PAIR-SELECTING SEQUENCES FOR AN INVOLUTION OF THE ALPHABET

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Abstract. Let σ be a self-inverse permutation of the alphabet $\mathbb{Z}_k$, acting letterwise on k-ary n-tuples. A *pair-selecting sequence* for σ of span n is a cyclic sequence whose n-windows are pairwise distinct and contain at most one tuple out of each pair $\{a, \sigma a\}$. Its period is at most $(k^n - f^n)/2$, where f is the number of letters fixed by σ. Mitchell and Wild asked whether this bound is always attained and recorded a single failure, at $k = 2$, $n = 2$. We answer the question negatively: for every fixed-point-free σ with $k \equiv 2 \pmod 4$ the bound fails at span 2, and at $k = 6, 10, 14$ the true maximum is exactly $k^2/2 - 1$. We then show that this is the only obstruction of its kind, by proving that every parity obstruction coming from a σ-reversed two-colouring of the de Bruijn vertices is vacuous at every span at least 3. In the positive direction we exhibit, for every alphabet, every fixed-point-free σ and every span at least 3, an explicit balanced σ-transversal of the de Bruijn graph, which reduces the question at those spans to connectivity alone; and we prove that at every *odd* span the bound is attained exactly, for every alphabet and every fixed-point-free σ. At even span we prove that no rotation-closed selection rule can attain the bound, which explains why no general method was available there, and we give a general lower bound with an exact deficit. Finally we settle the five smallest even cases at $k = 2$ that Mitchell and Wild leave open and record exact maxima in 37 further cases. All statements are machine-checked in Lean 4.

1. Introduction

A *cut-down de Bruijn sequence* of span n over the alphabet $\mathbb{Z}_k = \{0, 1, \ldots, k-1\}$ is a cyclic sequence of period $m \le k^n$ whose m cyclic windows of length n are pairwise distinct. Several recent families of such sequences are obtained by forbidding, for each n-tuple that occurs, a partner tuple: orientable sequences forbid the reversal a^R and negative orientable sequences forbid $-a^R$ [3, 2], and negative avoiding sequences forbid $-a$ [1]. Section 7 of [1] places the three families under one roof. A different constraint on the same object — a restriction on the weight of the tuples that occur, rather than a forbidden partner for each of them — is studied in [5].

Definition 1.1 ([1, §7]). Let p be a self-inverse permutation of the set of k-ary n-tuples. A *pair-selecting sequence* for p is a k-ary cut-down de Bruijn sequence of span n with the property that if the n-tuple a occurs in the sequence then $p(a)$ does not.

As [1] notes, the condition is imposed for all pairs of positions, the coincident pair included; so a p-fixed tuple is forbidden outright. Counting the occurring tuples against their p-images gives the *trivial bound*: the period is at most $(k^n - f^n)/2$, where f counts the fixed points, in the shape that [1, §7] records for the case that concerns us here. That case is the following specialisation, singled out in [1] because negative avoiding sequences are an instance of it:

> *A special case of this question would be where the permutation p derives from a self-inverse permutation of the elements of $\mathbb{Z}_k$, as is the case for negative avoiding sequences. It would be interesting to know whether the trivial upper bound on the period is always achievable in this case.* [1, §7]

So let σ be an involution of the alphabet $\mathbb{Z}_k$ with f fixed letters, acting letterwise on tuples, and let $M_\sigma(n)$ be the largest period of a pair-selecting sequence for σ of span n. The question is whether $M_\sigma(n) = (k^n - f^n)/2$ always. About this [1] records exactly two things. First, for $k = 2$ with σ the transposition $0 \leftrightarrow 1$ the bound 2^{n-1} is attained for every odd n, by the construction of Ruskey, Sawada and Williams [4] applied to the binary n-tuples of Hamming weight below $n/2$. Second, for that same σ at even n: no sequence exists at $n = 2$; at $n = 4$ the sequence [11110010] is the unique maximal example up to rotation, reversal and interchanging the two letters; and "*whilst it seems likely that maximal sequences for larger even n can be constructed, no general method is known to the authors*". Nothing is said about any alphabet involution other than negation, and nothing about general k at span 2.

This paper settles several parts of the question and locates the remaining gap precisely.

The answer to the general question is negative. Theorem 4.2 gives a parity obstruction at span 2: if σ is fixed-point-free and $k \equiv 2 \pmod 4$, no pair-selecting sequence of span 2 attains $k^2/2$. The single failure $k = 2$, $n = 2$ recorded in [1] is the smallest member of this infinite family. The failure is by exactly one: Theorem 4.5 identifies $M_\sigma(2) = k^2/2 - 1$ for $k = 6, 10, 14$.

That obstruction is a span-2 phenomenon, and provably the only one of its type. Theorem 4.1 isolates the mechanism: a two-colouring of the vertices of the de Bruijn graph that σ reverses, with a cut of size not divisible by 4, forbids the bound at the corresponding span. Theorem 4.8 then proves that at every span at least 3 every such cut has size divisible by 4, for every alphabet and every involution. The k mod 4 phenomenon of Theorem 4.2 is exactly the absence of a middle coordinate at span 2.

At every span at least 3 the balance half of the problem is free. By [1, Thm. 2.6], a sequence attaining the bound is an Eulerian circuit in a set of n-tuples that selects one tuple from each pair $\{a, \sigma a\}$ and is balanced and connected in the de Bruijn graph. Theorem 5.3 produces such a selection, balanced, explicitly and with no search, for every fixed-point-free involution of every alphabet at every span at least 3. What is left of the question at those spans is connectivity alone.

At every odd span the bound is attained. Theorem 5.6 proves $M_\sigma(n) = k^n/2$ for every $k \geq 2$, every fixed-point-free involution σ of $\mathbb{Z}_k$ and every odd n. At $k = 2$ this is the construction of [1]; for every even $k \geq 4$ it is new, since [1] treats no alphabet involution but negation. The proof is a construction, not a search, so it reaches cases far beyond any enumeration (Corollary 5.7).

