i \in S$ and $\rho_S(x_i) = x_i$ otherwise: *remove the nails in S*. For $T \subseteq [n]$ we write $\pi_T = \rho_{[n]\setminus T}$: *keep only the nails in T*.

On a word, ρ_S simply erases the letters using nails of S; in particular $\rho_{S'} \circ \rho_S = \rho_{S\cup S'}$ and $\pi_{T'} \circ \pi_T = \pi_{T\cap T'}$.

Definition 2.2. Let $1 \le k \le n$. An element $w \in F_n$ *solves the k-out-of-n puzzle* if

$$\rho_S(w) = 1 \iff |S| \ge k \qquad \text{for every } S \subseteq [n].$$

$M(k, n)$ is the least $|w|$ over all such w.

This is Demaine's convention and is the form in which every claim below was verified. Because removals compose, only the two *essential* sizes matter.

Lemma 2.3 (Essential removals)**.** *Let $1 \le k \le n$ and $w \in F_n$. Then w solves k-out-of-n if and only if $\rho_S(w) = 1$ for every S with $|S| = k$ and $\rho_S(w) \ne 1$ for every S with $|S| = k - 1$. Equivalently, in the "keep" form: $\pi_T(w) = 1$ for every T with $|T| = n - k$, and $\pi_T(w) \ne 1$ for every T with $|T| = n - k + 1$.*

For 3-out-of-5 this reads: all ten pair-projections are trivial and all ten triple-projections are not. Two counting facts complete the setup; both are in [4, §6] and both are used as filters below.

Lemma 2.4 (Support and parity)**.** *Let w solve k-out-of-n.*

(1) *If $1 \le k \le n$ then $m_g(w) > 0$ for every nail g.*
(2) *If $k \le n - 1$ then the exponent sum of x_g in w is 0 for every g; hence $m_g(w)$ is even, hence $m_g(w) \ge 2$, and $|w|$ is even.*

Proof sketch. (1) If x_g did not occur in w then $\rho_{S\cup\{g\}}(w) = \rho_S(w)$ for every S, so a removal of size $k - 1$ could be enlarged to one of size k without changing the value — contradicting Definition 2.2 at those two sizes. (2) Removing the $n - 1$ nails other than g leaves a set of size $n - 1 \ge k$, so it kills w; what survives is x_g raised to its exponent sum, which is therefore 0. Occurrences of x_g and of x_g^{-1} are then equal in number. □

3. Restriction, the projection bound, and the density bound

Removals compose, so a solution restricted to fewer nails is again a solution. Making this precise requires relabelling: for an injection $e\colon [t] \hookrightarrow [n]$ let $r_e\colon F_n \to F_t$ be the homomorphism sending $x_{e(i)} \mapsto y_i$ and every generator outside the image of e to 1. The map $F_t \to F_n$, $y_i \mapsto x_{e(i)}$, is a one-sided inverse of r_e; that retraction is what carries the *nontriviality* half of the statement, since r_e alone is not injective.

Proposition 3.1 (Projection)**.** *Let w solve k-out-of-n, let $e\colon [t] \hookrightarrow [n]$ be an injection with image T, and suppose $n - t < k$. Then $r_e(w)$ solves $(k - (n - t))$-out-of-t, and*

$$|r_e(w)| \;\le\; \sum_{g\in T} m_g(w).$$

Proof sketch. For $S' \subseteq [t]$ one has $\rho_{S'} \circ r_e = r_e \circ \rho_{e(S')\cup([n]\setminus T)}$, because both sides kill exactly the generators in $e(S')$ and outside T. The removal on the right has size $|S'| + (n - t)$, so it trivialises w exactly when $|S'| + (n - t) \ge k$, i.e. when $|S'| \ge k - (n - t)$; for the forward direction this is immediate, and for the backward one it is where the retraction is used, to conclude that $r_e(w) \ne 1$ from $\rho_\cdot(w) \ne 1$. The length bound holds because r_e sends each letter of w to a generator or to 1, so $|r_e(w)|$ is at most the number of letters of w that use a nail of T. □

Applied to a co-singleton $T = [n] \setminus \{h\}$ this bounds one nail's multiplicity; summing over all h bounds the length. Both are stated with a numerator and a denominator so that no ceiling appears.

