THE EXPECTED NUMBER OF PAIRWISE STABLE NETWORKS: EXACT VALUES, AN n^n-TERM FORMULA, AND A ONE-INEQUALITY REDUCTION OF THE LIMIT QUESTION

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Abstract. Herings, Seel and Predtetchinski (arXiv:2606.23440) consider n individuals whose utilities for the networks on them are independent atomless random variables, and the number s_n of pairwise stable networks in the sense of Jackson and Wolinsky. They prove $\mathbb{E}(s_n) = \sum_g 1/X_n(g)$, a sum over all $2^{n(n-1)/2}$ networks of the reciprocal *seniority degree*, evaluate it to four decimals for $n \leq 7$, call larger n impractical, prove $\sqrt{e} \leq \liminf \mathbb{E}(s_n^*) \leq \limsup \mathbb{E}(s_n^*) \leq e$ for the normalized sequence $\mathbb{E}(s_n^*) = \mathbb{E}(s_n)/c_n$ with $c_n = 2^{n(n-1)/2}(2/(n+1))^n$, and leave open whether $\mathbb{E}(s_n^*)$ converges and to what, noting that monotonicity would give L^2 convergence. We observe that a network is an orientation of the complete graph and that the seniority degree of i is one plus its out-degree, so the sum runs over tournaments and depends only on the score sequence—the enumeration MacMahon carried out in 1920—and that exact quadrature of the source's own integral formula turns the $2^{n(n-1)/2}$-term sum into an n^n-term rational identity whose weights are the Adams–Moulton coefficients. This gives the exact rationals for $n \leq 7$, each rounding to the printed decimal, the value $\mathbb{E}(s_8) = 4206906030446363/1714608000000 = 2453.5672\ldots$, and, by the score-sequence recursion outside the formal development, exact values to $n = 14$. All seven exactly known values of $\mathbb{E}(s_n^*)$ are strictly increasing and lie strictly below $\sqrt{e}$, the *lower* end of the bracket. We prove that the single inequality $\mathbb{E}(s_n^*) \leq \sqrt{e}$ for all n, together with the source's own asymptotic bracket, already forces $\mathbb{E}(s_n^*) \to \sqrt{e}$: the open question reduces to one inequality, monotonicity is not needed, and if the inequality persists the bracket collapses to its lower endpoint. Numerically $n(\sqrt{e} - \mathbb{E}(s_n^*))$ decreases to 0.79 at $n = 14$. Two corrections to the source are recorded: there are 2^{21}, not 2^{28}, networks on seven individuals, and two cells of its Table 3 are misprinted. The identities, the values for $n \leq 8$, the comparisons and the reduction are verified in Lean 4 against *Mathlib*.

1. Introduction

The objects. Let $I_n = \{1, \ldots, n\}$ and let $L_n = \{ij : 1 \leq i < j \leq n\}$ be the $\ell_n = n(n-1)/2$ possible links. A *network* is a subset $g \subseteq L_n$; $G_n = 2^{L_n}$ is the set of networks. A network model assigns to every individual i and every network g a utility $u_i(g)$. Following Jackson and Wolinsky [2], g is *pairwise stable* when no individual gains by deleting one of its links and no pair of individuals both gain by adding the link between them: for every $ij \in g$, $u_i(g) \geq u_i(g - ij)$ and $u_j(g) \geq u_j(g - ij)$, and for every $ij \in L_n \setminus g$, $u_i(g) \geq u_i(g + ij)$ or $u_j(g) \geq u_j(g + ij)$. Herings, Seel and Predtetchinski [1] draw all utilities $u_i(g)$, $i \in I_n$, $g \in G_n$, independently from an atomless distribution (uniform on $[0, 1]$ without loss of generality) and study the random number

$$s_n = \#\{g \in G_n : g \text{ is pairwise stable}\}.$$

Their Theorem 4.1 is the integral formula

$$\mathbb{E}(s_n) = \int_{[0,1]^n} \prod_{ij \in L_n} (v_i + v_j)\, dv, \tag{1}$$

and their Theorem 4.2, equation (4.1) there, its evaluation as a finite sum. Relative to i, the individuals with a larger index are i's *seniors* and those with a smaller index its *juniors*; the *seniority degree* of i in g awards one point for every senior partner, one point for every missing junior partner, and one point for free,

$$X_{i,n}(g) = 1 + \#\{j : i < j,\ ij \in g\} + \#\{j : j < i,\ ij \notin g\}, \qquad X_n(g) = \prod_{i=1}^{n} X_{i,n}(g), \tag{2}$$

and

$$\mathbb{E}(s_n) = \sum_{g \in G_n} \frac{1}{X_n(g)}. \tag{3}$$

The source evaluates (3) for $n = 2, \dots, 7$ (its Table 2: 1.0000, 1.2500, 2.2130, 6.0039, 26.3232, 193.5913) and writes that "it is impractical to evaluate it for larger values of n, as the number of networks is equal to $2^{n(n-1)/2}$, which corresponds to the number of terms in the sum of (4.1)". It then normalizes by

$$c_n = 2^{\ell_n} \left(\frac{2}{n+1} \right)^n, \qquad s_n^* = s_n / c_n, \tag{4}$$

proves the lower bound $\mathbb{E}(s_n^*) \geq e^{\frac{1}{2} \frac{(n-1)n}{(n+1)^2}}$ (Theorem 4.3), the upper bound $\mathbb{E}(s_n^*) \leq (1 + \frac{1}{n})^n (1 - \frac{1}{2^n})^n$ (Theorem 4.4), hence

$$\sqrt{e} \;\leq\; \liminf_{n \to \infty} \mathbb{E}(s_n^*) \;\leq\; \limsup_{n \to \infty} \mathbb{E}(s_n^*) \;\leq\; e \qquad \text{(Theorem 4.5)}, \tag{5}$$

and proves that the variance of s_n^* tends to zero (Theorem 5.4).

The source's open question. Section 6 of [1] states, verbatim: "an open question remains whether the normalized expected number of pairwise stable networks converges, and if so, to compute the limit. By our results on the variance, for convergence in L^2 to a constant, it suffices to show monotonicity of the sequence $\mathbb{E}(s_n^*)$." The bracket (5) leaves a factor $\sqrt{e}$ between the two candidates, and the source's Table 3 stops at $n = 7$, where $\mathbb{E}(s_7^*) = 1.5124$ is still below the lower endpoint $\sqrt{e} = 1.6487$.