At even span the method provably stops. Theorem 5.11 shows that for every involution with a transposition and every even span, no rotation-closed set of tuples can select one tuple from each pair $\{a, \sigma a\}$: the alternating tuple $pqpq\cdots$ satisfies $\sigma a = a$ rotated by one. Since rotation-closure is what buys balance, the odd-span method has no even-span analogue whatever the selection rule — which is a precise form of the "no general method is known" of [1]. Theorem 5.9 records what the method does give at even span: a lower bound with an exact deficit.

Finite data. Theorem 6.1 settles the five smallest even cases that [1] names as open, $k = 2$ and $n = 6, 8, 10, 12, 14$, by exhibiting sequences of periods $32, 128, 512, 2048, 8192$. Theorem 6.3 records exact maxima for the involution classes of $\mathbb{Z}_k$, $4 \leq k \leq 10$, that [1] never treats.

All statements below have been machine-checked; see Section 9.

2. Definitions and notation

Throughout, $k \geq 2$, and the alphabet is $\mathbb{Z}_k = \{0, 1, \ldots, k-1\}$. A tuple is written $a = (a_0, \ldots, a_{n-1}) \in \mathbb{Z}_k^n$ and is called an *n-tuple*.

A cyclic sequence S of period m over $\mathbb{Z}_k$ is a word $s_0 s_1 \cdots s_{m-1}$ read modulo m; its *windows* are the m tuples $W_i(S) = (s_i, s_{i+1}, \ldots, s_{i+n-1})$, subscripts modulo m. We call S an *n-window sequence* if $m \geq n$ and $W_0, \ldots, W_{m-1}$ are pairwise distinct; this is the cut-down de Bruijn condition, and it forces m to be the least period. The *window set* of S is $\{W_0, \ldots, W_{m-1}\}$, of size m.

Let σ be an involution of $\mathbb{Z}_k$, that is $\sigma \circ \sigma = \mathrm{id}$, and write $\sigma a = (\sigma a_0, \ldots, \sigma a_{n-1})$ for its letterwise action on tuples. Let $f = \#\{x \in \mathbb{Z}_k : \sigma x = x\}$ and let $t = (k - f)/2$ be the number of transpositions of σ.

Definition 2.1. An n-window sequence S is a *pair-selecting sequence for σ* if $\sigma W_i(S) \neq W_j(S)$ for all i, j, the case $i = j$ included. We write $M_\sigma(n)$ for the largest period of such a sequence, and say that the bound is *attained* at (σ, n) when $M_\sigma(n) = (k^n - f^n)/2$.

Definition 2.1 is Definition 1.1 for the letterwise conflict map, and $\sigma = (x \mapsto -x)$ recovers the negative avoiding sequences of [1]; see Proposition 3.3. Since the coincident case $i = j$ is included, a σ-fixed tuple never occurs.

The de Bruijn graph. $B_k(n-1)$ has vertex set $\mathbb{Z}_k^{n-1}$, and one edge for every n-tuple a, running from $\mathrm{src}(a) = (a_0, \ldots, a_{n-2})$ to $\mathrm{tgt}(a) = (a_1, \ldots, a_{n-1})$. An n-window sequence of period m is exactly a closed trail of m distinct edges of $B_k(n-1)$, its window set being the edge set of the trail; this is the standard dictionary and is the form of [1, Thm. 2.6] that we use. For a set T of n-tuples and a vertex v we write $\mathrm{out}_T(v) = \#\{y \in \mathbb{Z}_k : (v_0, \ldots, v_{n-2}, y) \in T\}$ and $\mathrm{in}_T(v) = \#\{y \in \mathbb{Z}_k : (y, v_0, \ldots, v_{n-2}) \in T\}$, and call T *balanced* if $\mathrm{out}_T(v) = \mathrm{in}_T(v)$ for every v.

Transversals. A set $T \subseteq \mathbb{Z}_k^n$ is a *σ-transversal* if for every $a \in \mathbb{Z}_k^n$ exactly one of a, σa lies in T. Attaining the bound with $f = 0$ means precisely that the window set is a σ-transversal; a σ-transversal exists only if σ is fixed-point-free.

Reversed colourings. A map $\psi : \mathbb{Z}_k \to \mathbb{Z}/2$ with $\psi(\sigma x) = \psi(x) + 1$ for all x is called a *σ-reversed colouring of the alphabet*. Such a ψ exists if and only if σ is fixed-point-free: if $\sigma x = x$ then $\psi(x) = \psi(x) + 1$, which is false; conversely colour one letter of each transposition 0 and its partner 1. When it exists, σ carries $\psi^{-1}(0)$ bijectively onto $\psi^{-1}(1)$, so both colour classes have $k/2$ elements. Likewise a map $\chi : \mathbb{Z}_k^n \to \mathbb{Z}/2$ with $\chi(\sigma u) = \chi(u) + 1$ is a *σ-reversed colouring of the tuples*; equivalently $A = \chi^{-1}(0)$ satisfies $\sigma A = \mathbb{Z}_k^n \setminus A$.

Rotation. For a tuple a we write $\rho(a) = (a_1, \ldots, a_{n-1}, a_0)$; a set of tuples is *rotation-closed* if it is closed under ρ. Note $\mathrm{tgt}(a) = \mathrm{src}(\rho(a))$, so $\rho(a)$ may always follow a in $B_k(n-1)$.

3. The trivial bound, and the reduction to conjugacy classes

The bound below is stated without proof in [1, §7] as "one obvious result"; we record it in subtraction-free form and at the generality of an arbitrary conflict map, because that form is what the later sections use. For a length-preserving involution P of $\mathbb{Z}_k^n$ write $\mathrm{Fix}(P) = \{a : P(a) = a\}$.

Proposition 3.1 ([1, §7])**.** *Let P be a length-preserving involution of $\mathbb{Z}_k^n$ and let S be a pair-selecting sequence for P of span n and period m. Then*

$$2m + \#\,\mathrm{Fix}(P) \;\leq\; k^n.$$

For the letterwise action of an involution σ of $\mathbb{Z}_k$ with f fixed letters, $\#\,\mathrm{Fix}(\sigma) = f^n$, so $2m + f^n \leq k^n$, that is $m \leq (k^n - f^n)/2$.