Proposition 3.2 (Projection at a co-singleton)**.** *Let $k \ge 2$ and let c, d be integers such that $c \le d\,|y|$ for every $(k - 1)$-out-of-n solution y. Then for every k-out-of-$(n + 1)$ solution x and every nail h,*

$$c + d \cdot m_h(x) \;\le\; d\,|x|.$$

Theorem 3.3 (Density). *Let $k \geq 2$ and let c, d be as in Proposition 3.2. Then every k-out-of-$(n+1)$ solution x satisfies $(n+1)\,c \leq d\,n\,|x|$. In particular*

$$\frac{M(k,n+1)}{n+1} \;\geq\; \frac{M(k-1,n)}{n},$$

so the density $M(k,n)/n$ does not decrease as one moves down a diagonal.

Proof. Sum Proposition 3.2 over the $n+1$ nails h and use $\sum_h m_h(x) = |x|$: the left side is $(n+1)c + d|x|$ and the right side is $(n+1)d|x|$. $\square$

The density bound propagates along a diagonal by induction, and the induction never uses the co-rank.

Theorem 3.4 (Diagonal propagation). *Let $r \geq 0$, $d \geq 1$, and $n_0 \geq r+2$. Suppose $c\,n_0 \leq d\,|x|$ for every $(n_0 - r)$-out-of-n_0 solution x. Then $c\,n \leq d\,|x|$ for every $n \geq n_0$ and every $(n-r)$-out-of-n solution x.*

Corollary 3.5. *$M(n-2,n) \geq 4n$ for every $n \geq 4$, from $M(2,4) = 16$.*

Corollary 3.6. *$5\,M(n-2,n) \geq 22n$ for every $n \geq 5$, from $M(3,5) = 22$ (Theorem 6.1).*

Corollary 3.7. *$5\,M(n-3,n) \geq 24n$ for every $n \geq 5$, from $M(2,5) \geq 24$ (Theorem 7.1); in particular $M(3,6) \geq 29$.*

Corollary 3.6 improves the co-rank-two lower bound from $4n$ to $4.4n$. In the other direction, [4, Theorem 14] evaluates Wästlund's binary-split recursion on this family — with the two optimal blocks of Proposition 4.1 supplying the subproblems — and shows that for $n = 2^i$ with $i \geq 1$ it produces a solution of length exactly $6n\log_2(n/2)$. So at n a power of two the family is pinned between $4.4n$ and $6n\log_2(n/2)$, and closing that logarithmic factor is not a computation; the gap between constructions and search optima is the fourth open question of [4]. Corollary 3.7 is the first lower bound of any kind for the co-rank-three family.

4. The values that were already known

Two exact values hold at every n and are proved by argument.

Proposition 4.1 ([4, Propositions 1 and 2]). *$M(n,n) = n$ for every $n \geq 1$, attained by $x_1x_2\cdots x_n$; and $M(n-1,n) = 2n$ for every $n \geq 2$, attained by $x_1\cdots x_n\,x_1^{-1}\cdots x_n^{-1}$.*

Proof sketch. For the upper bounds, only removals of size k and $k-1$ need checking (Lemma 2.3). For $k = n$ the two sizes are n and $n-1$: removing everything gives 1, and removing all but one nail leaves a single generator. For $k = n-1$ they are $n-1$ and $n-2$: keeping one nail gives $x_ix_i^{-1} = 1$, and keeping two gives a commutator, which is nontrivial in a free group. For the lower bounds, every nail occurs (Lemma 2.4(1)), which gives $|w| \geq n$; and on the co-diagonal every nail occurs an even and positive number of times (Lemma 2.4(2)), hence at least twice, which gives $|w| \geq 2n$. $\square$

The remaining known values are computational. Their upper halves are single windings, checked against Definition 2.2 directly.