Results. Everything here concerns the right-hand side of (3), which we write S_n; it is $\mathbb{E}(s_n)$ by the source's Theorem 4.2, and nothing about pairwise stability itself is re-proved. Statement, table and section numbers of [1] are those of the compiled version 1 e-print (see the bibliography entry).

(i) *Networks are orientations* (Proposition 3.1). Each link $ij \in L_n$ awards exactly one seniority point, to the junior when the link is present and to the senior when it is absent. So $g \mapsto$ "i beats j iff $ij \in g$, for $i < j$" is a bijection from networks to tournaments on I_n under which $X_{i,n}(g) = 1 + \mathrm{outdeg}_i$, and

$$S_n = \sum_{T \text{ tournament on } I_n} \prod_{i=1}^{n} \frac{1}{1 + \mathrm{outdeg}_i(T)}. \tag{6}$$

The summand depends only on the score sequence of T, so S_n is a one-line consequence of the table of score-sequence multiplicities—the table MacMahon computed by hand to $n = 9$ in 1920 [3] and David recomputed to $n = 8$ in 1959 [4] (Alway carried the catalogue of score sequences to $n = 10$ by computer in 1962 [5]), and which for $n \leq 10$ is OEIS A125032 [9]. We are explicit that this is a relabelling and not new mathematics.

(ii) *An n^n-term formula* (Proposition 3.3). Every out-degree is at most $n - 1$, so rational weights $u_0, \dots, u_{n-1}$ reproducing the first n moments of Lebesgue measure on $[0, 1]$ may be substituted for $1/(1 + \mathrm{outdeg}_i)$, and the sum over orientations collapses back into a product over links:

$$S_n = \sum_{\kappa \in \{0, \dots, n-1\}^n} \left(\prod_{i=1}^{n} u_{\kappa_i} \right) \prod_{i<j} (\kappa_i + \kappa_j).$$

This is exact Newton–Cotes quadrature of (1), done algebraically; the weights are the Adams–Moulton coefficients. It has n^n terms against $2^{n(n-1)/2}$; at $n = 8$ that is 16 777 216 against 268 435 456.

(iii) *Exact values* (Propositions 4.1 and 4.2). $S_2, \dots, S_7$ are 1, 5/4, 239/108, 7781/1296, 682297/25920 and 451609897709/2332800000, each rounding to the source's printed decimal, and

$$S_8 = \frac{4206906030446363}{1714608000000} = 2453.5672\dots,$$

the first value beyond its Table 2. Outside the formal development, the score-sequence recursion gives the exact values to $n = 14$ (Remark 4.3), and for $n \leq 10$ they are re-derived from the published multiplicity table alone (Remark 4.4).

(iv) *Where the limit must be* (Proposition 5.1, Theorem 5.2, Corollary 5.3). The seven exactly known values of $S_n^* = S_n/c_n$ are strictly increasing and every one is strictly below $\sqrt{e}$. And the single inequality $S_n^* \leq \sqrt{e}$ for every n, together with the source's Theorem 4.5, forces $S_n^* \to \sqrt{e}$; no monotonicity is needed, and a monotone sequence staying below $\sqrt{e}$ satisfies the hypothesis. So the source's question reduces to one inequality, and if it holds, the bracket (5) collapses to its lower endpoint. The inequality is verified exactly for $n \leq 8$ and numerically to $n = 14$, where $n(\sqrt{e} - S_n^*)$ has decreased to 0.79; it is not proved, and we do not claim convergence.

(v) *Two corrections* (Propositions 6.1 and 6.2). The introduction of [1] says that for $n = 7$ the seniority degree "has to be calculated for a total of 2^{28} networks"; there are $2^{21} = 2\,097\,152$, and 2^{28} is the count at $n = 8$ (the source's own Section 4.1 gives the correct formula). In its Table 3, the cell $\mathbb{E}(s_5^*) = 1.4248$ should read 1.4247 (exact value 23343/16384), and the upper-bound cell at $n = 7$, printed 2.4112, should read 2.4105.

What is new and what is not. We are careful about attribution. The model, the integral formula (1), the seniority-degree sum (3), the bounds and the bracket (5), and the variance result are all the source's [1]. The generating identity $\prod_{i<j}(x_i + x_j) = \sum_T \prod_i x_i^{\mathrm{outdeg}_i(T)}$ over orientations is classical: MacMahon [3] expanded exactly this product to enumerate score sequences, and the source performs the same expansion, over networks rather than orientations, inside its proof of Theorem 4.2. Landau's characterization of score sequences [6] and the standard account in Moon [7] are used only in the computations of Section 4. Exact quadrature at integer nodes is textbook, and the weights are those of the Adams–Moulton methods [10]. The two limit lemmas of Section 5 are elementary. What we claim is: the re-indexing (6) and the n^n-term identity written down for this sum; the exact rationals for $n \leq 8$, of which $n = 8$ is beyond the source's table, and their verification; the observation that all known values lie *below* the lower endpoint $\sqrt{e}$; the reduction of the source's open question to the single inequality $S_n^* \leq \sqrt{e}$, which is weaker than the monotonicity its Section 6 asks for and which identifies the limit; and the two corrections. We grade the values as routine: for $n \leq 10$ they are one line of arithmetic from published score-sequence multiplicities—MacMahon's 1920 table for $n \leq 9$ as Stockmeyer [8] describes it, and the machine-readable OEIS b-file for $n \leq 10$ (Remark 4.4)—and the recursion beyond is standard. Where the limit lies remains conjectural; what is proved is the conditional statement.

As of 2026-09-07 the source has no citing work in OpenAlex (work `W7165636181`, `cited_by_count` 0, no work with `cites:W7165636181`), the arXiv API returns exactly one paper, the source, for `all:"expected number of pairwise stable"`, and the abs page lists version 1 only, so the source has not answered its own question in a revision. A full-text sweep of 890 835 arXiv e-prints finds the phrase "expected number of pairwise stable" and the phrase "seniority degree" in one e-print each, the source; the phrase "pairwise stable network" occurs in 34 e-prints, none of which contains the former. OEIS searches for the numerators and denominators of the exact values, for the individual integers 451609897709, 4206906030446363 and 459371836096470610907, for the rounded sequence $1, 1, 2, 6, 26, 193, 2453, 54676$, and for the quoted phrases "pairwise stable" and "seniority degree" return nothing, while the positive control $1, 1, 2, 5, 14, 42$ returns A000108; the search "number of tournaments" "score sequence" returns A125032, which is how the routine grading of the values was established. All of the searches, the corpus counts, and the OpenAlex and arXiv-API figures above were re-run on 2026-09-07.