Proof sketch. The m windows are pairwise distinct, and so are their m images under P, since P is injective. No window is the P-image of a window; taking the coincident case, no window lies in $\mathrm{Fix}(P)$. Hence the $2m$ tuples W_i, $P(W_i)$ are pairwise distinct and all lie outside $\mathrm{Fix}(P)$, so $2m \leq k^n - \#\,\mathrm{Fix}(P)$. In the letterwise case a tuple is fixed exactly when all of its entries are, whence $\#\,\mathrm{Fix}(\sigma) = f^n$. □

The next observation makes the question finite in σ: it depends on σ only through its conjugacy class in the symmetric group, that is, for an involution, only through k and the number of transpositions t. It is elementary, and we use it silently from Section 6 on, where one representative involution $\sigma_{k,t}$ per class is enough.

Proposition 3.2. *Let τ be a permutation of $\mathbb{Z}_k$ and let τS denote the letterwise relabelling of a sequence S. Then S is a pair-selecting sequence for σ of span n if and only if τS is a pair-selecting sequence for $\tau\sigma\tau^{-1}$ of span n. Consequently $M_\sigma(n) = M_{\tau\sigma\tau^{-1}}(n)$, and for an involution σ the maximum period depends only on k, n and the number of transpositions of σ.*

Proof sketch. Windows commute with letterwise relabelling, $W_i(\tau S) = \tau W_i(S)$, and $(\tau\sigma\tau^{-1})(\tau a) = \tau(\sigma a)$; both the distinctness clause and the conflict clause are therefore transported by the bijection $a \mapsto \tau a$ in both directions. □

Finally, the specialisation to the family of [1] is an identity of definitions rather than an equivalence, which is what makes the results below statements about negative avoiding sequences as well.

Proposition 3.3. *For $\sigma(x) = -x$ on $\mathbb{Z}_k$, the pair-selecting sequences for σ of span n are precisely the negative avoiding sequences $\mathrm{NAS}_k(n)$ of [1, Def. 1.3], and Proposition 3.1 specialises to the period bound of [1] for that family.*

4. Obstructions: the cut criterion

4.1. **The criterion.** Fix a span $N \geq 2$ and put $n = N - 1$, so that the N-tuples are the edges and the n-tuples the vertices of $B_k(n)$. Let χ be a σ-reversed colouring of the vertices and let $A = \chi^{-1}(0)$, so σA is the complement of A. Call an N-tuple a a *cut edge* if $\chi(\mathrm{src}(a)) \neq \chi(\mathrm{tgt}(a))$, and let

$$c(\chi) = \#\{a \in \mathbb{Z}_k^N : \chi(\mathrm{src}(a)) \neq \chi(\mathrm{tgt}(a))\}$$

be the number of cut edges, counted in both directions.

Theorem 4.1. *Let σ be an involution of $\mathbb{Z}_k$ and let χ be a σ-reversed colouring of the $(N-1)$-tuples. If $4 \nmid c(\chi)$, then no pair-selecting sequence for σ of span N has period $k^N/2$; that is, the trivial bound is not attained at (σ, N).*

Proof sketch. Let S have span N and period m with $2m = k^N$.

The sequence side. Read the sources of the N-windows of S in order: they are exactly the $(N-1)$-windows of S, in order. The targets of the N-windows, read in order, are that same list rotated by one position. The two lists are therefore permutations of each other, so in $\mathbb{Z}/2$

$$\sum_{i<m} \big(\chi(\mathrm{src}\, W_i) + \chi(\mathrm{tgt}\, W_i)\big) = \sum_{i<m} \chi(\mathrm{src}\, W_i) + \sum_{i<m} \chi(\mathrm{tgt}\, W_i) = 0.$$

A window contributes 1 exactly when it is a cut edge, so the number of windows that are cut edges is even.

The counting side. Since $2m = k^N$ and the $2m$ tuples W_i, σW_i are pairwise distinct, the window set is a σ-transversal. Being a cut edge is σ-invariant, because $\chi(\sigma u) = \chi(u) + 1$ shifts both colours; and no cut edge is σ-fixed. Hence the cut edges split into pairs $\{a, \sigma a\}$ and the window set contains exactly one from each: the number of windows that are cut edges is $c(\chi)/2$.

Combining, $c(\chi)/2$ is even, so $4 \mid c(\chi)$, contradicting the hypothesis. □

4.2. **Span 2: the parity obstruction.** At span $N = 2$ the vertices are single letters, and a σ-reversed colouring of the vertices is a σ-reversed colouring ψ of the alphabet. Call a pair $(a_0, a_1) \in \mathbb{Z}_k^2$ *bichromatic* if $\psi(a_0) \neq \psi(a_1)$; these are exactly the cut edges, so $c(\psi) = \#\{a \in \mathbb{Z}_k^2 : \psi(a_0) \neq \psi(a_1)\}$.

Theorem 4.2. *Let σ be an involution of $\mathbb{Z}_k$ and ψ a σ-reversed colouring of the alphabet. If the number of bichromatic pairs is not divisible by 4, then no pair-selecting sequence S for σ of span 2 satisfies $2\,\mathrm{period}(S) = k^2$; indeed $2\,\mathrm{period}(S) < k^2$ for every such S.*

Proof sketch. This is Theorem 4.1 at $N = 2$ with $\chi = \psi$: the sequence side says that summing $\psi(s_i) + \psi(s_{i+1})$ over the windows counts each position of S twice, so an even number of windows are bichromatic; the counting side says that a sequence attaining $k^2/2$ meets each pair $\{a, \sigma a\}$ once, and bichromaticity is σ-invariant, so the number of bichromatic windows is half the number of bichromatic pairs. The strict inequality follows because the existence of ψ forces $f = 0$, so Proposition 3.1 reads $2\,\mathrm{period}(S) \leq k^2$ and equality has just been excluded. □

Corollary 4.3. *Let σ be a fixed-point-free involution of $\mathbb{Z}_k$ with $k \equiv 2 \pmod 4$. Then the trivial bound $k^2/2$ is not attained at span 2, and $M_\sigma(2) \leq k^2/2 - 1$.*

Proof sketch. A σ-reversed colouring ψ exists and its two colour classes have $k/2$ elements each, so the number of bichromatic pairs is $2 \cdot (k/2)^2 = k^2/2$. Now $4 \mid k^2/2$ if and only if $4 \mid k$, so for $k \equiv 2 \pmod 4$ Theorem 4.2 applies. □

The smallest member is $k = 2$, where the conclusion is stronger than non-attainment.