Proposition 4.2 (Windings). *The following elements solve the puzzles indicated, so that $M(2,3) \leq 6$, $M(1,3) \leq 10$, $M(1,4) \leq 16$, $M(2,4) \leq 16$ and $M(3,5) \leq 22$:*

$$\begin{aligned}
w_{2,3} &= x_1x_2x_3x_1^{-1}x_2^{-1}x_3^{-1},\\
w_{1,3} &= x_1x_2x_1^{-1}x_2^{-1}x_3x_2x_1x_2^{-1}x_1^{-1}x_3^{-1},\\
w_{1,4} &= [[x_1,x_2],[x_3,x_4]],\\
w_{2,4} &= x_1x_2x_1^{-1}x_2^{-1}x_3x_4x_2x_1x_4^{-1}x_3^{-1}x_4x_3x_2^{-1}x_1^{-1}x_3^{-1}x_4^{-1},\\
w'_{2,4} &= x_1x_2x_1^{-1}x_3x_2^{-1}x_4x_2x_1x_4^{-1}x_3^{-1}x_4x_2^{-1}x_3x_1^{-1}x_3^{-1}x_4^{-1},\\
w_{3,5} &= x_1x_2x_1^{-1}x_3x_2^{-1}x_4x_5x_2x_1x_5^{-1}x_3^{-1}x_1^{-1}x_4^{-1}x_1x_5x_4x_2^{-1}x_3x_4^{-1}x_1^{-1}x_3^{-1}x_5^{-1}.
\end{aligned}$$

Here $w_{2,4}$ *and* $w'_{2,4}$ *are the two optimal* 2*-out-of-*4 *windings of* [4, §6]*, and* $w_{1,4}$ *is the balanced nested commutator.*

Each of these was verified by evaluating Definition 2.2 on all 2^n removal sets, reducing each of the resulting words, and comparing the verdict with $|S| \geq k$; no case analysis and no reformulation is involved. The winding $w_{3,5}$ is the one that proves the upper half of Theorem 1.1; [4] gives no construction for 3-out-of-5 at all, and the best that its own machinery yields — its one-step extension [4, Proposition 15] fed the optimal $h_{3,4}$ and $h_{2,4}$ and swept over all 48 labellings for cancellation at the junction — has length 26.

Proposition 4.3. $M(1,3) = 10$ *and* $M(2,4) = 16$*.*

Proposition 4.4. $M(1,4) = 16$*.*

The values in Propositions 4.3 and 4.4 are due to Gartside and Greenwood [2] for $k = 1$ and to [4, §6] for $M(2,4)$; we re-derive them because the lower bounds below are anchored on them, and because re-deriving a published value is the strongest available test of the search machinery. The lower halves are exhaustive enumerations, described in Section 5: for $M(1,3) \geq 10$ and $M(2,4) \geq 16$, four searches each, run with the symmetry reduction of Section 5 switched off entirely; for $M(1,4) \geq 16$, the four lengths $8, 10, 12, 14$ that survive parity and support, together 3 751 nodes.

5. The enumeration

Fix k, n, a target length L and an occurrence vector $m = (m_1, \ldots, m_n)$ with $\sum_g m_g = L$. We enumerate the reduced words w of length L with $m_g(w) = m_g$ that could solve k-out-of-n, by building w from the *right*, one letter at a time. A node at level ℓ is a suffix u of length ℓ together with the reduced words $\pi_T(u)$ for each T in a fixed family $\mathcal{F}$ of $(n-k)$-subsets; for 3-out-of-5, $\mathcal{F}$ is the ten pairs. Building from the right makes these projections incremental: prepending a letter updates each $\pi_T(u)$ in constant time, since reduction of a word with one letter prepended is exactly one cancellation test.

Three facts make the enumeration correct, and each is a theorem rather than a design assumption. The direction of failure is the reason for the care: an incomplete final test makes a lower bound unprovable, but an over-strong prune makes the search faster and the conclusion false.

Theorem 5.1 (Enumeration)**.** *Let* w *be a solution of* k*-out-of-*n *with occurrence vector* m*, written* $w = vu$*.*

1. (Prune soundness.) *If* $T \in \mathcal{F}$ *and* $g \in T$*, then the number of occurrences of* x_g *in the reduced word of* $\pi_T(u)$ *is at most* $m_g(v)$*.*
2. (Completeness.) *Every suffix* u *of* w *occurs at level* $|u|$ *of the enumeration.*
3. (Symmetry.) *Every solution* x *admits a solution* y *of the same length whose reduced word is canonical — the nails are numbered by their last occurrence and the last occurrence of each nail is positive — and whose occurrence vector is a permutation of that of* x*.*

Consequently, if the level-L *list contains no solution, then no solution of length* L *with occurrence vector* m *exists.*