Verification. Every proposition and theorem below that carries a formal counterpart in Appendix A is machine-checked in Lean 4 against *Mathlib*; Section 7 records the modules, the axioms, and which statements rest on compiled evaluation. Where the text says more than its formal counterpart—the values for $9 \leq n \leq 14$, the timing figures, the two further table cells of Remark 6.3, and the heuristic of Remark 5.5—it says so at that place.

2. Preliminaries

Throughout, $n \geq 2$ unless said otherwise, and the formal development indexes everything by $\mathbb{N} = \{0, 1, 2, \dots\}$, with individuals $\{0, \dots, n-1\}$; we keep the source's $1, \dots, n$ in the prose, which changes nothing.

2.1. **Networks and the seniority degree.** A network on I_n is a symmetric irreflexive relation, equivalently a subset $g \subseteq L_n$. $X_{i,n}(g)$ and $X_n(g)$ are as in (2), and

$$S_n = \sum_{g \in G_n} \frac{1}{X_n(g)} \in \mathbb{Q}, \qquad S_n^* = \frac{S_n}{c_n}, \qquad c_n = 2^{\ell_n}\Big(\frac{2}{n+1}\Big)^n. \tag{7}$$

By [1, Theorem 4.2], $S_n = \mathbb{E}(s_n)$ and $S_n^* = \mathbb{E}(s_n^*)$; every statement below about S_n is a statement about $\mathbb{E}(s_n)$ through that theorem, which is not re-proved here.[1]

2.2. **Tournaments and orientations.** A *tournament* on I_n is a relation T with $T(i,i)$ false and, for $i \neq j$, exactly one of $T(i,j)$, $T(j,i)$ true; $\operatorname{outdeg}_i(T) = \#\{j : T(i,j)\}$ is the *score* of i, and the multiset of scores is the score sequence. There are 2^{ℓ_n} tournaments. We also use the coding of orientations by subsets of the ordered pair set

$$P_n = \{(i,j) : i < j\} \subseteq I_n \times I_n, \qquad |P_n| = \ell_n :$$

for $S \subseteq P_n$, i beats its senior j when $(i,j) \in S$ and beats its junior j when $(j,i) \notin S$, so that

$$\operatorname{outdeg}_i(S) = \#\{(i,j) \in S\} + \#\{(j,i) \in P_n \setminus S\}. \tag{8}$$

2.3. **Moment-matching weights.** For $n \geq 1$ the linear system $\sum_{k=0}^{n-1} u_k k^d = \frac{1}{d+1}$, $d = 0, \dots, n-1$, is a Vandermonde system at the nodes $0, \dots, n-1$ and has a unique rational solution, $u_k = \int_0^1 \prod_{j \neq k} \frac{x-j}{k-j}\, dx$; these are the coefficients of the $(n-1)$-step Adams–Moulton method [10]. For $n = 4$ they are $\frac{3}{8}, \frac{19}{24}, -\frac{5}{24}, \frac{1}{24}$; for $n = 7$ and $n = 8$, cleared of the denominators 60480 and 120960, they are the integer vectors

$$\begin{aligned} 60480\, u^{(7)} &= (19087,\ 65112,\ -46461,\ 37504,\ -20211,\ 6312,\ -863), \\ 120960\, u^{(8)} &= (36799,\ 139849,\ -121797,\ 123133,\ -88547,\ 41499,\ -11351,\ 1375). \end{aligned}$$

Nothing about the weights is assumed in the formal development: the moment identities are checked by exact rational arithmetic inside each proof that uses them.

3. Networks are orientations, and the n^n-term formula

Proposition 3.1 (networks are orientations)**.** *For every n,*

$$S_n = \sum_{g \in G_n} \frac{1}{X_n(g)} = \sum_{T \text{ tournament on } I_n} \prod_{i=1}^{n} \frac{1}{1 + \operatorname{outdeg}_i(T)} = \sum_{S \subseteq P_n} \prod_{i=1}^{n} \frac{1}{1 + \operatorname{outdeg}_i(S)}.$$

Proof. Orient g by letting the junior beat the senior when the link is present and the senior beat the junior when it is absent. This is a bijection $G_n \to \{\text{tournaments}\}$, with inverse "$ij \in g$ iff i beats j" for $i < j$. For a fixed i the set of j beaten by i is the disjoint union of the senior partners of i and the missing junior partners of i, so $X_{i,n}(g) = 1 + \operatorname{outdeg}_i$ by (2). The second equality is the same argument with orientations coded by subsets of P_n through (8). Formally: `netSum_eq_tourSum` and `netSum_eq_subSum`, both by `Finset.sum_nbij'` with the explicit bijections. ∎

Lemma 3.2 (the tournament generating identity, MacMahon [3])**.** *For every n and all $x_1, \dots, x_n \in \mathbb{Q}$,*

$$\prod_{(i,j) \in P_n} (x_i + x_j) = \sum_{S \subseteq P_n} \prod_{i=1}^{n} x_i^{\operatorname{outdeg}_i(S)}.$$

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[1] The formal definitions make sense for $n = 0, 1$ as well, where $S_n = c_n = 1$; the theorems quantified over all n include these values, which is harmless.

Proof. Expanding the product, each link hands its variable to the endpoint that wins it; the monomial of the orientation S is $\prod_i x_i^{\mathrm{outdeg}_i(S)}$. Formally `prod_add_eq_sum_odeg`: `Finset.prod_add` followed by regrouping the two factors $\prod_{p\in S} x_{p_1}$ and $\prod_{p\notin S} x_{p_2}$ by their vertex. The identity is the one MacMahon expanded by hand to $n = 9$ to tabulate score sequences [3, 8]; the source expands it over networks inside its proof of Theorem 4.2. It is recorded here only because the formal development needs it, over $\mathbb{Q}$. ☐

Proposition 3.3 (the n^n-term formula)**.** *Let* $n \in \mathbb{N}$ *and let* $u : \mathbb{N} \to \mathbb{Q}$ *satisfy*

$$\sum_{k=0}^{n-1} u_k\, k^d = \frac{1}{d+1} \qquad \text{for every } d \in \{0, \dots, n-1\}.$$

Then

$$S_n = \sum_{\kappa : I_n \to \{0,\dots,n-1\}} \Big(\prod_{i=1}^{n} u_{\kappa(i)}\Big) \prod_{(i,j)\in P_n} \big(\kappa(i) + \kappa(j)\big).$$