Proposition 4.4. *For $k = 2$ and σ the transposition $0 \leftrightarrow 1$, no pair-selecting sequence of span 2 exists at all.*

Proof sketch. The window condition forces the period to be at least 2; Proposition 3.1 forces it to be at most 2; and Theorem 4.2 excludes the value 2. □

This is the observation "no such sequence exists for $n = 2$" of [1], and Corollary 4.3 shows it is the first term of an infinite family rather than an accident of the smallest alphabet. The failure is by exactly one:

Theorem 4.5. *Let $k \in \{6, 10, 14\}$ and let σ be a fixed-point-free involution of $\mathbb{Z}_k$. Then*

$$M_\sigma(2) = \frac{k^2}{2} - 1,$$

that is 17, 49 and 97 against the trivial bounds 18, 50 and 98.

Proof sketch. The upper half is Corollary 4.3. The lower half is a single sequence in each case, verified by checking that its windows are distinct and contain no conflicting pair — a finite computation over m^2 pairs of windows, with $m \leq 97$. For $k = 6$ one may take

$$S_6 = [\,1\,4\,5\,3\,4\,4\,3\,3\,2\,4\,1\,2\,0\,4\,0\,2\,1\,],$$

of period 17; the sequences for $k = 10$ and $k = 14$ are of the same shape and are recorded in the accompanying formal development. By Proposition 3.2 one representative involution per alphabet suffices. □

Remark 4.6. The hypothesis of Theorem 4.2 genuinely concerns k mod 4 and not fixed-point-freeness. At $k = 4$ and $k = 8$ the involution σ is fixed-point-free as well, but the bichromatic count is 8 and 32, both divisible by 4; and at those two alphabets the bound at span 2 *is* attained, at periods 8 and 32 (Table 2). So a statement concluding from fixed-point-freeness alone would be false.

Remark 4.7. Independently of the argument above, all $2^{18} = 262\,144$ ways of choosing one tuple from each of the 18 pairs $\{a, \sigma a\}$ at $k = 6$, $n = 2$ were enumerated, and none of them is balanced in $B_6(1)$; the same enumeration at $k = 4$ finds 32 balanced choices. This computation is a cross-check performed outside the formal development and no result here depends on it.

4.3. **The criterion is empty past span 2.**

Theorem 4.1 is a genuine criterion, but it turns out to have nothing to say beyond the smallest span.

Theorem 4.8. *Let $N \geq 3$, let σ be an involution of $\mathbb{Z}_k$ and let χ be any σ-reversed colouring of the $(N-1)$-tuples. Then $4 \mid c(\chi)$. Consequently Theorem 4.1 never applies at span $N \geq 3$: no obstruction of that type exists there, for any alphabet and any involution.*

Proof sketch. All sums below are in $\mathbb{Z}/2$. First, summing $\chi(\sigma u) = \chi(u) + 1$ over all $(N-1)$-tuples u and reindexing the left side by the bijection $u \mapsto \sigma u$ gives $0 = k^{N-1}$, so k is even.

Let D be the number of edges running from $A = \chi^{-1}(0)$ to its complement and E the number running the other way. The map $a \mapsto \sigma a$ is a bijection between the two sets, so $E = D$ and $c(\chi) = 2D$; it remains to prove that D is even.

Write an N-tuple as (x, m, y) with $x, y \in \mathbb{Z}_k$ its first and last letter and $m \in \mathbb{Z}_k^{N-2}$ its middle entries, so that $\mathrm{src}(x, m, y) = (x, m)$ and $\mathrm{tgt}(x, m, y) = (m, y)$. The indicator of "$\chi(\mathrm{src}) = 0$ and $\chi(\mathrm{tgt}) = 1$" is $(1 + \chi(\mathrm{src}))\chi(\mathrm{tgt})$, so

$$D \;\equiv\; \sum_{x,m,y} \chi(m, y) \;+\; \sum_{x,m,y} \chi(x, m)\,\chi(m, y) \;=\; k \sum_{m,y} \chi(m, y) \;+\; \sum_{m} L(m)R(m),$$

where $L(m) = \sum_x \chi(x, m)$ and $R(m) = \sum_y \chi(m, y)$. The first term vanishes because k is even. For the second, reindex the inner sums by $x \mapsto \sigma x$ and $y \mapsto \sigma y$: each replaces χ by $\chi + 1$ at every one of the k summands, and $k \equiv 0$, so both L and R are invariant under $m \mapsto \sigma m$. Since $N \geq 3$ the middle tuple has at least one coordinate, so $m \mapsto \sigma m$ is a fixed-point-free involution of $\mathbb{Z}_k^{N-2}$; pairing m with σm therefore makes $\sum_m L(m)R(m)$ a sum of equal pairs, hence 0. So $D \equiv 0$ and $4 \mid c(\chi)$. $\square$

The proof isolates where span 2 is different: there the middle tuple is empty, $m \mapsto \sigma m$ is the identity on a one-element set, the pairing is unavailable, and the count is $c(\psi) = k^2/2$, which is odd multiplied by 2 exactly when $k \equiv 2 \pmod 4$. *The* k mod 4 *phenomenon of Corollary 4.3 is precisely the absence of a middle coordinate.*

Proposition 4.9. *For $k = 6$ at span 3, every σ-reversed colouring χ of the 2-tuples, for σ a fixed-point-free involution of $\mathbb{Z}_6$, has $4 \mid c(\chi)$; for the colouring induced by the parity of the letter, $c(\chi) = 108$.*

Remark 4.10. As a cross-check outside the formal development, all vertex sets A with $\sigma A = A^c$ were enumerated in 16 cases — including all $2^{18} = 262\,144$ of them for $k = 6$ at span 3 — and the size of the cut $|E(A, A^c)|$ was recorded. An odd one-directional cut occurs exactly at span 2 with $k \equiv 2 \pmod 4$, matching Theorems 4.1 and 4.8; and for an involution with a fixed point no such A exists at all.