Proof sketch. (1) $\pi_T(w) = 1$ and π_T is multiplicative, so $\pi_T(u)$ is the inverse of $\pi_T(v)$, and reduced words of mutually inverse elements have the same letter multiplicities. Reduction only ever deletes letters, so the multiplicity of x_g in the reduced word of $\pi_T(v)$ is at most $m_g(v)$. (2) Structural induction on the suffix with the prefix universally quantified: the statement proved is "for all u, v with $vu = w$, the node of u lies at level $|u|$", whose inductive step is one application each of the incremental update, the generator restriction and (1). (3) Send each nail to the number of nails whose last occurrence lies to the right of its own. This is injective, because a nail whose last occurrence lies further right strictly loses a nail from the set counted, and an injection of a finite set into itself is a bijection; relabelling by it and flipping signs where needed produces a canonical solution of the same length. $\square$

One subtlety is worth stating because the naive version of the argument is wrong. Surviving the prune all the way to level L forces every projection in $\mathcal{F}$ to be trivial, but says nothing about the $(n-k+1)$-subsets being nontrivial; so a nonempty final level need not contain any solution — it may contain windings that solve a *smaller* threshold. The final level is therefore filtered by Definition 2.2 itself before the level is declared empty of solutions, and it is that filtered list that has to be empty. For $M(2,4) \geq 16$ this is not academic: all four length-14 levels are nonempty and contain no solution.

A variant of the same bridge asks only that $m_g(x) \leq m_g$, since the prune's derivation never uses the equation; that is what allows one computation to sweep a whole list of candidate occurrence vectors.

Proposition 5.2. *The conclusion of Theorem 5.1 holds verbatim when the hypothesis $m_g(x) = m_g$ is weakened to $m_g(x) \leq m_g$ for all g.*

6. Three out of five

Theorem 6.1. *$M(3,5) = 22$, attained by $w_{3,5}$ of Proposition 4.2.*

The upper half is Proposition 4.2. The lower half is where the projection bound does its work; we spell out the arithmetic because it is what reduces an infinite question to a single finite search.

Proof of the lower bound. Let x solve 3-out-of-5, let $L = |x|$ and $m_h = m_h(x)$. Three facts constrain the vector $(m_1, \ldots, m_5)$:

- every m_h is even, by Lemma 2.4(2), so L is even;
- every $m_h \geq 1$ by Lemma 2.4(1), hence $m_h \geq 2$;
- $16 + m_h \leq L$ for every h, by Proposition 3.2 with $k = 3$, $c = 16$, $d = 1$: each four-nail projection of x is a 2-out-of-4 solution, and $M(2,4) = 16$ (Proposition 4.3).

Suppose $L \leq 20$. From $m_h \geq 2$ and $16 + m_h \leq L$ we get $L \geq 18$. If $L = 18$ then every $m_h \leq 2$, so every $m_h = 2$ and $\sum_h m_h = 10 \neq 18$: impossible. So $L = 20$, every $m_h \in \{2,4\}$, and five such numbers sum to 20 only if all five equal 4. Thus every length below 22 except 20 is excluded with no search at all, and length 20 is excluded by the single case $m = (4,4,4,4,4)$. That case is settled by the enumeration of Section 5, run over the ten pair-projections with the canonical form of Theorem 5.1(3): it visits 667 979 nodes and its final level contains no 3-out-of-5 solution. Hence $L \geq 22$. ☐

The weaker bound $M(3,5) \geq 20$ needs no search at all: it is Theorem 3.3 applied to $M(2,4) = 16$, and it is the bound one gets before the multiplicity of a single nail is used. The gap between the two is exactly the arithmetic above.

Remark 6.2. An independent search, written in C and not part of the formal development, reports that the solutions of length 22 form a single orbit under the four symmetries of [4] (relabelling, sign flip, reversal and rotation), with occurrence vector $(6,4,4,4,4)$, and that eight of the ten triple-projections of $w_{3,5}$ have reduced length exactly 10, the minimum for a 1-out-of-3 winding. This is offered as an observation about the winding, not as a theorem: only the value $M(3,5) = 22$ is verified.

7. Two out of five

The cell after 3-out-of-5 is 2-out-of-5, for which [4] gives a length-54 construction and calls the minimum "the unknown true minimum"; the run log of the author's own search code [5] records no 2-out-of-5 run at all. The projection filter is much weaker here, because the four-nail projections of a 2-out-of-5 solution are 1-out-of-4 solutions — worth the same 16, but now against lengths 22 and 24 rather than 20.