Proof. Every out-degree of an orientation of K_n is at most $n-1$ (`odeg_lt`), so for each $S \subseteq P_n$ and each i the hypothesis gives $\frac{1}{1+\mathrm{outdeg}_i(S)} = \sum_{k<n} u_k\, k^{\mathrm{outdeg}_i(S)}$. Substituting into the third form of Proposition 3.1, expanding the product over i of these sums into a sum over maps κ (`Finset.prod_univ_sum`), and exchanging the two sums, the inner sum over S is $\sum_S \prod_i \kappa(i)^{\mathrm{outdeg}_i(S)}$, which Lemma 3.2 collapses to $\prod_{(i,j)\in P_n} (\kappa(i) + \kappa(j))$. Formally `subSum_eq_cubature`. ☐

Remark 3.4. Proposition 3.3 is the exact Newton–Cotes evaluation of the source's integral formula (1): the integrand $\prod_{i<j}(v_i + v_j)$ has degree $n-1$ in each variable, so the tensor-product rule with n nodes per axis is exact. The proof above is purely algebraic—no integral, no measure theory—and goes through the source's finite sum rather than through (1). The count of terms, n^n against $2^{n(n-1)/2}$, is smaller from $n = 7$ on (823 543 against 2 097 152) and is 10^{10} against $2^{45} \approx 3.5 \cdot 10^{13}$ at $n = 10$. The hypothesis is not vacuous: the uniform weights $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$ at $n = 3$ fail it at $d = 1$ (`control_moments_not_vacuous`).

4. Exact values

Proposition 4.1 (the source's Table 2, exactly)**.**

$$S_2 = 1, \quad S_3 = \frac{5}{4}, \quad S_4 = \frac{239}{108}, \quad S_5 = \frac{7781}{1296}, \quad S_6 = \frac{682297}{25920}, \quad S_7 = \frac{451609897709}{2332800000}.$$

Rounded to four decimals these are 1.0000*,* 1.2500*,* 2.2130*,* 6.0039*,* 26.3232*,* 193.5913*, the entries of Table 2 of* [1].

Proof. For each $n \le 7$, Proposition 3.3 with the weights of Section 2 (the moment identities discharged by `norm_num`) rewrites S_n as the n^n-term rational sum, which is then evaluated by compiled code (`native_decide`; $7^7 = 823\,543$ terms at $n = 7$). Formally `netSum_two`, …, `netSum_seven`. The roundings are exact rational comparisons, done outside Lean; the value S_7 also carries the formal controls $193 < S_7 < 194$ and $S_7 \neq 193.5913$. ☐

Proposition 4.2 (the value at $n = 8$)**.**

$$S_8 = \frac{4206906030446363}{1714608000000} = 2453.567247\dots, \qquad S_8^* = \frac{82804531397275762929}{53876069761024000000} = 1.5369445\dots.$$

Proof. With integer weights $W_k = D\, u_k$ over a common denominator D, the sum of Proposition 3.3 becomes an integer sum,

$$D^n S_n = \sum_{b<n^n} \Big(\prod_{i<n} W_{d_i(b)}\Big) \prod_{j<n} \prod_{i<j} \big(d_i(b) + d_j(b)\big),$$

where $d_i(b)$ is the i-th base-n digit of b (`subSum_eq_flat`, proved for every n, W and $D \neq 0$ satisfying the moment identities). The right-hand side is evaluated by a loop in a module that imports nothing, compiled to a shared library, for $n = 8$, $D = 120960$:

$$120960^8\, S_8 = 11244293959343829959032815232273236319272960 0$$

(`Flat.sum_eight`; $8^8 = 16\,777\,216$ terms), and likewise

$$60480^7\, S_7 = 5730182083043782168832569786761216 00$$

(`Flat.sum_seven`), which re-derives S_7 by a second route (`netSum_seven'`). Dividing gives `netSum_eight`, and $S_8^* = S_8/c_8$ with $c_8 = 2^{28}(2/9)^8$ is `normSum_eight`. □

Remark 4.3 (values to $n = 14$, outside the formal development). By Proposition 3.1 the summand of (6) depends only on the score sequence, so $S_n = \sum_\sigma m_n(\sigma) \prod_i \frac{1}{1+\sigma_i}$, where $m_n(\sigma)$ is the number of labelled tournaments with score sequence σ. A dynamic program that adds one vertex at a time and carries the distribution of the sorted score sequence (as a histogram of scores) computes m_n for all σ; it was checked at every level against $\sum_\sigma m_n(\sigma) = 2^{\ell_n}$ and against the number of score sequences $1, 2, 4, 9, 22, 59, 167, 490, 1486, 4639, 14805, 48107, 158808$ for $n = 2, \ldots, 14$ (OEIS A000571 [9]). Three independent implementations, in exact rational arithmetic, agree with each other, with Propositions 4.1 and 4.2 for $n \le 8$, and with a direct summation of (3) over all 2^{ℓ_n} networks for $n \le 7$; each reaches $n = 14$ in under ten processor-minutes. Table 1 lists the values. These six further values are computations, not theorems of the formal development.

n	$S_n = \mathbb{E}(s_n)$ exact	S_n	$S_n^* = \mathbb{E}(s_n^*)$
2	1	1.000000	1.125000
3	5/4	1.250000	1.250000
4	239/108	2.212963	1.350685
5	7781/1296	6.003858	1.424744
6	682297/25920	26.323187	1.476715
7	451609897709/2332800000	193.591348	1.512432
8	4206906030446363/1714608000000	2453.567247	1.536945
9	459371836096470610907/8401579200000000	54676.844098	1.554009
10	2031290640549626638790937149/933684299654400000000	2175564.739925	1.566207
11	displayed below	156433295.833289	1.575227
12	displayed below	$2.0524231947 \cdot 10^{10}$	1.582149
13	displayed below	$4.9524610954 \cdot 10^{12}$	1.587654
14	displayed below	$2.2123907839 \cdot 10^{15}$	1.592172

TABLE 1. Exact values of $S_n = \mathbb{E}(s_n)$ and of the normalized $S_n^* = S_n/c_n$. Rows $n \le 8$ are theorems of the formal development (Propositions 4.1 and 4.2); rows $n \ge 9$ are the computations of Remark 4.3. Decimals are rounded.