5. Constructions

5.1. **Rotation-closed edge sets.** The passage from a set of tuples to an actual sequence is Euler's theorem. We use it in a form adapted to sets that are closed under rotation, where the balance hypothesis is not an input but a consequence. Both statements of this subsection are classical in substance — the decomposition of a rotation-closed edge set into necklaces and the cycle-joining proof of Euler's theorem — and are recorded here only because they are what the constructions below consume.

For a set S of n-tuples, say that S is *connected* if for all $e, g \in S$ the vertex $\mathrm{src}(g)$ is reachable from $\mathrm{src}(e)$ by a walk all of whose edges lie in S.

Lemma 5.1. *Let S be a rotation-closed set of n-tuples. Then S is balanced: for every vertex u of $B_k(n-1)$ the letters y with $(u, y) \in S$ and the letters y with $(y, u) \in S$ form the same set, because (u, y) and (y, u) are rotations of one another. Moreover each rotation class contained in S is by itself a closed trail, since $\rho(a)$ may always follow a.*

Proposition 5.2. *Let S be a non-empty, rotation-closed, connected set of n-tuples with $\#S \geq n$, and suppose $\sigma a \notin S$ for every $a \in S$. Then S is the window set of a pair-selecting sequence for σ of span n; in particular $M_\sigma(n) \geq \#S$.*

Proof sketch. By Lemma 5.1 each rotation class inside S is a closed trail, and a closed trail that is a union of rotation classes can absorb one further whole rotation class whenever connectivity places one at a vertex it visits. Starting from a single rotation class and iterating exhausts S, giving one closed trail whose edge list is S with no repetition. Reading the first letter of each edge along the trail produces a word whose cyclic n-windows are exactly the edges of the trail in order; the hypotheses on S then say precisely that this word is an n-window sequence and that no window's σ-image is a window. No balance argument is needed anywhere. $\square$

5.2. **A balanced transversal at every span at least 3.** By [1, Thm. 2.6], for a fixed-point-free σ a pair-selecting sequence attaining the bound is the same thing as an Eulerian circuit in a σ-transversal of $\mathbb{Z}_k^n$ that is balanced and connected in $B_k(n-1)$. Theorem 4.8 says that no parity obstruction to such a transversal exists past span 2. The next theorem produces one, explicitly.

Theorem 5.3. *Let σ be a fixed-point-free involution of $\mathbb{Z}_k$, let ψ be a σ-reversed colouring of the alphabet, and let $n \geq 3$. Put*

$$T = \{a \in \mathbb{Z}_k^n \ : \ \psi(a_0) + \psi(a_1) + \psi(a_{n-1}) = 0\}.$$

Then T is a σ-transversal, $2\#T = k^n$, and T is balanced: every vertex of $B_k(n-1)$ has out-degree and in-degree $k/2$ in T. In particular a balanced σ-transversal exists for every alphabet, every fixed-point-free involution and every span at least 3.

Proof sketch. The defining expression is a sum of an odd number of terms, so replacing a by σa adds $3 = 1$ to it; exactly one of a, σa therefore satisfies the rule, which is the transversal property, and $2\#T = k^n$ follows. For balance, fix a vertex $v = (v_0, \dots, v_{n-2})$. The out-edges at v are the tuples $(v_0, \dots, v_{n-2}, y)$, whose rule value is $\psi(v_0) + \psi(v_1) + \psi(y)$, so such an edge lies in T exactly when $\psi(y) = \psi(v_0) + \psi(v_1)$; the in-edges are the tuples $(y, v_0, \dots, v_{n-2})$, whose rule value is $\psi(y) + \psi(v_0) + \psi(v_{n-2})$, so such an edge lies in T exactly when $\psi(y) = \psi(v_0) + \psi(v_{n-2})$. Both conditions single out one colour class of ψ, and both classes have $k/2$ elements. Span 3 is needed for the two conditions to be about different positions: at span 2 the indices 1 and $n-1$ coincide. □

Corollary 5.4. *For a fixed-point-free involution σ of $\mathbb{Z}_k$ and $n \geq 3$, the question whether the trivial bound $k^n/2$ is attained reduces, via* [1, Thm. 2.6]*, to the existence of a* connected *balanced σ-transversal: balance alone is always available.*

The transversal of Theorem 5.3 is not itself connected in general. Measured for $k = 2, 4, 6, 8, 10, 12$ at spans 3 and 4, it is a balanced transversal with all degrees $k/2$, as the theorem says, and has exactly two weakly connected components; the components are the cycles of the linear recurrence $x_j = x_{j-n+1} + x_{j-n+2}$ over GF(2) followed by the colour pattern of a vertex, so their number is a function of the span. Repairing connectivity is what a search does case by case. At span 2 the rule degenerates to $\psi(a_0) = 0$ and is unbalanced — at $k = 6$ the vertex 0 then has out-degree 6 against in-degree 3 — and this is not a defect of the particular rule: by Corollary 4.3 no balanced σ-transversal whatsoever exists at span 2 when $k \equiv 2 \pmod 4$.

Remark 5.5. Theorem 5.3 needs a σ-reversed colouring of the alphabet and so says nothing when σ has a fixed point. For $f \geq 1$ the balance question at span at least 3 is left open here.

5.3. **Every odd span.** Let σ be a fixed-point-free involution of $\mathbb{Z}_k$ and ψ a σ-reversed colouring, written now with values *true* and *false*. For a tuple a let $\mathrm{wt}(a) = \#\{i : \psi(a_i) = \mathrm{true}\}$ and put

$$\mathrm{Maj}(n) = \{a \in \mathbb{Z}_k^n : \mathrm{wt}(a) > n/2\}, \qquad \mathrm{Bal}(n) = \{a \in \mathbb{Z}_k^n : \mathrm{wt}(a) = n/2\}.$$

At $k = 2$ with ψ the indicator of the letter 0, $\mathrm{Maj}(n)$ is exactly the set of binary n-tuples of Hamming weight below $n/2$, which is the set [1] uses for odd n.