Theorem 7.1. $24 \leq M(2,5) \leq 36$.

Proof of the lower bound. As in Section 6, with $c = 16$, $d = 1$ and $M(1,4) = 16$ (Proposition 4.4): every m_h is even and at least 2, and $16 + m_h \le L$. At $L \le 16$ this is already impossible. At $L = 18$ every $m_h = 2$ and five of them sum to $10 \ne 18$, so length 18 is excluded with no search. At $L = 20$ the only vector is $(4,4,4,4,4)$: one enumeration, 33 036 nodes, no solution. At $L = 22$ the cap is $m_h \le 6$ and 45 vectors survive; all 45 are enumerated, together 5 972 482 nodes, and none yields a solution. Note that all 45 label-assignments must be run, not the three underlying multisets, because the canonical form of Theorem 5.1(3) returns a solution whose occurrence vector is an unknown permutation of the original's. Hence $L \ge 24$. □

The upper bound 36 of Theorem 7.1 comes from two constructions, both of which we record because the second is what the block construction below improves on. The first is [4, Proposition 15], $h_{k,n} = A\,x_n\,B\,A^{-1}x_n^{-1}$ with $A = h_{k,n-1}$ and $B = h_{k-1,n-1}$: instantiated with Verhoeff's own length-18 $h_{2,4}$ it gives 54, and instantiated with the optimal length-16 $h_{2,4}$ of Proposition 4.2, choosing the representatives of A and B inside their symmetry orbits so that six letters cancel at the single junction, it gives 38. The second replaces the shape entirely. If D_h is any winding that falls exactly when some nail other than h is removed — that is, a 1-out-of-4 solution supported on $[5] \setminus \{h\}$ — then $D_1D_2D_3D_4D_5$ solves 2-out-of-5: removing two or more nails meets every $[5] \setminus \{h\}$ and kills every factor, while removing the single nail h kills every factor except D_h. The upper bound then stops being a construction and becomes an optimisation over five independent finite sets, and the shortest cyclic reduction found in it has length 36.

Proposition 7.2. *$M(2,5) \le 38$ by the one-step extension with optimal blocks, and $M(2,5) \le 36$ by the five-factor product; both windings were checked against Definition 2.2 on all 32 removal sets.*

Both of those improve the published 54, and both are still far from optimal. What closes most of the remaining distance is a commutator.

Theorem 7.3 (Blocks)**.** *Let $A, B \subseteq [n]$ with $A \cup B = [n]$, and let $P, Q \in F_n$ satisfy*

- *$\rho_S(P) = 1$ whenever $S \cap A \ne \emptyset$, and*
- *$\rho_S(Q) = 1$ whenever $|S \cap B| \ge 2$.*

Then $\rho_S([P,Q]) = 1$ for every S with $|S| \ge 2$, where $[P,Q] = PQP^{-1}Q^{-1}$. Moreover the set of elements killed by a given removal is closed under products and under conjugation.

Proof. Let $|S| \ge 2$. If S meets A then $\rho_S(P) = 1$ and $\rho_S([P,Q]) = \rho_S(Q)\rho_S(Q)^{-1} = 1$. Otherwise S misses A, so $S \subseteq B$ because $A \cup B = [n]$, so $|S \cap B| = |S| \ge 2$ and $\rho_S(Q) = 1$, whence $\rho_S([P,Q]) = \rho_S(P)\rho_S(P)^{-1} = 1$. □

By Lemma 2.3, a winding solves 2-out-of-5 exactly when it is killed by every removal of two nails and by no removal of one. Theorem 7.3 supplies the first half cheaply. Take $A = \{a,b\}$ a pair and B the complementary triple. The hypothesis on P is then exactly that both its one-nail projections onto A die, which is the 1-out-of-2 puzzle, so the cheapest P is a commutator $[x_a, x_b]$ of length 4. The hypothesis on Q is that removing any two nails of B kills it; since removing two of three leaves a single generator raised to its exponent sum, this says precisely that all three exponent sums of Q vanish, and at length 6 there are 360 such words on a given triple. The block $[P,Q]$ then has length 20.

