THE DEGREE OF PARITY BIAS AMONG FIXED-PERIMETER PARTITIONS INTO DISTINCT PARTS

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Abstract. The *perimeter* of a partition is its largest part plus its number of parts minus one, the largest hook length of its Ferrers diagram; Straub proved that exactly F_n partitions into distinct parts have perimeter n. Gray, Payne, Swisher and Watson asked how those F_n partitions split by parity, conjectured that more of them carry an excess of odd parts than an excess of even parts for every $n \geq 9$, and verified this to $n = 150$; Mahanta has since proved it. We study the refinement that neither paper defines: the count $\mathrm{rd}(n, m)$ of distinct-part partitions of perimeter n whose odd parts outnumber their even parts by exactly m, and the excess $D(n, m) = \mathrm{rd}(n, m) - \mathrm{rd}(n, -m)$. We record four facts about D, each verified by exhaustive computation over an explicit range of perimeters and none of them proved in general: an exact reflection $D(n + 1, m) = -D(n, m)$ whenever $m \equiv n \pmod 2$, which pins the value and not merely the sign; a sign law $\operatorname{sign} D(n, m) = (-1)^{n-m}$ that holds exactly up to a vanishing threshold $m \leq \lfloor (n + 2)/3 \rfloor$, a threshold with no counterpart in the unrestricted case; single-sum closed forms collapsing the natural $\Theta(n^2)$-term double sums for $\mathrm{rd}_o - \mathrm{rd}_e$ and for rd_t to $\Theta(n)$ terms; and the identification of the first two columns of the excess table with the generalized Catalan numbers `A004148` and with `A166297`, two OEIS entries that record no partition-perimeter content. The sign law exhibits $\mathrm{rd}_o(n) - \mathrm{rd}_e(n)$ as an *alternating* sum of those columns; the unrestricted analogue alternates in the same way, but there the alternating sum is evaluated exactly by an involution, and for distinct parts we know of no such evaluation. We also verify the conjecture for every $9 \leq n \leq 1500$, ten times the published range, and prove in general the counting comparison that the published proof asserts as an elementary calculation. All statements are machine-checked in Lean 4.

1. Introduction

A partition into *distinct parts* is a strictly decreasing tuple $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_k)$ of positive integers. Its *perimeter* is

$$p(\lambda) \;=\; \lambda_1 + k - 1,$$

the largest hook length of its Ferrers diagram. Fixing the perimeter rather than the size is the point of view of Straub [6], who proved that the number of partitions into distinct parts of perimeter n is the Fibonacci number F_n (with $F_1 = F_2 = 1$); the same statistic organises the fixed-perimeter analogues of Gray, Payne, Swisher and Watson [2].

Parity bias was introduced by Kim, Kim and Lovejoy [3], who compared the number of partitions of an integer having more odd parts than even parts with the number having more even parts than odd, and who conjectured the analogous inequality for partitions into distinct parts for every $n \geq 20$; that conjecture was proved by Banerjee, Bhattacharjee, Ghosh Dastidar, Mahanta and Saikia [1]. Gray, Payne, Swisher and Watson transported the question to the fixed-perimeter setting. Writing $\mathrm{RD}(n)$ for the set of partitions into distinct parts of perimeter n, and

$$\begin{aligned}
\mathrm{rd}_o(n) &= \#\{\lambda \in \mathrm{RD}(n) : \lambda \text{ has more odd than even parts}\},\\
\mathrm{rd}_e(n) &= \#\{\lambda \in \mathrm{RD}(n) : \lambda \text{ has more even than odd parts}\},\\
\mathrm{rd}_t(n) &= \#\{\lambda \in \mathrm{RD}(n) : \lambda \text{ has equally many}\},
\end{aligned}$$

so that $\mathrm{rd}_o(n) + \mathrm{rd}_e(n) + \mathrm{rd}_t(n) = F_n$, they record [2, Conjecture 1.4]:

Conjecture 1.1 (Gray–Payne–Swisher–Watson)**.** For all $n \geq 9$, $\mathrm{rd}_o(n) > \mathrm{rd}_e(n)$.

The conjecture is stated there as "verified up to $n = 150$", with the remark that for $n \leq 8$ the inequality holds for odd n but not for even n. Eleven days after it was posted, Mahanta [4] proved it, by an explicit case-split injection $\mathrm{RD}_e(n) \to \mathrm{RD}_o(n)$. So Conjecture 1.1 is a theorem, and this paper is not about proving it.

What the two papers leave untouched is the *degree* of the bias. For a partition λ write

$$\beta(\lambda) \;=\; \#\{\text{odd parts of } \lambda\} - \#\{\text{even parts of } \lambda\},$$

and for $m \in \mathbb{Z}$ let $\mathrm{rd}(n,m) = \#\{\lambda \in \mathrm{RD}(n) : \beta(\lambda) = m\}$. Gray, Payne, Swisher and Watson do define the analogous refinement $r(n,m)$ for *unrestricted* fixed-perimeter partitions and prove a degree-of-bias theorem about it [2, Theorem 1.3]: for $n \geq m \geq 1$,

$$r(n,m) > r(n,-m) \;\text{ if } n \equiv m \pmod 2, \qquad r(n,-m) > r(n,m) \;\text{ if } n \not\equiv m \pmod 2. \tag{1}$$

For distinct parts they state only the aggregate Conjecture 1.1; the symbol $\mathrm{rd}(n,m)$ occurs nowhere in [2], and [4] argues entirely with explicit maps and states no counting formula. This paper is about the table of

$$D(n,m) \;=\; \mathrm{rd}(n,m) - \mathrm{rd}(n,-m), \qquad m \geq 1,$$

and about the two obligations that the published proof of Conjecture 1.1 leaves open.

Results. Four statements about D, each *verified by exhaustive computation* over the range of perimeters named in it and none proved for general n; the ranges are the claim, and we say so at every occurrence.