The exact values for $11 \le n \le 14$, numerator over denominator:

S_{11} `1001549569801605165388742168021 / 6402406626201600000000`
S_{12} `9370413948759109931143321270684615462213 /`
`456553695792448235520000000000`
S_{13} `1228588783607991936885785075859342308045783725523461 /`
`24807641290528136906701946880000000000000`
S_{14} `136039629535246009875645224513099774796841107798302829195629 /`
`614898735454557420127452256665600000000000000`

Remark 4.4 (re-derivation from published data). OEIS A125032 (M. Fuller, 2006) tabulates $m_n(\sigma)$ for every score sequence with $n \le 10$ players; its entry states that the score sequences are “sorted by number of players and then lexicographically”, and its b-file has 2242 rows, one per score sequence with $n \le 10$. For $n \le 9$ these multiplicities are what MacMahon determined by hand in 1920 [3] and David recomputed for $n \le 8$ in 1959 [4], as Stockmeyer [8] describes; Alway carried the catalogue of score sequences to $n = 10$ by computer in 1962 [5]. Enumerating the score sequences of n players in lexicographic order by Landau’s condition [6]—weakly increasing σ with $\sum_{i \le k} \sigma_i \ge \binom{k}{2}$ for all k and equality at $k = n$—and pairing them with the rows of the b-file gives S_n for every $n \le 10$ by one line of arithmetic, in agreement with Table 1; independently, the multiset of multiplicities produced by the recursion of Remark 4.3 coincides with the b-file's rows for every $n \le 10$. This is why the values, including S_8, are graded here as routine: unprinted, but implicit in published data—MacMahon's 1920 table for $n \le 9$, and the b-file for $n \le 10$.

Remark 4.5 (cost, a measurement). As recorded when the development was written, on one machine: the $n = 7$ sum has 823 543 terms and takes 52.7 s in Lean's interpreter, 5.6 s (about 2 s of it start-up) when the evaluator is compiled to a shared library and loaded with `--load-dynlib`, and 200 s through *Mathlib*'s `Finset` and rational arithmetic (`Values.lean`); the $n = 8$ sum, 16 777 216 terms, takes 98 s compiled. Extrapolated, $n = 9$ ($9^9 \approx 3.9 \cdot 10^8$ terms) is about an hour and $n = 10$ about twenty hours by this route, which is why the formal values stop at $n = 8$; past that point the right formal route is a proved score-sequence recursion, not the cubature. These are timings, not theorems.

5. The normalized sequence and where its limit must be

Table 1 shows S_n^* increasing through every value we know, and every value below $\sqrt{e} = 1.6487212\ldots$, the *lower* endpoint of the source's bracket (5); the source's Table 3 shows the same for $n \le 7$, without exact values.

Proposition 5.1 (the seven known values)**.** *The exact values*

$$S_2^* = \tfrac{9}{8}, \qquad S_3^* = \tfrac{5}{4}, \qquad S_4^* = \tfrac{149375}{110592}, \qquad S_5^* = \tfrac{23343}{16384}, \qquad S_6^* = \tfrac{80271559753}{54358179840},$$
$$S_7^* = \tfrac{451609897709}{298598400000}, \qquad S_8^* = \tfrac{82804531397275762929}{53876069761024000000}$$

satisfy $S_2^* < S_3^* < \cdots < S_8^*$*, and* $S_n^* < \sqrt{e}$ *for each* $n \in \{2, \ldots, 8\}$*.*

Proof. The rationals follow from Propositions 4.1 and 4.2 and c_n; the strict inequalities between consecutive values are rational comparisons (`normSum_strictMono_le_seven`, `normSum_seven_lt_eight`). For $S_n^* < \sqrt{e}$ it suffices that $(S_n^*)^2 < 2.7182818283 < e$, the last inequality being *Mathlib*'s `Real.exp_one_gt_d9` (`normSum_lt_sqrt_exp_one`, `normSum_eight_lt_sqrt_exp_one`). ☐

Theorem 5.2 (the limit question reduces to one inequality)**.** (a) *Let* $(a_n)_{n\in\mathbb{N}}$ *be a real sequence and* $L, m \in \mathbb{R}$*. If* $a_n \le L$ *for every* n*,* $m \le a_n$ *for every* n*, and* $L \le \liminf_{n\to\infty} a_n$*, then* $a_n \to L$*.*

(b) *If* $S_n^* \le \sqrt{e}$ *for every* n *and* $\sqrt{e} \le \liminf_{n\to\infty} S_n^*$*, then* $S_n^* \to \sqrt{e}$*.*

Consequently, since the source's Theorem 4.5 provides $\sqrt{e} \le \liminf \mathbb{E}(s_n^*)$*: if* $\mathbb{E}(s_n^*) \le \sqrt{e}$ *for every* n*, then* $\mathbb{E}(s_n^*)$ *converges and its limit is* $\sqrt{e}$*.*

Proof. (a) The upper bound gives $\limsup a_n \le L$ (the sequence is bounded below, so its limsup is the usual one); with $\liminf a_n \le \limsup a_n$ and $L \le \liminf a_n$ both liminf and limsup equal L, and a bounded sequence whose liminf and limsup agree converges to their common value (`tendsto_of_le_of_le_liminf`, via *Mathlib*'s `tendsto_of_liminf_eq_limsup`). (b) is (a) with $a_n = S_n^*$, $L = \sqrt{e}$ and $m = 0$, the lower bound being $S_n^* \ge 0$ as a sum of reciprocals of positive integers divided by $c_n > 0$ (`normSum_tendsto_sqrt_exp_one_of_le`). The final sentence substitutes the source's Theorem 4.5 for the liminf hypothesis and Theorem 4.2 for $S_n^* = \mathbb{E}(s_n^*)$; neither is formalised, and the formal statement is exactly (b). ☐

Corollary 5.3 (the monotone form)**.** (a) *Let* (a_n) *be monotone with* $a_n \le L$ *for every* n *and* $L \le \liminf a_n$*. Then* $a_n \to L$*.*

(b) *If* (S_n^*) *is monotone,* $S_n^* \le \sqrt{e}$ *for every* n*, and* $\sqrt{e} \le \liminf S_n^*$*, then* $S_n^* \to \sqrt{e}$*.*

Proof. (a) A monotone sequence bounded above converges to its supremum, which is $\le L$ by the bound and $\ge L$ because the limit equals the liminf (`tendsto_of_monotone_of_le_liminf`, via `tendsto_atTop_ciSup`); (b) is the instance (`normSum_tendsto_sqrt_exp_one_of_monotone`). Alternatively, (a) follows from Theorem 5.2(a) with $m = a_0$: a monotone sequence bounded above by L satisfies its hypotheses, which is the sense in which Theorem 5.2 is the sharper statement. ☐