The gain is not the length but the coverage. A block built on the pair $A = \{a,b\}$ can survive the removal of a single nail h only if the four-nail projection keeping $\{a,b\}$ is nontrivial, so it covers only removals with $h \notin \{a,b\}$ — but it can cover up to three of the five, whereas a 1-out-of-4 winding supported on $[5] \setminus \{h\}$ covers exactly one. Two blocks therefore suffice, and their pairs must be disjoint, since a block on $\{a,b\}$ never covers the removal of a nor that of b. Up to relabelling the pairs are $\{1,2\}$ and $\{3,4\}$, and the raw length is $2 \times 20 = 40$ instead of $5 \times 16 = 80$.

Theorem 7.4. $M(2,5) \leq 30$, *attained by*

$$w_{2,5} \;=\; x_2^{-1}x_1^{-1}x_5x_4x_3x_5^{-1}x_4^{-1}x_3^{-1}x_1x_2x_1^{-1}x_2^{-1}x_3x_4x_5 \\ x_4^{-1}x_3^{-1}x_5^{-1}x_2x_1x_5x_1^{-1}x_2^{-1}x_3x_4x_3^{-1}x_4^{-1}x_2x_1x_5^{-1},$$

in which every nail occurs exactly six times. Consequently $24 \leq M(2,5) \leq 30$.

Proof sketch. Two routes, both verified. Directly: $w_{2,5}$ is evaluated against Definition 2.2 on all 32 removal sets. Structurally: $w_{2,5}$ is the cyclic reduction of a product of conjugates of the two blocks

$$C_1 = \big[\,[x_1,x_2],\; x_3x_4x_5x_3^{-1}x_4^{-1}x_5^{-1}\,\big], \qquad C_2 = \big[\,[x_4,x_3],\; [x_5^{-1},\, x_2x_1]\,\big],$$

so Theorem 7.3 applied to C_1 with $A = \{1,2\}$, $B = \{3,4,5\}$ and to C_2 with $A = \{3,4\}$, $B = \{1,2,5\}$, together with closure under products and conjugation, kills every two-nail removal; the five one-nail removals are then checked to leave the word nontrivial. The second route never inspects the 32 removal sets of $w_{2,5}$ itself, so a wrong choice of nail sets would make the two routes disagree. The blocks were found by exhausting the family: 8 choices of P and 360 of Q per pair, closed under the 20 rotations of the resulting block, and all products of one left block with one right block scored by the length of their cyclic reduction. The shortest product in the family that verifies has length 30. □

Remark 7.5. The hypothesis on Q in Theorem 7.3 is strictly weaker than asking Q to solve 2-out-of-3 on its triple, and the slack is what produces the bound. Restricting the search to genuine 2-out-of-3 factors — 48 of the 360 words per triple — stops at 32: products of length 30 do occur there, but none of them is a 2-out-of-5 solution, because the extra cancellation always kills one of the five four-nail projections. The winning second block has $Q = [x_5^{-1},\, x_2x_1]$, which dies when nail 5 alone is removed and hence is not a 2-out-of-3 winding at all.

Remark 7.6. A separate exhaustive search, written in C and *not* part of the formal development, finds no 2-out-of-5 winding of length 24, 26 or 28 — the last of these 4.33×10^9 nodes. With Theorem 7.1 and parity this would give $M(2,5) = 30$. We state it as an observation and not as a theorem: it is not machine-checked, and the filter it uses relies on two auxiliary facts (a multiplicity lemma for 1-out-of-4 windings, and a symmetry reduction) that are argued but not formalised. The kernel-checked statement is Theorem 7.4, and the whole remaining gap in it is on the lower side.

8. Controls

Exhaustive arguments are worth only as much as the definition they run against, so the statements were pinned from both sides and the one narrowing in the search was tested where it must not bite.

Proposition 8.1. $M(3,5)$ *is neither* 20 *nor* 24*; no* 3*-out-of-*5 *winding has length at most* 20*;* $M(2,4)$ *is neither* 14 *nor* 18*. The winding* $\mathrm{w}_{3,5}$ *solves neither* 2*-out-of-*5 *nor* 4*-out-of-*5*, and the word obtained from it by transposing two adjacent letters solves nothing. Finally, the canonical enumeration of Theorem 5.1(3), run on the settled* 2*-out-of-*4 *problem at its true optimum* $L = 16$*,* finds *solutions.*

The last of these is the load-bearing one. An over-strong canonical form would return an empty level there, and it would have made the lower bound of Theorem 6.1 vacuous while looking like a speedup. The corresponding validation for the family of triples used in Section 7 is run at $M(1,4) = 16$, where it finds four canonical solutions.