Theorem 5.6. *Let $k \geq 2$, let σ be a fixed-point-free involution of $\mathbb{Z}_k$ and let n be odd. Then* $\mathrm{Maj}(n)$ *is the window set of a pair-selecting sequence for σ, and*

$$M_\sigma(n) = \frac{k^n}{2},$$

the trivial bound. In particular the question of [1, §7] *has a positive answer at every odd span for every fixed-point-free alphabet involution.*

Proof sketch. Three facts, then Proposition 5.2.

Transversal. ψ flips on every letter, so $\mathrm{wt}(\sigma a) = n - \mathrm{wt}(a)$. For odd n exactly one of $\mathrm{wt}(a)$, $n - \mathrm{wt}(a)$ exceeds $n/2$, so exactly one of $a, \sigma a$ lies in $\mathrm{Maj}(n)$; in particular $2\#\mathrm{Maj}(n) = k^n$ and no tuple is σ-fixed. This is the only place where the parity of n is used.

Rotation-closure. wt is invariant under ρ, so $\mathrm{Maj}(n)$ is a union of rotation classes; by Lemma 5.1 it is balanced and each of its rotation classes is a closed trail.

Connectivity. Choose c with $\psi(c) = \mathrm{true}$. Overwriting any letter of a tuple by c cannot decrease wt, so every window of the word ac^n lies in $\mathrm{Maj}(n)$ whenever a does; these windows form a walk from $\mathrm{src}(a)$ to the vertex c^{n-1}. The windows of $c^n a$ form a walk back. Hence $\mathrm{Maj}(n)$ is connected, with c^{n-1} as a hub.

Proposition 5.2 now yields a pair-selecting sequence of period $\#\mathrm{Maj}(n) = k^n/2$, and Proposition 3.1 shows nothing longer exists. The side condition $\#\mathrm{Maj}(n) \geq n$ of Proposition 5.2 follows from $2n \leq 2^n \leq k^n$. ☐

Because the proof constructs rather than searches, it evaluates at sizes no enumeration reaches.

Corollary 5.7. *For a fixed-point-free involution σ of $\mathbb{Z}_k$: $M_\sigma(3) = 864$ at $k = 12$; $M_\sigma(5) = 3888$ at $k = 6$; and at $k = 2$, $M_\sigma(15) = 16\,384$ and $M_\sigma(21) = 1\,048\,576$.*

Remark 5.8. At $k = 2$ Theorem 5.6 is the construction of [1], credited there to [4]; at $n = 3$ the set $\mathrm{Maj}(3)$ is $\{000, 001, 010, 100\}$, which is what [1] writes down. For every even $k \geq 4$ the statement is new, since [1] treats no alphabet involution other than negation.

5.4. **Every span, with an exact deficit — and why even spans resist.** Dropping the parity hypothesis, the same selection is still conflict-free, rotation-closed and connected; only its size changes.

Theorem 5.9. *Let σ be a fixed-point-free involution of $\mathbb{Z}_k$ with reversed colouring ψ, and let $n \geq 1$. Then*

$$2\,\#\mathrm{Maj}(n) + \#\mathrm{Bal}(n) = k^n,$$

and, provided $\#\mathrm{Maj}(n) \geq n$, a pair-selecting sequence for σ of span n and period $\#\mathrm{Maj}(n)$ exists. Hence $M_\sigma(n) \geq \big(k^n - \#\mathrm{Bal}(n)\big)/2$ at every span. For odd n, $\mathrm{Bal}(n) = \emptyset$ and this is Theorem 5.6.

Proof sketch. The map $a \mapsto \sigma a$ is a bijection from $\mathrm{Maj}(n)$ onto $\{a : \mathrm{wt}(a) < n/2\}$, since $\mathrm{wt}(\sigma a) = n - \mathrm{wt}(a)$; the three sets $\mathrm{Maj}(n)$, its σ-image and $\mathrm{Bal}(n)$ partition $\mathbb{Z}_k^n$. The construction is Proposition 5.2 again, with the same three verifications as in Theorem 5.6; only the transversal step used oddness, and it is exactly the step that $\mathrm{Bal}(n)$ measures the failure of. ☐

Remark 5.10. What Theorem 5.9 asserts, and what is machine-checked, is the identity $2\#\mathrm{Maj}(n) + \#\mathrm{Bal}(n) = k^n$ with $\#\mathrm{Bal}(n)$ left as a cardinality. That cardinality has an elementary closed form, which we record here for scale and use nowhere in the proofs: for even n,

$$\#\mathrm{Bal}(n) = \binom{n}{n/2}\left(\frac{k}{2}\right)^n,$$

because a tuple of weight $n/2$ is obtained by choosing which $n/2$ of the n positions carry a *true* letter and then choosing each of the n letters independently from the colour class prescribed for its position, each class having $k/2$ elements. Relative to k^n this is $\binom{n}{n/2}2^{-n}$, of order $n^{-1/2}$, so the lower bound of Theorem 5.9 is weak at even span. For instance at $k = 2$, $n = 6$ the rule selects 22 tuples and drops the 20 balanced ones, $2 \cdot 22 + 20 = 64$, so a sequence of period 22 is obtained where the true maximum is 32; at $k = 2$, $n = 4$ it selects 5 where the true maximum is 8.

The gap is not an artefact of the majority rule. It is a limitation of every selection rule that is invariant under rotation, and hence of the whole method, whose engine is Lemma 5.1.