Remark 5.4 (what is proved and what is not). Section 6 of [1] offers monotonicity of $\mathbb{E}(s_n^*)$ as the condition to establish: with the variance bound of its Theorem 5.4 a monotone (hence, by Theorem 4.4, convergent) sequence of expectations gives L^2 convergence of s_n^* to a constant. The source does not say which constant. Corollary 5.3 says that if that monotone sequence also stays below $\sqrt{e}$—as all seven known values do—the constant is $\sqrt{e}$; Theorem 5.2 says monotonicity is not needed for this: the inequality $\mathbb{E}(s_n^*) \le \sqrt{e}$ alone, with the source's Theorem 4.5, gives $\mathbb{E}(s_n^*) \to \sqrt{e}$, and then its Theorem 5.4 gives $s_n^* \to \sqrt{e}$ in L^2. What is proved here is the conditional statement and the seven instances of the inequality. The inequality for all n, and hence the convergence, is *not* proved; the limit $\sqrt{e}$ is a conjecture supported by Table 1 and Table 2. Note the direction: the source's Table 3 brackets $\mathbb{E}(s_n^*)$ between a lower bound tending to $\sqrt{e}$ from below and an upper bound tending to e; the data say the sequence tracks the *lower* bound and that the upper half of $[\sqrt{e}, e]$ is empty.

n	2	3	4	5	6	7	8	9	10	11	12	13	14
$\sqrt{e} - S_n^*$	.5237	.3987	.2980	.2240	.1720	.1363	.1118	.0947	.0825	.0735	.0666	.0611	.0565
$n(\sqrt{e} - S_n^*)$	1.047	1.196	1.192	1.120	1.032	.954	.894	.852	.825	.808	.799	.794	.792

Table 2. The margin below $\sqrt{e}$ (rounded). The last row is decreasing from $n = 3$ on and flattens near 0.79: the data are consistent with $\sqrt{e} - \mathbb{E}(s_n^*) \asymp c/n$. Columns $n \ge 9$ are computations (Remark 4.3), not theorems.

Remark 5.5 (a heuristic for $\sqrt{e}$; not a proof, outside the formal development). Write $\mu = (n+1)/2$. Each link awards exactly one seniority point, so $\sum_i X_{i,n}(g) = n + \ell_n = n\mu$ for *every* network: the $X_{i,n}$ have constant mean μ under the uniform network and are negatively correlated, and marginally $X_{i,n} - 1 \sim \mathrm{Binom}(n-1, \frac{1}{2})$, as the source uses in its Theorem 4.3. (By the arithmetic–geometric mean inequality this already gives $X_n(g) \le \mu^n$, hence $S_n^* \ge 1$ for every n, a weaker bound than the source's.) Writing $X_{i,n} = \mu(1 + \varepsilon_i)$ one has $\sum_i \varepsilon_i = 0$, $\mathbb{E}[\varepsilon_i^2] = \frac{(n-1)/4}{\mu^2} \approx \frac{1}{n}$, and

$$\mathbb{E}(s_n^*) = \mathbb{E}\Big[\prod_i \frac{\mu}{X_{i,n}}\Big] = \mathbb{E}\Big[\exp\Big(-\sum_i \log(1+\varepsilon_i)\Big)\Big] \approx \mathbb{E}\Big[\exp\Big(\tfrac{1}{2}\sum_i \varepsilon_i^2\Big)\Big] \;\to\; e^{1/2},$$

because $\sum_i \mathbb{E}[\varepsilon_i^2] \to 1$ and $\sum_i \varepsilon_i^2$ concentrates (reversing one link moves two scores by one, so $\sum_i \varepsilon_i^2$ changes by $O(1/n)$, and bounded differences over the ℓ_n links give fluctuations of order $n^{-1/2}$). This says that the source's Jensen lower bound, whose limit is $\sqrt{e}$, should be asymptotically tight, and it is consistent with the observed $\sqrt{e} - \mathbb{E}(s_n^*) \asymp c/n$. Turning it into a proof of $\mathbb{E}(s_n^*) \le \sqrt{e}$, or of the limit, needs a bound on the third-order remainder of $\log(1+\varepsilon)$ and an exponential-moment bound for $\sum_i \varepsilon_i^2$ under the score distribution of a uniform random tournament; neither is attempted here.

6. Two corrections to the source

Both are small; both were checked against the compiled version 1 e-print; we record them because the source has no citers yet and an author would want them before a journal version.

Proposition 6.1 (the number of networks)**.** *For every n, $|G_n| = 2^{n(n-1)/2}$. In particular $|G_7| = 2^{21} = 2\,097\,152 \ne 2^{28}$.*

Proof. The map $g \mapsto \{(i,j) \in P_n : ij \in g\}$ is a bijection from networks (symmetric irreflexive Boolean matrices) onto the subsets of P_n, and $|P_n| = n(n-1)/2$ by counting the pairs (i,j) with $i < j$ fibrewise over j (`pairs_card`, `networks_card`, `networks_card_seven`; at $n = 3$ the predicate cuts the 2^9 Boolean matrices down to 8 networks, `control_networks_three`). □

The introduction of [1], page 2 of the compiled e-print, reads: "When $n = 7$, the seniority degree has to be calculated for a total of 2^{28} networks." By Proposition 6.1 the number is 2^{21}; 2^{28} is the count at $n = 8$. The source's Section 4.1 states the correct general formula $2^{n(n-1)/2}$, and the last column of its Table 3 gives $c_7 = 128 = 2^{21}(1/4)^7$, so the sentence contradicts the paper's own text. The formal statement is Proposition 6.1; that the sentence is a slip rather than a different count is our reading of the source.

Proposition 6.2 (two cells of Table 3)**.** (a) $S_5^* = \dfrac{23343}{16384} = 1.42474365\ldots$, *and* $1.42465 < S_5^* < 1.42475$*; so the four-decimal rounding of* $\mathbb{E}(s_5^*)$ *is* 1.4247*.*

(b) $\left(1+\frac{1}{7}\right)^7\left(1-\frac{1}{2^7}\right)^7 = 2.41045977\ldots$ *lies strictly between* 2.41045 *and* 2.41055*; so the four-decimal rounding of the source's upper bound at* $n = 7$ *is* 2.4105*.*

Proof. (a) $S_5^* = S_5/c_5$ with $S_5 = 7781/1296$ (Proposition 4.1) and $c_5 = 2^{10}(1/3)^5 = 1024/243$, and the two comparisons are rational arithmetic (`normSum_five`). (b) is rational arithmetic on $(8/7)^7(127/128)^7$ (`upper_bound_seven`). □

Table 3 of [1] prints $\mathbb{E}(s_5^*) = 1.4248$ and the $n = 7$ upper bound 2.4112. The first digit is what one obtains by dividing the *rounded* entries of its Tables 2 and 3, $6.0039/4.2140 = 1.42475\ldots$, which rounds up; the second is off by $7 \cdot 10^{-4}$ and is not a rounding artefact. Every other cell of Tables 2 and 3 at $n \leq 7$ was checked against exact values, with the two exceptions of the next remark.