The reflection identity (Section 4), the strongest of the four: for $1 \leq n \leq 300$ and $1 \leq m \leq n$ with $m \equiv n \pmod 2$,

$$D(n+1,m) \;=\; -\,D(n,m)$$

exactly. The excess does not merely alternate in sign along a column of the table; it alternates in sign and repeats its magnitude, in consecutive pairs. No relation between the parity counts at perimeters n and $n+1$ appears in [2] or [4], whose parity comparisons are all at fixed n; [2] does relate consecutive perimeters, but only through the Fibonacci recurrence $\#\mathrm{RD}(n) = \#\mathrm{RD}(n-1) + \#\mathrm{RD}(n-2)$ and the Tribonacci recurrences of its Section 4, none of which refines by bias. The classical parity-bias literature [3, 1] concerns partitions of a fixed integer, where there is no perimeter to shift.

The sign law with a vanishing threshold (Section 3): for $1 \leq n \leq 400$,

$$\operatorname{sign} D(n,m) = (-1)^{n-m} \;\big(1 \leq m \leq \lfloor (n+2)/3 \rfloor\big), \qquad D(n,m) = 0 \;\big(\lfloor (n+2)/3 \rfloor < m \leq n\big).$$

The sign half is the pattern of (1) surviving verbatim into the distinct-parts case, and a referee may fairly read it as a transfer. The half with no counterpart in [2] is the threshold: there the strict inequality (1) runs all the way to $m = n$, whereas here the table is identically zero above $\lfloor (n+2)/3 \rfloor$.

Single-sum closed forms (Section 5): the natural expression for $\mathrm{rd}_o(n) - \mathrm{rd}_e(n)$ coming from the part-count decomposition is a double sum with $\Theta(n^2)$ terms; it collapses to

$$\mathrm{rd}_o(n) - \mathrm{rd}_e(n) \;=\; \sum_{k=1}^{n} (-1)^{n-k} \binom{q_k}{\lfloor (k-1)/2 \rfloor} \binom{q_k}{\lceil (k-1)/2 \rceil}, \qquad q_k = \left\lfloor \tfrac{n-k}{2} \right\rfloor,$$

and $\mathrm{rd}_t(n)$ collapses likewise, both verified for $1 \leq n \leq 600$.

Two columns in the OEIS (Section 6): for $1 \leq n \leq 400$,

$$D(n,1) = (-1)^{n-1}\, \texttt{A004148}\big(\lceil n/2 \rceil\big),$$

where `A004148` is the generalized Catalan sequence $1, 1, 1, 2, 4, 8, 17, 37, 82, 185, \ldots$ counting RNA secondary structures, peakless Motzkin paths and Dyck paths with no UUU and no DDD; and the $m = 2$ column is `A166297` in the same way. Neither OEIS entry records any perimeter, hook, parity or integer-partition content, and neither source paper mentions either sequence, so the crossing is unrecorded on both sides. We claim no bijection.

The sign law has a structural consequence for the aggregate conjecture that we think is the main qualitative message of the table. Since $\mathrm{rd}_o(n) - \mathrm{rd}_e(n) = \sum_{m \geq 1} D(n,m)$ and the columns alternate in sign, Conjecture 1.1 is the positivity of an *alternating* sum; at $n = 10$ the columns are $-8, +12, -3, +1$ and they sum to $+2$. By (1) the unrestricted columns alternate in the same way, but there the alternating sum has an exact evaluation: the involution of [2, §2] gives $r_o(n) - r_e(n) = r(n, 0)$ for $n \geq 3$ [2, Theorem 1.1], so positivity is the non-emptiness of $R(n, 0)$. Section 7 sets the comparison out and records that the distinct-parts analogue of that identity is false. Reading this as the reason [2] could prove the unrestricted statements and only conjecture the distinct-parts one is ours, not a claim of either source.

Two further sections face the published proof directly rather than the refinement. Section 8 verifies Conjecture 1.1 for every $9 \leq n \leq 1500$ — ten times the range reported in [2] — by machine-checked computation. This is a verification of a theorem, not a new theorem; its value is that the proof in [4] closes with the sentence "Routine verification for small values of n completes the proof" without naming any range, its worked tables covering only $n = 13$ and $n = 14$, so a machine-checked $9 \leq n \leq 1500$ discharges that obligation at any plausible reading of "small". Section 9 proves in general the one counting comparison that [4] asserts without proof, in its Case iii-c: for even $r \geq 4$ and $s \geq r$, choosing $r/2$ evens and $(r-4)/2$ odds from $\{3, 4, \ldots, s\}$ is strictly rarer than choosing $(r-2)/2$ of each. The source calls this "an elementary calculation"; it is, and Theorem 9.2 is that calculation, for all r and s, with no appeal to computation anywhere in it.

Method. Every statement below has been checked in Lean 4 (Section 10). The statements of Sections 3–7 and 8 are exhaustive finite computations, and we are explicit about what was computed and over what range; the statements of Section 9 are proofs in the ordinary sense, quantified over all parameters. Where an argument is sketched but not formalised — the elementary reason for the threshold in Section 3, and the cancellation behind the single sums in Section 5 — we say so in the statement of the remark that contains it.

2. Preliminaries

Definition 2.1. For $n \geq 1$ let $\mathrm{RD}(n)$ denote the set of partitions into distinct parts of perimeter n, and for $\lambda \in \mathrm{RD}(n)$ let $\beta(\lambda)$ be the number of odd parts of λ minus the number of even parts. For $m \in \mathbb{Z}$ set

$$\mathrm{rd}(n,m) = \#\{\lambda \in \mathrm{RD}(n) \;:\; \beta(\lambda) = m\}, \qquad D(n,m) = \mathrm{rd}(n,m) - \mathrm{rd}(n,-m) \quad (m \geq 1).$$

Then $\mathrm{rd}_o(n) = \sum_{m\geq 1} \mathrm{rd}(n,m)$, $\mathrm{rd}_e(n) = \sum_{m\geq 1} \mathrm{rd}(n,-m)$, $\mathrm{rd}_t(n) = \mathrm{rd}(n,0)$, and $\mathrm{rd}_o(n) - \mathrm{rd}_e(n) = \sum_{m\geq 1} D(n,m)$.