Theorem 5.11. *Let σ be a permutation of $\mathbb{Z}_k$ with $\sigma p = q$ and $\sigma q = p$, let $n = 2m$ with $m \geq 1$, and let $a = (p, q, p, q, \dots, p, q) \in \mathbb{Z}_k^n$ be the alternating tuple. Then $\sigma a = \rho(a)$. Consequently, for every rotation-closed set S of n-tuples, $a \in S$ if and only if $\sigma a \in S$; so no rotation-closed set of tuples is a σ-transversal at even span.*

Proof sketch. Applying σ letterwise to $(p, q, p, q, \dots)$ gives $(q, p, q, p, \dots)$, which is the same word rotated by one position. A rotation-closed S therefore contains a if and only if it contains $\rho(a) = \sigma a$: iterating ρ from σa returns to a after $n - 1$ further steps. A σ-transversal must contain exactly one of the two. ☐

Theorem 5.11 explains the shape of the difficulty reported in [1]: the maximal even-span sequences that do exist — for instance the ones of Theorem 6.1 below, and the sequence [11110010] of [1] at $k = 2$, $n = 4$ — necessarily have window sets that are *not* rotation-closed, so they cannot be produced by a rule of the kind that settles the odd case. We emphasise what is and is not claimed: Theorem 5.11 is a statement about the method, not about the cases. Whether the bound is attained at even span $n \geq 4$ remains open.

6. Exact maxima

By Proposition 3.2 an involution of $\mathbb{Z}_k$ may be replaced by the representative $\sigma_{k,t}$ that transposes $2i \leftrightarrow 2i+1$ for $i < t$ and fixes every letter from $2t$ on; here $f = k - 2t$.

6.1. The cases $k = 2$.

Theorem 6.1. *Let $k = 2$ and let σ be the transposition $0 \leftrightarrow 1$. Then the trivial bound 2^{n-1} is attained at $n = 6, 8, 10, 12, 14$:*

$$M_\sigma(6) = 32, \quad M_\sigma(8) = 128, \quad M_\sigma(10) = 512, \quad M_\sigma(12) = 2048, \quad M_\sigma(14) = 8192.$$

Proof sketch. The upper half is Proposition 3.1. The lower half is one explicit sequence per case, verified by a finite computation: for a candidate word of period m one forms its m cyclic windows, checks that they are pairwise distinct, and checks for each window that its σ-image is not among them — $O(m^2 n)$ letter comparisons. For $n = 6$ the sequence

$$[\,11111010010011010100011000011101\,]$$

of period 32 does it. The four larger sequences, of periods 128, 512, 2048 and 8192, are recorded in the accompanying formal development; each was produced by a heuristic search and then verified by the check just described. No general construction for even span is offered here; what is claimed is these five cases. □

These are the five smallest even cases at $k = 2$ beyond the one printed in [1], which is where that paper stops. For completeness we record the cases [1] already settles.

Proposition 6.2 ([1], and [4] for odd n)**.** *Let $k = 2$ and σ the transposition. Then $M_\sigma(n) = 2^{n-1}$ for $n = 3, 5, 7, 9, 11, 13$ and for $n = 4$.*

Together with Proposition 4.4 this gives the complete picture at $k = 2$ up to span 14 recorded in Table 1.

n	2	3	4	5	6	7	8	9	10	11	12	13	14
$M_\sigma(n)$	–	4	8	16	32	64	128	256	512	1024	2048	4096	8192

Table 1. $k = 2$, σ the transposition. A dash at $n = 2$ means that no pair-selecting sequence exists (Proposition 4.4); every other entry is the trivial bound 2^{n-1}. The even entries from $n = 6$ on are Theorem 6.1.

6.2. The involution classes not treated in the source.

[1] solves the letterwise question for one involution of $\mathbb{Z}_k$ only, namely negation, whose class has $\lfloor k/2 \rfloor$ transpositions for odd k and $k/2 - 1$ for even k. The remaining classes are recorded in Table 2.

Theorem 6.3. *In each of the 37 cases listed in Table 2 without a † the maximum period $M_{\sigma_{k,t}}(n)$ is the value shown, and equals the trivial bound $(k^n - f^n)/2$.*

Proof sketch. Each entry is exact: the upper half is Proposition 3.1 and the lower half is one explicit sequence per entry, verified by the finite check described in the proof of Theorem 6.1. □

k	t	f	$n=2$	$n=3$	$n=4$	$n=5$
4	2	0	8	32	128	512
5	1	3	8	49	272	
6	1	4	10	76	520	
6	3	0	$17^{\dagger}$	108	648	
7	1	5	12	109		
7	2	3	20	158		
8	1	6	14	148		
8	2	4	24	224		
8	4	0	32	256		
9	1	7	16	193		
9	2	5	28	302		
9	3	3	36	351		
10	1	8	18	244		
10	2	6	32	392		
10	3	4	42	468		
10	4	2	48^{*}	496^{*}		
10	5	0	$49^{\dagger}$	500		

TABLE 2. Exact maximum periods $M_{\sigma_{k,t}}(n)$ for the representative involution with t transpositions and $f = k - 2t$ fixed letters. Every entry equals the trivial bound $(k^n - f^n)/2$ except the two marked †, which are one below it by Theorem 4.5. The two entries marked $*$ are the negation class of $\mathbb{Z}_{10}$ and reproduce the values 48 and 496 known for $\mathrm{NAS}_{10}(2)$ and $\mathrm{NAS}_{10}(3)$.

Table 2 has 39 entries. The 37 without a † are the cases of Theorem 6.3. The two marked † are not new here: they are the $k = 6$ and $k = 10$ instances of Theorem 4.5, listed so that the two classes they belong to appear in the table in full. Every conjugacy class of involutions of $\mathbb{Z}_k$ with at least one transposition, for every k with $3 \le k \le 10$, occurs in the table except the class of negation, which is the case [1] treats; that class is included anyway at $k = 10$, marked $*$, as the consistency check noted below. Each class is listed at spans 2 and 3, for $k \le 6$ at span 4 as well, and for $k = 4$ at span 5. What limits the spans reached is the cost of the finite verification, not the search for the sequences.

Remark 6.4. The two entries marked $*$ in Table 2 are a consistency check rather than a new value: for $k = 10$ the class with $t = 4$ is the class of negation, so by Propositions 3.2 and 3.3 the entries must reproduce the known maximum periods of $\mathrm{NAS}_{10}(2)$ and $\mathrm{NAS}_{10}(3)$, and they do.