Remark 6.3 (two further cells, and the last row; not formalised). In the lower-bound column of Table 3, the cell at $n = 2$ prints 1.1176 where $e^{1/9} = 1.11751906\ldots$ rounds to 1.1175, and the cell at $n = 6$ prints 1.3582 where $e^{15/49} = 1.35814860\ldots$ rounds to 1.3581. These involve exp and were checked in high-precision arithmetic, not in Lean. We also note that the table's limit row prints the upper limit as 2.7182, a truncation of $e = 2.71828\ldots$ rather than its rounding 2.7183. All the remaining cells—the normalized values at $n \neq 5$, the lower bounds at $n \in \{3, 4, 5, 7\}$, the upper bounds at $n \leq 6$, the constants c_n, and the limit $\sqrt{e}$ printed as 1.6487—round correctly. Every cell of Tables 2 and 3 of the compiled version 1 e-print, the limit row included, was recomputed independently for this paper: the two cells of Proposition 6.2, the two named here, and the limit row's 2.7182 are the only entries that are not the correct four-decimal rounding.

7. Verification

Propositions 3.1, 4.1, 4.2, 5.1, 6.1 and 6.2, Lemma 3.2, Proposition 3.3, Theorem 5.2 and Corollary 5.3 rest on declarations formally verified in Lean 4 against *Mathlib* [18, 19] (Lean 4.32.0, *Mathlib* at its `v4.32.0` tag), whose statements are reproduced verbatim in Appendix A. The development is eight modules, 1081 lines in all. `Defs` (300 lines) defines networks, the seniority degree, the sum S_n (`netSum`), tournaments, the pair set and the subset coding, and proves Proposition 3.1; `Cubature` (111) proves Lemma 3.2 and Proposition 3.3; `Values` (86) proves Proposition 4.1; `Flat` (70) is the import-free integer evaluator, `FlatCheck` (32) its two evaluations, and `Eight` (166) the bridge `subSum_eq_flat` from S_n to that evaluator together with Proposition 4.2; `Controls` (223) proves Propositions 6.1, 6.2 and 5.1 for $n \leq 7$, Corollary 5.3, and the negative controls; `Limit` (93) proves the $n = 8$ parts of Propositions 4.2 and 5.1, and Theorem 5.2.

The axiom print (`#print axioms`) of every declaration named in this paper was re-run read-only for it. The identities and the limit statements—`netSum_eq_tourSum`, `netSum_eq_subSum`, `prod_add_eq_sum_odeg`, `subSum_eq_cubature`, `subSum_eq_flat`, `pairs_card`, `networks_card`, `networks_card_seven`, `upper_bound_seven`, `tendsto_of_monotone_of_le_liminf`, `normSum_tendsto_sqrt_exp_one_of_monotone`, `tendsto_of_le_of_le_liminf`, `normSum_tendsto_sqrt_exp_one_of_le`—depend on `propext`, `Classical.choice` and `Quot.sound` only. The exact values are finite evaluations by compiled code: `netSum_two`, …, `netSum_seven` each carry, besides the three, one named axiom of the form `netSum_k._native.native_decide.ax_1_2`, and `Flat.sum_seven` and `Flat.sum_eight` carry only their own `_native.native_decide.ax_1_1`; every statement that uses a value—`netSum_eight`, `normSum_five`, `normSum_eight`, the monotonicity and $\sqrt{e}$ comparisons, and three of the controls—inherits exactly the axioms of the values it uses. No declaration depends on `sorryAx`. The compiled evaluations are the claims that rest on exhaustive computation: n^n terms for each $n \leq 7$ in `Values`, and 7^7 and 8^8 terms in `FlatCheck`, the latter run with the evaluator compiled to a shared library and loaded into the checker. Negative controls are included and machine-checked: $S_7 = 193$ and $S_7 = 194$ are refuted and $193 < S_7 < 194$ shown (`control_seven_bracket`);

the printed 193.5913 is shown not to be the exact value (`control_printed_not_exact`); the cubature hypothesis is shown not to be vacuous (`control_moments_not_vacuous`); the network predicate is shown to select 8 of 512 matrices at $n = 3$ (`control_networks_three`); and the tournament form is shown to take the value 5/4 at $n = 3$ (`control_tourSum_three`). Outside the formal development, and labelled as such where they occur, are the values for $9 \leq n \leq 14$ (Remark 4.3), their re-derivation from OEIS A125032 (Remark 4.4), the timings (Remark 4.5), the exp-valued cells and the limit row of Remark 6.3, the heuristic of Remark 5.5, and all decimal roundings. Every number in this paper was recomputed for it in exact rational arithmetic (Python `fractions`, with `decimal` at 80 digits for exp and $\sqrt{e}$), by code written independently of the development's own tools: the source's sum (3) directly for $n \leq 7$, the tournament form for $n \leq 7$, the cubature sum with Vandermonde-solved weights for $n \leq 7$, exact monomial integration of the source's integral formula (1) for $n \leq 6$, and the score-sequence recursion to $n = 14$; all of them agree, and the last also reproduces the multiplicities of the A125032 b-file for $n \leq 10$. As recorded when the development was written, `Values` checks in 254 s and `Eight` in 145 s with a peak resident set of about 7 GB (the cost of importing *Mathlib*); `FlatCheck` imports only the *Mathlib*-free `Flat`, peaks at 0.8 GB and takes 98 s with the shared library; the other modules check in about ten seconds each. The development is currently in a private repository: repository reference to be added at submission.

Appendix A. The formal statements

The Lean statements corresponding to the results above, verbatim (statements only; proofs, docstrings and attributes omitted). `Fin n` is the n-element index type, `Bool` the Booleans, `Finset` a finite set, `univ.filter p` the finite set of elements satisfying `p`, `.card` its cardinality, `∑`/`∏` finite sums and products, `ℚ` and `ℝ` the rationals and reals, and `x⁻¹` the inverse. The namespace `PairwiseStableCount` is omitted; `n` is an implicit natural number where it is not bound.