Everything computed in this paper rests on one elementary decomposition, which we record with its proof because it is what makes the counts computable at all.

Lemma 2.2 (Part-count decomposition)**.** *Fix $n \geq 1$. A partition $\lambda \in \mathrm{RD}(n)$ with exactly k parts has largest part $a = n - k + 1$, and its remaining $k - 1$ parts form a $(k-1)$-element subset of $\{1, 2, \ldots, a-1\}$; such λ exist precisely when $1 \leq k \leq a$. Write $m = n - k = a - 1$. The set $\{1, \ldots, m\}$ contains $\lceil m/2 \rceil$ odd and $\lfloor m/2 \rfloor$ even elements, so for each t there are exactly*

$$\binom{\lceil m/2 \rceil}{t}\binom{\lfloor m/2 \rfloor}{k-1-t}$$

partitions in $\mathrm{RD}(n)$ with k parts, of which exactly t non-maximal parts are odd, and each of them has bias

$$\beta(\lambda) = t + [a \text{ odd}] - (k-1-t) - [a \text{ even}].$$

Consequently $\mathrm{rd}(n,\cdot)$, and with it $\mathrm{rd}_o, \mathrm{rd}_e, \mathrm{rd}_t$ and D, is a double sum of $\Theta(n^2)$ binomial products.

Proof. If λ has k parts and perimeter n then $\lambda_1 + k - 1 = n$, so $\lambda_1 = a = n - k + 1$ is determined by k; the parts $\lambda_2 > \cdots > \lambda_k$ are distinct and lie in $\{1, \ldots, a-1\}$, and conversely any $(k-1)$-subset of $\{1, \ldots, a-1\}$ together with a is a partition of $\mathrm{RD}(n)$ with k parts. Such a subset exists iff $k-1 \le a-1$. The count of odd and even elements of $\{1,\dots,m\}$ and the bias formula are immediate. $\square$

The formal development evaluates the double sum of Lemma 2.2. Because every statement in this paper is read off that evaluation, the model is cross-checked against two literal enumerations that share no code with it and no code with each other: one sweeping all 2^n subsets of $\{1,\dots,n\}$ and keeping those whose largest element plus cardinality minus one equals n, the other organised by part count as in Lemma 2.2 but enumerating the subsets rather than counting them.

Proposition 2.3 (Model agreement)**.** *The bitmask enumeration, the part-count enumeration and the binomial double sum produce the same triple* $(\mathrm{rd}_o(n), \mathrm{rd}_e(n), \mathrm{rd}_t(n))$ *for every* $1 \le n \le 18$*. Moreover the integer-indexed excess* $\mathrm{rd}(n,m) - \mathrm{rd}(n,-m)$ *of Definition 2.1 agrees with the array reading of* $D(n,m)$ *used in the sweeps below, for every* $1 \le n \le 60$ *and* $1 \le m \le n$*.*

Verification. Two exhaustive evaluations. The first compares three independently written counting procedures at each $n \le 18$; the bitmask procedure alone visits 2^{18} subsets at the top of that range, which is what limits it. The second compares two array indexings of the same distribution for all $1 \le m \le n \le 60$. Both are decided by compiled evaluation. $\square$

Next, the numbers the sources themselves report.

Proposition 2.4 (The sources' counts, recomputed)**.** (i) $\#\mathrm{RD}(n) = F_n$ *for every* $1 \le n \le 22$*, computed from the literal enumeration of* $\mathrm{RD}(n)$ *— Straub's theorem* [6]*, as quoted in* [2, §1].

(ii) $\mathrm{rd}_o(n) + \mathrm{rd}_e(n) + \mathrm{rd}_t(n) = F_n$ *for every* $1 \le n \le 300$*.*

(iii) *The triples* $(\mathrm{rd}_o(n), \mathrm{rd}_e(n), \mathrm{rd}_t(n))$ *for* $1 \le n \le 8$ *are*

$$(1,0,0),\ (0,1,0),\ (1,0,1),\ (1,1,1),\ (2,1,2),\ (3,3,2),\ (6,3,4),\ (8,8,5),$$

so that $\mathrm{rd}_e(n) < \mathrm{rd}_o(n)$ *at* $n = 1,3,5,7$*, the inequality is reversed at* $n=2$*, and* $\mathrm{rd}_o(n) = \mathrm{rd}_e(n)$ *at* $n = 4,6,8$*.*

Verification. Exhaustive evaluation over the stated ranges: (i) enumerates $\mathrm{RD}(n)$ literally and compares its length with F_n; (ii) and (iii) evaluate the double sum of Lemma 2.2. Part (iii) is the content of the remark in [2] that for $n \le 8$ the inequality of Conjecture 1.1 holds for odd n but not for even n; the failures at even n are one reversal and three exact ties. The row sums $1,1,2,3,5,8,13,21$ are $F_1,\dots,F_8$. $\square$

3. The excess table and the sign law

The table of $D(n,m)$ begins as follows (blank entries are 0).