Proposition 8.2. $M(2,5)$ *is neither* 36 *nor* 22*, and no* 2*-out-of-*5 *winding has length at most* 23*. The winding* $w_{2,5}$ *solves neither* 1*-out-of-*5 *nor* 3*-out-of-*5*, and transposing two of its adjacent letters — which leaves the word reduced, of length* 30*, and with the same occurrence vector — destroys the solution. The two hypotheses of Theorem 7.3 are not vacuous: the factor* $[x_1,x_2]$ *fails the hypothesis on* P *at* $A = \{1,3\}$*, and the factor* $x_3x_4x_5x_3^{-1}x_4^{-1}x_5^{-1}$ *fails the hypothesis on* Q *at* $B = \{1,2,5\}$.

9. Verification

Every statement in this paper has been formally verified in Lean 4 against *Mathlib*; the development is free of `sorry`, and a repository reference is to be added at submission. The upper bounds carry no compiled evaluation of any kind: Proposition 4.2, Proposition 7.2 and Theorem 7.4, together with the definitional controls in Propositions 8.1 and 8.2, are decided by the proof kernel itself, evaluating Definition 2.2 on all 2^n removal sets — about two seconds for the 22-letter winding. So are the general statements: Propositions 3.1 and 3.2, Theorems 3.3, 3.4, 5.1 and 7.3, Proposition 4.1 and the completeness and symmetry lemmas depend on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`. What is delegated to compiled evaluation, through Lean's `native_decide`, is exactly the exhaustive enumerations, one compiled Boolean per search: four for $M(1,3) \geq 10$ and four more for $M(2,4) \geq 16$, both run with the symmetry reduction switched off entirely; one for the 667 979-node run behind Theorem 6.1, which therefore rests on nine compiled evaluations in total, its upper half on none; and six for the lower half of Theorem 7.1, a disjoint set, since the 2-out-of-5 lower bound goes through $M(1,4) = 16$ and not through $M(1,3)$ or $M(2,4)$. Every node count quoted was reproduced beforehand by an independent implementation in C that mirrors the enumeration exactly — same prune, same generator, same canonical form — and the windings of Proposition 4.2 were re-checked by a separately written program testing Definition 2.2 in its "remove S" form rather than the projection form.

References

[1] E. D. Demaine, M. L. Demaine, Y. N. Minsky, J. S. B. Mitchell, R. L. Rivest and M. Pătraşcu, *Picture-Hanging Puzzles*, Theory of Computing Systems **54**(4) (2014) 531–550; `doi:10.1007/s00224-013-9501-0`; arXiv:1203.3602.

[2] P. Gartside and S. Greenwood, *Brunnian links*, Fundamenta Mathematicae **193** (2007) 259–276. Theorem 7 gives $l(w) = 2^m(3k + 2^m)$ for a minimal Brunnian n-word with $n = 2^m + k$, $k < 2^m$.

[3] OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, `https://oeis.org`, entry A073121. The entry carries the sequence $1, 4, 10, 16, 28, 40, \ldots$ under a different description and does not cite [2].

[4] T. Verhoeff, *The k-out-of-n picture-hanging puzzle: shorter solutions for small k and $n - k$*, arXiv:2605.27617v1 (26 May 2026). All proposition, theorem and section numbers cited here are those of v1, which is the only version posted.

[5] T. Verhoeff, *Picture-Hanging Puzzles: Search and Verification Code*, source repository and run log (`runs/runs.md`), `gitlab.tue.nl/t-verhoeff-software/picture-hanging-puzzles`.

[6] J. Wästlund, *Faulty picture-hanging improved*, arXiv:2102.00984 (2021). This paper gives 1-out-of-n constructions — of lengths $16, 28, 40, 52, 64$ at $n = 4, \ldots, 8$, which are exactly the minima of [2] — and calls their minimality an open problem; it states no numeric minimum of its own. The length 66 for 2-out-of-5 sometimes attached to it is *Verhoeff's* evaluation of its recursion ([4, §5.2]), not a figure it records.