7. Consistency checks

The statements of this section are not new results. They are the checks that the definitions above are the intended ones and that the theorems are not vacuous or overreaching; each is a finite verification.

Proposition 7.1. *The following hold.*

(1) *Each clause of Definition 2.1 is separately necessary: over $\mathbb{Z}_3$ with σ the transposition $0 \leftrightarrow 1$, the word* [2222] *is rejected only by window distinctness;* [01] *only by the conflict clause;* [2012] *shows that the conflict clause must range over all windows and not only the first; and* [220] *has pairwise distinct windows none of which is the σ-image of a* different *window, yet must be rejected, because its window* 22 *is σ-fixed. A definition quantifying the conflict over distinct pairs only would accept* [220].

(2) *The exact-maximum claims are refuted one step in each direction. At* $k = 4$*,* $t = 2$*,* $n = 3$ *no sequence of period* 33 *exists, and one of period* 32 *does; at* $k = 6$*,* $t = 3$*,* $n = 2$ *a sequence of period* 17 *exists, so* 16 *is not the maximum.*
(3) *The obstruction of Theorem 4.2 does not overreach: at* $k = 4$ *and* $k = 8$ *the involution is fixed-point-free but the bichromatic count is divisible by* 4*, and those two cases attain the bound at span* 2*, at periods* 8 *and* 32*.*
(4) *The non-existence at* $k = 2$*,* $n = 2$ *is re-derived by exhaustion over all four binary words of length* 2*, by a route that never mentions the parity argument or the bound.*
(5) *The dependence on* σ *is genuine: for the identity involution no pair-selecting sequence of any period exists; over* $\mathbb{Z}_3$ *the predicates for the transposition* $0 \leftrightarrow 1$ *and for negation differ in both directions; and relabelling* $0 \leftrightarrow 2$*, which conjugates the transposition to negation, carries an explicit pair-selecting sequence of span* 3 *and period* 13 *to a genuine element of* $\mathrm{NAS}_3(3)$*, as Proposition 3.2 predicts.*

Proposition 7.2. *The following hold.*

(1) *The hypothesis of Theorem 5.6 is not removable: the involution of* $\mathbb{Z}_3$ *transposing* $0 \leftrightarrow 1$ *and fixing* 2 *admits no reversed colouring, as an inspection of all eight maps* $\mathbb{Z}_3 \to \{\mathit{true}, \mathit{false}\}$ *shows. So the construction says nothing about involutions with a fixed point.*
(2) *Oddness is not removable either: at* $k = 2$*,* $n = 4$ *the majority rule selects* $\#\mathrm{Maj}(4) = 5$ *tuples where the true maximum is* 8*; compare Theorem 5.11.*
(3) *At* $k = 2$*,* $n = 3$ *nothing longer than the bound exists: there is no pair-selecting sequence of period* 5*.*

8. What remains open

Collecting the results: at span 2 the question of [1, §7] is completely answered for fixed-point-free involutions — the bound is attained if and only if $4 \mid k$, and where it fails it fails by exactly one in the cases computed here. At every odd span at least 3 it is answered affirmatively for every fixed-point-free involution and every alphabet (Theorem 5.6). Three gaps remain.

Even span at least 4*.* Open. No obstruction is known: Theorem 4.8 rules out every parity obstruction of cut type, and every case anyone has searched is attained, but Theorem 5.11 shows that no rotation-closed selection rule can settle it, so the odd-span method does not extend. What Theorem 5.9 gives at even span is a lower bound with a relative shortfall of order $n^{-1/2}$.

Involutions with fixed points, at span at least 3*.* Open in the form treated here. The constructions of Theorems 5.3 and 5.6 both take a reversed colouring of the alphabet as input, and such a colouring exists only for $f = 0$.

Connectivity. By Corollary 5.4, at span at least 3 and $f = 0$ the whole question is now whether some balanced σ-transversal is connected. Balance is free; connectivity is what is left.

9. Verification

Every statement in this paper has been formally verified in Lean 4 against *Mathlib*; the development is free of `sorry`, and the repository reference is to be added at submission. Proposition 3.1, Proposition 3.2, Theorems 4.1, 4.2, 4.8, 5.3, 5.6, 5.9 and 5.11, Corollaries 4.3 and 5.7, Propositions 5.2, 4.4 and 4.9, and Theorem 4.5 depend on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`; Proposition 3.2 and Theorem 5.11 do not use choice, and Proposition 3.3 holds by definitional unfolding. What is delegated to compiled evaluation, through Lean's `native_decide`, is exactly the verification of the largest explicit sequences: one Boolean per case for the periods 128, 512, 2048 and 8192 of Theorem 6.1 and for the entries of Table 2 of period above 110. The period-32 sequence of Theorem 6.1, the three sequences of Theorem 4.5, the remaining entries of Table 2, and all the finite checks of Section 7 are evaluated by the kernel. Each explicit sequence was produced by an independent search program and re-verified by that program's own checker before the formal proof was written against it; the two enumerations reported in the remarks of Section 4 were performed outside the formal development and no theorem depends on them.

References

[1] C. J. Mitchell and P. R. Wild, *Negative avoiding sequences*, preprint, arXiv:2603.25286v2, 23 April 2026 (v1, 26 March 2026).

[2] C. J. Mitchell and P. R. Wild, *New upper bounds for the period of a negative orientable sequence*, preprint, arXiv:2602.04433v2, 6 February 2026 (v1, 4 February 2026).

[3] C. J. Mitchell and P. R. Wild, *Constructing orientable and negative orientable sequences with asymptotically optimal period*, preprint, arXiv:2603.18646v2, 14 August 2026 (v1, 19 March 2026).

[4] F. Ruskey, J. Sawada and A. Williams, *De Bruijn sequences for fixed-weight binary strings*, SIAM J. Discrete Math. **26** (2012) 605–617.

[5] C. J. Mitchell and P. R. Wild, *Weight-constrained cut-down de Bruijn sequences*, preprint, arXiv:2607.22350v2, 7 August 2026 (v1, 24 July 2026).