$n\backslash m$	1	2	3	4	5	6
7	4	−2	1			
8	−4	5	−1			
9	8	−5	3			
10	−8	12	−3	1		
11	17	−12	9	−1		
12	−17	28	−9	4		
13	37	−28	25	−4	1	
14	−37	66	−25	14	−1	
15	82	−66	66	−14	5	
16	−82	156	−66	44	−5	1

Theorem 3.1 (Sign law with a vanishing threshold)**.** *For every* $1 \le n \le 400$ *and every* $1 \le m \le n$*:*

$$D(n,m) > 0 \quad \text{if } n \equiv m \pmod 2, \qquad D(n,m) < 0 \quad \text{if } n \not\equiv m \pmod 2,$$

whenever $m \leq \lfloor (n+2)/3 \rfloor$*; and* $D(n,m) = 0$ *whenever* $\lfloor (n+2)/3 \rfloor < m \leq n$*. Equivalently,* $\operatorname{sign} D(n,m) = (-1)^{n-m}$ *below the threshold and* $D(n,m) = 0$ *above it.*

Verification. Exhaustive computation. For each perimeter n in $1 \leq n \leq 400$ the full bias distribution of $\mathrm{RD}(n)$ is built once from Lemma 2.2, and the displayed three-way test — positive, negative, or zero according to the position of m relative to $\lfloor (n+2)/3 \rfloor$ and to the parity of $n-m$ — is evaluated at every $1 \leq m \leq n$. That is $\sum_{n \leq 400} n = 80\,200$ cells, each read off a distribution obtained from $\Theta(n^2)$ binomial products. No general-n proof is offered, and the range $n \leq 400$ is the claim. The per-cell statements ($D(n,m) > 0$, < 0, $= 0$ under the corresponding hypotheses) follow from the sweep by an axiom-free argument. □

Remark 3.2 (Where the threshold comes from). The vanishing half of Theorem 3.1 has an elementary reason, which we give because it explains the shape of the table; it is a sketch, it is *not* part of the formal development, and Theorem 3.1 remains asserted only over $n \leq 400$. Suppose $\lambda \in \mathrm{RD}(n)$ has p odd and q even parts, so $j = p + q$ parts in all and largest part $a = n - j + 1$. The odd parts are distinct odd numbers in $\{1, \ldots, a\}$, so $p \leq \lceil a/2 \rceil$, and likewise $q \leq \lfloor a/2 \rfloor$. If $\beta(\lambda) = m > 0$ then $p \geq m$ and, taking $q = 0$ as the least constrained case, $m \leq \lceil (n-m+1)/2 \rceil$, i.e. $3m \leq n+2$. Hence $\mathrm{rd}(n,m) = 0$ for $m > \lfloor (n+2)/3 \rfloor$. The same computation on the even side gives $\mathrm{rd}(n,-m) = 0$ already for $m > \lfloor (n+1)/3 \rfloor$, so the top band $\lfloor (n+1)/3 \rfloor < m \leq \lfloor (n+2)/3 \rfloor$ is where the positive column survives alone.

Proposition 3.3 (Both hypotheses are load-bearing)**.** $D(9,1) = 8$ *and* $D(9,2) = -5$*, so no single sign is correct at a fixed perimeter and the parity condition in Theorem 3.1 cannot be dropped. At* $n = 20$ *the threshold* $\lfloor (n+2)/3 \rfloor$ *equals* 7*, and* $D(20,7) = -1 \neq 0$ *while* $D(20,8) = 0$*; so the threshold can be neither raised nor lowered by one.*

Verification. Four cell evaluations of the distribution of Lemma 2.2. □

Remark 3.4 (Comparison with the unrestricted case). Theorem 3.1 should be read against [2, Theorem 1.3], quoted as (1) above. For unrestricted partitions of perimeter n the strict inequality between $r(n,m)$ and $r(n,-m)$ holds for *every* $n \geq m \geq 1$; the distinct-parts table instead becomes identically zero once m exceeds $\lfloor (n+2)/3 \rfloor$, because a distinct-part partition cannot carry many odd parts without a large largest part, and a large largest part costs perimeter. The sign pattern itself is the same in both cases.

4. The reflection identity

Reading the excess table down a column shows more than an alternation of signs: the magnitudes repeat in consecutive pairs. At $m = 1$ they are $4, -4$, then $8, -8$, then $17, -17$, then $37, -37$; at $m = 2$ they are $1, -1$, $2, -2$, $5, -5$, $12, -12$.

Theorem 4.1 (Reflection)**.** *For every* $1 \leq n \leq 300$ *and every* $1 \leq m \leq n$ *with* $m \equiv n \pmod 2$*,*

$$D(n+1, m) \;=\; -\,D(n,m).$$

Verification. Exhaustive computation. For each $n \leq 300$ the bias distributions of $\mathrm{RD}(n)$ and of $\mathrm{RD}(n+1)$ are built once from Lemma 2.2 and the identity is evaluated at every $m \leq n$ of the right parity; cells of the wrong parity are passed over, so the sweep asserts nothing about them. The range $1 \leq n \leq 300$ is the claim; no proof for general n is offered. □

Proposition 4.2 (The parity hypothesis is load-bearing)**.** *At* $n = 10$*,* $m = 1$ *— where* $n \not\equiv m \pmod 2$ *— the identity fails:* $D(11,1) = 17$ *while* $-D(10,1) = 8$*.*

Verification. Two cell evaluations. □

Remark 4.3. Theorem 4.1 is stronger than the sign half of Theorem 3.1 in two senses. It pins the value rather than the sign; and, over the range where both are checked, it transports the sign law from the parity class $n \equiv m$ to the class $n \not\equiv m$, since $D(n, m) > 0$ then forces $D(n+1, m) < 0$ with $n+1 \not\equiv m$. It is also what makes the indexing of Section 6 what it is: because the magnitudes come in consecutive pairs, a column of the table is a single sequence read at $\lceil n/2 \rceil$ rather than at n. We know of nothing of this shape in the literature: the parity comparisons of [2] and [4] are all at fixed perimeter — what [2] does prove across consecutive perimeters, in its Section 4, are the Fibonacci and Tribonacci recurrences for counts that carry no bias refinement — and the classical parity-bias results [3, 1] concern partitions of a fixed integer, where there is no perimeter to shift. A bijective proof would presumably be a sign-reversing involution between the two perimeters; we have not attempted one.

5. Single-sum closed forms

Lemma 2.2 expresses $\mathrm{rd}_o(n) - \mathrm{rd}_e(n)$ and $\mathrm{rd}_t(n)$ as double sums with $\Theta(n^2)$ terms. Both collapse to $\Theta(n)$ terms.

Theorem 5.1 (Single sums)**.** *For every* $1 \le n \le 600$*, with* $q_k = \lfloor (n-k)/2 \rfloor$*,*

$$\mathrm{rd}_o(n) - \mathrm{rd}_e(n) = \sum_{k=1}^{n} (-1)^{n-k} \binom{q_k}{\lfloor \frac{k-1}{2} \rfloor} \binom{q_k}{\lceil \frac{k-1}{2} \rceil}, \tag{2}$$

$$\mathrm{rd}_t(n) = \sum_{\substack{k=2 \\ k \text{ even}}}^{n} \binom{\lceil (n-k)/2 \rceil}{k/2} \binom{\lfloor (n-k)/2 \rfloor}{k/2 - 1}. \tag{3}$$

Verification. Exhaustive computation: for each $n \le 600$ both sides of both identities are evaluated independently — the left sides through the double sum of Lemma 2.2, the right sides through the displayed single sums — and compared. The range $n \le 600$ is the claim. Identity (3) is in fact an immediate reading of Lemma 2.2, since a tie forces k even and leaves exactly one admissible split of the non-maximal parts; (2) is the one that requires cancellation, and Remark 5.3 sketches it. ☐

Proposition 5.2 (Summing the refinement)**.** *For every* $1 \le n \le 300$*,*

$$\sum_{m=1}^{n} D(n, m) = \mathrm{rd}_o(n) - \mathrm{rd}_e(n).$$

Verification. Exhaustive computation over $n \le 300$. This is a cross-check rather than an unfolding: the left side is accumulated from the bias distribution array and the right side from the separately written three-way tally of Lemma 2.2, so an indexing error in either fold would show up here. ☐

Remark 5.3 (How (2) was found). The following argument is a sketch. It has not been formalised and it is not offered as a proof of Theorem 5.1, which we assert only over $1 \le n \le 600$. Fix n and a part count k, and let $m = n - k$, $a = m + 1$; write i and j for the numbers of odd and even non-maximal parts, so $i + j = k - 1$.

- If a is odd, then $m = 2q$ is even, both pools {odds} and {evens} of $\{1, \ldots, m\}$ have size q, and $N(i, j) = \binom{q}{i}\binom{q}{j}$ is symmetric in $i \leftrightarrow j$. The bias is $i + 1 - j$, so the contribution to $\mathrm{rd}_o - \mathrm{rd}_e$ is $\sum_{i \ge j} N - \sum_{j \ge i+2} N$. By symmetry $\sum_{j \ge i+2} N = \sum_{i \ge j+2} N$, so everything cancels except the terms with $i - j \in \{0, 1\}$; at fixed $i + j = k - 1$ exactly one such term exists, and it equals $\binom{q}{\lfloor (k-1)/2 \rfloor}\binom{q}{\lceil (k-1)/2 \rceil}$.
- If a is even, then $m = 2q+1$, the pools have sizes $q+1$ and q, and one splits $\binom{q+1}{i} = \binom{q}{i} + \binom{q}{i-1}$. The first half is the symmetric case at $i + j = k - 1$ and contributes the same product with the opposite sign, because the bias is now $i - j - 1$; the second half is the symmetric case at $(i - 1) + j = k - 2$ and contributes 0.

In both cases the magnitude is $\binom{q}{\lfloor (k-1)/2\rfloor}\binom{q}{\lceil (k-1)/2\rceil}$ with $q = \lfloor (n-k)/2\rfloor$, and the sign is $(-1)^{n-k}$. The technique — collapsing a double binomial sum by a symmetry of the summand — is the one [2, §3] uses for the unrestricted counts, where Theorem 1.3 is proved by comparing the two sums summand by summand; what appears to be unrecorded is the resulting identity for distinct parts.

Remark 5.4 (A regrouped form, and why positivity is not termwise). Substituting $j = n-k$ in (2) and grouping by $\lfloor j/2\rfloor$ turns the identity into

$$\mathrm{rd}_o(n) - \mathrm{rd}_e(n) = \begin{cases} \displaystyle\sum_{r\geq 0} \binom{r}{s-r}\left[\binom{r}{s-r} - \binom{r}{s-r-1}\right], & n = 2s+1, \\ \displaystyle\sum_{r\geq 0} \binom{r}{s-r-1}\left[\binom{r}{s-r} - \binom{r}{s-r-1}\right], & n = 2s, \end{cases}$$

with the convention $\binom{a}{b} = 0$ outside $0 \leq b \leq a$. This regrouping is a rearrangement of (2); it is outside the formal development, and we verified it numerically for $n \leq 200$ only. The bracketed factor is a ballot number and is *negative* for small r — at $s = 10$ it takes the values -4 and -5 at $r = 5, 6$ before turning positive — so a proof of Conjecture 1.1 through (2) would have to be a genuine estimate and not a termwise positivity argument. We have not attempted one; such a proof would be an independent route to Mahanta's theorem, by formula rather than by injection.

Proposition 5.5 ((2) computes the difference, not either count)**.** *At $n = 12$ the right-hand side of* (2) *equals* 6*, which is* $\mathrm{rd}_o(12) - \mathrm{rd}_e(12)$*, whereas* $\mathrm{rd}_o(12) = 59$.

Verification. Three evaluations at $n = 12$. □

6. Two columns in the OEIS

Let A denote the sequence `A004148` of the On-Line Encyclopedia of Integer Sequences [5], defined there by the recurrence of its name line

$$A(n+1) = A(n) + \sum_{k=1}^{n-1} A(k)\, A(n-1-k), \qquad A(0) = A(1) = 1,$$

so that $A = 1, 1, 1, 2, 4, 8, 17, 37, 82, 185, 423, 978, 2283, 5373, 12735, 30372, \ldots$ on the entry's own offset 0. It is the sequence of generalized Catalan numbers; among many other readings the entry records that it arises in enumerating RNA secondary structures, that $A(n)$ enumerates the peakless Motzkin paths of length n, and that for $n \geq 1$ it counts the Dyck paths of semilength $n-1$ with no UUU and no DDD. We quote the entry at version 407 (16 August 2026).

Theorem 6.1 (The $m = 1$ column)**.** *For every* $1 \leq n \leq 400$, $D(n,1) = (-1)^{n-1} A\big(\lceil n/2\rceil\big)$.

Theorem 6.2 (The $m = 2$ column)**.** *For every* $2 \leq n \leq 400$,

$$D(n,2) = (-1)^n\, B\big(\lfloor n/2\rfloor + 1\big), \qquad B(j) \;=\; A(j+1) - A(j) - A(j-1).$$

Here B is `A166297`*, read on that entry's own offset* 0*, where*

$$B \;=\; 0, 0, 0, 1, 2, 5, 12, 28, 66, 156, 370, 882, 2112, 5079, 12264, \ldots,$$

and the displayed $B(j)$ is that entry's own formula line $a(n) =$ A004148$(n+1) -$ A004148$(n) -$ A004148$(n-1)$. The entry asserts that line for $n \geq 3$; it also reproduces the entry's value at $n = 2$, which is the one further index this identification uses, the argument $\lfloor n/2\rfloor + 1$ being at least 2 *throughout the stated range. It does not extend below that: at $n = 1$ it returns -1 while the entry's value is* 0*.* A166297 *counts the occurrences of $UUDUDD$ starting at level* 0 *in all Dyck paths of semilength n with no UUU and no DDD; we quote the entry at version 21 (24 July 2022).*

Verification of Theorems 6.1 and 6.2. Exhaustive computation. The sequence A is generated inside the formal development from the recurrence displayed above — that is, from the OEIS name line, not from a table of values — far enough to cover the stated ranges, and the identity is evaluated at every n in the stated range against $D(n,1)$, resp. $D(n,2)$, read off the distribution of Lemma 2.2. The ranges are the claim; no proof for general n is offered, and in particular we exhibit no bijection. □

Proposition 6.3 (The reading, spelled out)**.** *The values of* $D(n,1)$ *for* $n = 1,\dots,10$ *are*

$$1,\ -1,\ 1,\ -1,\ 2,\ -2,\ 4,\ -4,\ 8,\ -8,$$

and $A(0),\dots,A(8)$ *are* $1,1,1,2,4,8,17,37,82$.

Verification. Nineteen evaluations. □

Remark 6.4 (The crossing is unrecorded on both sides)*.* `A004148` is a heavily developed entry — RNA secondary structures, peakless Motzkin paths, UUU- and DDD-free Dyck paths, antidiagonal sums of Narayana numbers, q-deformed rationals — and a search of its full entry text, at the two versions cited above and run on 28 August 2026, finds no occurrence of *perimeter*, *hook*, *parity*, *distinct part*, or of [6] or [2]; the only occurrences of the word "partition" are noncrossing *set* partitions. In `A166297` none of those words, "partition" included, occurs at all. Conversely neither [2] nor [4] mentions either sequence. Two things make the crossing more than an inspection of the first few terms: the columns had to be extracted from a refinement that neither source defines, and the argument $\lceil n/2\rceil$ — rather than n — is forced by the pairing of Theorem 4.1, so it cannot be read off the raw values. The obvious next question is whether there is a bijection between the distinct-part partitions of perimeter n and bias ± 1 and the peakless Motzkin paths counted by $A(\lceil n/2\rceil)$. We do not answer it.

7. The aggregate inequality as an alternating sum

This section states the structural payoff of Theorem 3.1 for Conjecture 1.1. By Definition 2.1,

$$\mathrm{rd}_o(n) - \mathrm{rd}_e(n) \;=\; \sum_{m\ge 1} D(n,m), \tag{4}$$

and by Theorem 3.1 the summands on the right alternate in sign, up to the vanishing threshold, since consecutive m have opposite parity. So over the range where the sign law has been checked, the conjectured inequality $\mathrm{rd}_o(n) > \mathrm{rd}_e(n)$ is the positivity of an alternating sum.

Proposition 7.1 (The alternation, at $n = 10$)**.** $D(10,1) = -8$*,* $D(10,2) = 12$*,* $D(10,3) = -3$*,* $D(10,4) = 1$*,* $D(10,5) = 0$*, and* $\mathrm{rd}_o(10) - \mathrm{rd}_e(10) = 2$*.*

Verification. Six evaluations at $n = 10$. The first four sum to 2, as (4) requires; note that the leading column is the negative one and is larger in magnitude than any single positive column except the second. □

Remark 7.2 (The contrast with the unrestricted case, and our reading of it)*.* The first three sentences below are read off [2] or computed here; the last is our own inference and is not a claim either source makes.

By [2, Theorem 1.3], quoted as (1), the unrestricted excesses $r(n,m) - r(n,-m)$ alternate in sign with m exactly as the distinct-parts excesses do, so the unrestricted aggregate $r_o(n) - r_e(n) = \sum_{m\ge1}\big(r(n,m) - r(n,-m)\big)$ is an alternating sum too. What separates the two cases is that there the alternating sum is *evaluated*: the involution Θ of [2, §2] carries $R_o(n)$ onto $R(n,0) \sqcup R_e(n)$, giving $r_o(n) - r_e(n) = r(n,0)$ for $n \ge 3$ [2, Theorem 1.1], so that positivity there is the non-emptiness of $R(n,0)$. The distinct-parts analogue of that identity is false: at $n = 9$ we have $\mathrm{rd}_o(9) - \mathrm{rd}_e(9) = 6$ while $\mathrm{rd}(9,0) = \mathrm{rd}_t(9) = 10$, by Proposition 8.2(ii). We know of no substitute for it, and that, we believe, is why [2] could prove the unrestricted statements and could only conjecture the distinct-parts one, and why the proof that eventually settled it [4] is an injection built by cases rather than a computation with formulas.

8. The conjecture verified to perimeter 1500

Conjecture 1.1 is a theorem of [4], so nothing in this section is a new mathematical statement. What it adds is quantitative. The conjecturers report a verification to $n = 150$. The published proof closes with the sentence "Routine verification for small values of n completes the proof" without naming a range, and its worked tables cover only $n = 13$ and $n = 14$; the base-case obligation is therefore open text.

Theorem 8.1. $\mathrm{rd}_e(n) < \mathrm{rd}_o(n)$ *for every* $9 \le n \le 1500$.

Verification. Exhaustive computation, in eight blocks — 9–200, 201–400, 401–600, 601–800, 801–1000, 1001–1200, 1201–1350, 1351–1500 — each a single compiled Boolean evaluation, combined into the stated range by a case split that itself uses no computation. For each n, $\mathrm{rd}_o(n)$ and $\mathrm{rd}_e(n)$ are computed from the double sum of Lemma 2.2, at a cost of $\Theta(n^2)$ binomial products per perimeter and $\Theta(n^3)$ over the sweep. Note that this is an independent verification, not a re-run of the argument of [4]: the counts come from Lemma 2.2, whose agreement with two literal enumerations sharing no code with it is Proposition 2.3. The range covers ten times the $n = 150$ reported in [2], and discharges the unquantified small-n obligation of [4] at any plausible reading of "small". □

Proposition 8.2 (Sharpness and non-vacuity)**.**

(i) *The hypothesis* $n \ge 9$ *cannot be weakened:* $\mathrm{rd}_o(2) < \mathrm{rd}_e(2)$*, and* $\mathrm{rd}_o(n) = \mathrm{rd}_e(n)$ *at* $n = 4, 6, 8$*, so the strict inequality fails at every threshold* ≤ 8.

(ii) *At* $n = 9$ *all three classes are nonempty —* $(\mathrm{rd}_o, \mathrm{rd}_e, \mathrm{rd}_t)(9) = (15, 9, 10)$*, out of* $\#\mathrm{RD}(9) = 34 = F_9$ *— so the first instance of the conjecture is a statement about actual partitions.*

(iii) *The inequality does not strengthen to a constant factor:* $\mathrm{rd}_o(10) = 22$ *and* $\mathrm{rd}_e(10) = 20$*, so* $\mathrm{rd}_o(10) > 2\,\mathrm{rd}_e(10)$ *is false.*

(iv) *The bias is not monotone:* $\mathrm{rd}_o(11) - \mathrm{rd}_e(11) = 13$ *while* $\mathrm{rd}_o(12) - \mathrm{rd}_e(12) = 6$.

Verification. Direct evaluation at the named perimeters. □

9. The counting step of the published proof

The proof in [4] is an injection $\mathrm{RD}_e(n) \to \mathrm{RD}_o(n)$ built by a case split on the smallest part of the partition — at least 3, equal to 1, equal to 2 — with the third of those cases split further: first by the bias $\beta(\pi)$, then by the gap between the two largest parts and the parity of the largest, and once more by the bias. Every case is an explicit map except the last, labelled Case iii-c there, which is closed by a counting comparison stated without proof:

> *An elementary calculation shows that the number of ways to choose $\frac{r}{2}$ even numbers and $\frac{r-4}{2}$ odd numbers from $\{3, 4, \dots, s\}$, where $s \ge r$ is a positive integer, is less than the number of ways to choose $\frac{r-2}{2}$ even numbers and $\frac{r-2}{2}$ odd numbers from the same set.*

No proof is given there and no reference is cited for it. We quote [4] at v2 throughout, and the case labels are that version's. This section is that calculation. Unlike everything above, Lemma 9.1 and Theorem 9.2 are proved for all values of their parameters, with no appeal to computation; only the controls in Proposition 9.3 are finite checks.

Write E_s and O_s for the numbers of even and of odd integers in $\{3, 4, \dots, s\}$, so that $E_s = \lfloor s/2 \rfloor - 1$ (read as 0 when $s \le 1$, where the set is empty) and $O_s = \lfloor (s-1)/2 \rfloor$, and note $E_s \le O_s$ for every s (an integer interval beginning at an odd number never holds more evens than odds).

Lemma 9.1 (Pascal-ratio comparison)**.** *Let* E, O, u *be integers with* $1 \le u \le E \le O$*. Then*

$$\binom{E}{u+1}\binom{O}{u-1} < \binom{E}{u}\binom{O}{u}.$$

Proof. Both $\binom{E}{u}$ and $\binom{O}{u-1}$ are positive because $u \le E \le O$. Pascal's ratio in the form $\binom{N}{v+1}(v+1) = \binom{N}{v}(N-v)$ gives

$$\binom{E}{u+1}(u+1) = \binom{E}{u}(E-u), \qquad \binom{O}{u}\, u = \binom{O}{u-1}(O-u+1).$$

Multiplying the claimed inequality by $u(u+1) > 0$ and substituting both identities turns it into

$$\binom{E}{u}\binom{O}{u-1}(E-u)\,u \;<\; \binom{E}{u}\binom{O}{u-1}(O-u+1)(u+1),$$

so after cancelling the positive factor $\binom{E}{u}\binom{O}{u-1}$ it suffices to see $(E-u)u < (O-u+1)(u+1)$. Since $O \ge E$ we have $O-u+1 \ge (E-u)+1$, hence $(O-u+1)(u+1) \ge \big((E-u)+1\big)(u+1) > (E-u)u$. ☐

Theorem 9.2 (Mahanta's Case iii-c, in general)**.** *Let $r \ge 4$ be even and let $s \ge r$. Then*

$$\binom{E_s}{r/2}\binom{O_s}{(r-4)/2} \;<\; \binom{E_s}{(r-2)/2}\binom{O_s}{(r-2)/2}.$$

Proof. Put $u = (r-2)/2$, so that $r/2 = u+1$ and $(r-4)/2 = u-1$, and $u \ge 1$ because $r \ge 4$. The hypothesis $s \ge r$ gives $u = (r-2)/2 \le \lfloor s/2 \rfloor - 1 = E_s$, and $E_s \le O_s$ always. Now apply Lemma 9.1 with $E = E_s$, $O = O_s$. ☐

Proposition 9.3 (Supporting checks and controls)**.** (i) *$E_s = \lfloor s/2 \rfloor - 1$ and $O_s = \lfloor (s-1)/2 \rfloor$ really count the even and the odd members of $\{3, \ldots, s\}$, for every $s \le 300$; and the binomial table used elsewhere in this development agrees entrywise with $\binom{a}{b}$ for $a \le 22$, $b \le 23$. Both are finite checks, not proofs.*

(ii) *Theorem 9.2 agrees with direct evaluation of both sides for every even $4 \le r \le 60$ and every $r \le s \le 300$.*

(iii) *The inequality is not vacuous. At $r = 4$, $s = 6$ the two sides are 1 and 4; at $r = 10$, $s = 30$ they are $\binom{14}{5}\binom{14}{3} = 728\,728$ and $\binom{14}{4}^2 = 1\,002\,001$.*

(iv) *It is not an inequality between the individual binomial factors: at $r = 10$, $s = 30$ the left factor $\binom{14}{5} = 2002$ exceeds the right factor $\binom{14}{4} = 1001$, so only the products compare.*

Verification. Exhaustive evaluation over the stated ranges for (i) and (ii); direct evaluation for (iii) and (iv). An independent search in a separate language over all even $4 \le r \le 400$ and all $r \le s \le 1200$ found no violation of Theorem 9.2 and no instance with a vanishing right-hand side. ☐

10. Verification

Every statement of this paper has been checked in Lean 4 (toolchain `v4.32.0`), against *Mathlib* where Mathlib is used; the development contains no `sorry`, introduces no axiom of its own, and is organised so that the two kinds of claim are visibly separated. Lemma 9.1 and Theorem 9.2 — the statements quantified over all parameters — are ordinary proofs, depending on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`, with no appeal to compiled evaluation anywhere in them. Everything else is a finite search discharged by compiled evaluation (`native_decide`), and the range named in each statement is exactly the range that was evaluated: $n \le 18$ and $n \le 60$ in Proposition 2.3; $n \le 22$, $n \le 300$ and $n \le 8$ in Proposition 2.4; $1 \le m \le n \le 400$ in Theorem 3.1; $1 \le m \le n \le 300$ in Theorem 4.1; $n \le 600$ in Theorem 5.1 and $n \le 300$ in Proposition 5.2; $n \le 400$ in Theorems 6.1 and 6.2; $9 \le n \le 1500$, in eight blocks, in Theorem 8.1; $s \le 300$ and (r, s) with $r \le 60$, $s \le 300$ in Proposition 9.3; and the individual cells named in Propositions 3.3, 4.2, 5.5, 6.3, 7.1 and 8.2. The step from a block of evaluations to the statement about an individual n or m — the case split in Theorem 8.1, and the per-cell corollaries of Theorems 3.1 and 4.1 — is a proof in the ordinary sense and uses no axioms at all. All counts are computed from Lemma 2.2, whose agreement with two literal enumerations sharing no code with it is Proposition 2.3; the numerical claims about Theorem 9.2 in Proposition 9.3 were additionally reproduced by an independent implementation.

The sketches in Remarks 3.2, 5.3 and 5.4 are *not* part of the formal development and are labelled as such where they occur. Repository reference to be added at submission.

References

[1] K. Banerjee, S. Bhattacharjee, M. Ghosh Dastidar, P. J. Mahanta and M. P. Saikia, *Parity biases in partitions and restricted partitions*, European J. Combin. **103** (2022), 103522; `doi:10.1016/j.ejc.2022.103522`; arXiv:2111.10590.

[2] G. Gray, E. Payne, H. Swisher and R. Watson, *Fixed Perimeter Analogues of Several Partition Results Related to Parity*, arXiv:2608.00421v1 (1 August 2026). Every numbered statement cited from it here is numbered as in v1, the only version as of 28 August 2026.

[3] B. Kim, E. Kim and J. Lovejoy, *Parity bias in partitions*, European J. Combin. **89** (2020), 103159; `doi:10.1016/j.ejc.2020.103159`.

[4] P. J. Mahanta, *A Proof of a Conjecture on Fixed Perimeter Partitions*, arXiv:2608.11684v2 (19 August 2026; v1 12 August 2026). The case labels and the sentence quoted in Section 9 are v2's.

[5] OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, `https://oeis.org`; entry `A004148` (version 407, 16 August 2026) and entry `A166297` (version 21, 24 July 2022), both read on 28 August 2026.

[6] A. Straub, *Core partitions into distinct parts and an analog of Euler's theorem*, European J. Combin. **57** (2016), 40–49; `doi:10.1016/j.ejc.2016.04.002`; arXiv:1601.07161.