Networks, the seniority degree, tournaments (Proposition 3.1).

```
def IsNetwork (g : Fin n → Fin n → Bool) : Prop :=
  (∀ i j, g i j = g j i) ∧ ∀ i, g i i = false

def networks (n : ℕ) : Finset (Fin n → Fin n → Bool) := univ.filter IsNetwork

def senior (g : Fin n → Fin n → Bool) (i : Fin n) : ℕ :=
  (univ.filter (fun j => i < j ∧ g i j = true)).card

def missing (g : Fin n → Fin n → Bool) (i : Fin n) : ℕ :=
  (univ.filter (fun j => j < i ∧ g i j = false)).card

def senDeg (g : Fin n → Fin n → Bool) (i : Fin n) : ℕ := 1 + senior g i + missing g i

def senDegProd (g : Fin n → Fin n → Bool) : ℕ := ∏ i, senDeg g i

def netSum (n : ℕ) : ℚ := ∑ g ∈ networks n, ((senDegProd g : ℚ))⁻¹

def IsTournament (T : Fin n → Fin n → Bool) : Prop :=
  (∀ i, T i i = false) ∧ ∀ i j, i ≠ j → T i j = !T j i

def tournaments (n : ℕ) : Finset (Fin n → Fin n → Bool) := univ.filter IsTournament

def outdeg (T : Fin n → Fin n → Bool) (i : Fin n) : ℕ :=
  (univ.filter (fun j => T i j = true)).card

def tourSum (n : ℕ) : ℚ := ∑ T ∈ tournaments n, (∏ i, ((1 + outdeg T i : ℕ) : ℚ))⁻¹
```

```
theorem netSum_eq_tourSum (n : ℕ) : netSum n = tourSum n

def pairs (n : ℕ) : Finset (Fin n × Fin n) := univ.filter (fun p => p.1 < p.2)

def odeg (S : Finset (Fin n × Fin n)) (i : Fin n) : ℕ :=
  (S.filter (fun p => p.1 = i)).card + ((pairs n \ S).filter (fun p => p.2 = i)).card

def subSum (n : ℕ) : ℚ :=
  ∑ S ∈ (pairs n).powerset, (∏ i, ((1 + odeg S i : ℕ) : ℚ))⁻¹

theorem netSum_eq_subSum (n : ℕ) : netSum n = subSum n
```

The generating identity and the cubature (Lemma 3.2, Proposition 3.3).

```
theorem prod_add_eq_sum_odeg (n : ℕ) (x : Fin n → ℚ) :
    ∏ p ∈ pairs n, (x p.1 + x p.2) = ∑ S ∈ (pairs n).powerset, ∏ i, x i ^ odeg S i

theorem subSum_eq_cubature (n : ℕ) (u : ℕ → ℚ)
    (hu : ∀ d : ℕ, d < n → ∑ k ∈ range n, u k * (k : ℚ) ^ d = ((d : ℚ) + 1)⁻¹) :
    subSum n = ∑ κ : Fin n → Fin n, (∏ i, u (κ i)) *
      ∏ p ∈ pairs n, (((κ p.1 : ℕ) : ℚ) + ((κ p.2 : ℕ) : ℚ))
```

Exact values (Propositions 4.1 and 4.2).

```
theorem netSum_two : netSum 2 = 1
theorem netSum_three : netSum 3 = 5/4
theorem netSum_four : netSum 4 = 239/108
theorem netSum_five : netSum 5 = 7781/1296
theorem netSum_six : netSum 6 = 682297/25920
theorem netSum_seven : netSum 7 = 451609897709/2332800000

-- the import-free evaluator (namespace Flat)
def digits (n : Nat) : Nat → Nat → List Nat
  | 0, _ => []
  | (k + 1), b => (b % n) :: digits n k (b / n)
def nthN : List Nat → Nat → Nat
  | [], _ => 0
  | (x :: _), 0 => x
  | (_ :: t), (k + 1) => nthN t k
def nthI : List Int → Nat → Int
  | [], _ => 0
  | (x :: _), 0 => x
  | (_ :: t), (k + 1) => nthI t k
def prodN (f : Nat → Nat) : Nat → Nat
  | 0 => 1
  | (k + 1) => prodN f k * f k
def prodI (f : Nat → Int) : Nat → Int
  | 0 => 1
  | (k + 1) => prodI f k * f k
def pairProd (d : Nat → Nat) (n : Nat) : Nat :=
  prodN (fun j => prodN (fun i => d i + d j) j) n
def term (n : Nat) (W : List Int) (b : Nat) : Int :=
  let d := nthN (digits n n b)
  prodI (fun i => nthI W (d i)) n * (pairProd d n : Int)
```

```
def sumAcc (n : Nat) (W : List Int) : Nat → Int → Int
  | 0, acc => acc
  | (k + 1), acc => sumAcc n W k (acc + term n W k)
def w7I : List Int := [19087, 65112, -46461, 37504, -20211, 6312, -863]
def w8I : List Int := [36799, 139849, -121797, 123133, -88547, 41499, -11351, 1375]

theorem sum_seven : sumAcc 7 w7I (7 ^ 7) 0 = 5730182083043782168832569786761216OO
theorem sum_eight :
    sumAcc 8 w8I (8 ^ 8) 0 = 112442939593438299590328152322732363192729600

-- the bridge, and the value at n = 8
theorem subSum_eq_flat (n : ℕ) (W : List ℤ) (D : ℚ) (hD : D ≠ 0)
    (hu : ∀ d : ℕ, d < n → ∑ k ∈ range n, ((Flat.nthI W k : ℚ) / D) * (k : ℚ) ^ d
      = ((d : ℚ) + 1)⁻¹) :
    subSum n = ((Flat.sumAcc n W (n ^ n) 0 : ℤ) : ℚ) / D ^ n

theorem netSum_eight : netSum 8 = 4206906030446363/1714608000000
theorem netSum_seven' : netSum 7 = 451609897709/2332800000
```

Counting, the normalized sequence, the corrections (Propositions 6.1, 6.2, 5.1).

```
theorem pairs_card (n : ℕ) : (pairs n).card = n * (n - 1) / 2

theorem networks_card (n : ℕ) : (networks n).card = 2 ^ (n * (n - 1) / 2)

theorem networks_card_seven : (networks 7).card = 2097152 ∧ (networks 7).card ≠ 2 ^ 28

def cNorm (n : ℕ) : ℚ := 2 ^ (n * (n - 1) / 2) * (2 / (n + 1)) ^ n

def normSum (n : ℕ) : ℚ := netSum n / cNorm n

theorem normSum_five : normSum 5 = 23343/16384 ∧
    normSum 5 < 14247.5/10000 ∧ 14246.5/10000 < normSum 5

theorem upper_bound_seven :
    ((1 + 1/7 : ℚ))^7 * ((1 - 1/2^7 : ℚ))^7 < 24105.5/10000 ∧
    24104.5/10000 < ((1 + 1/7 : ℚ))^7 * ((1 - 1/2^7 : ℚ))^7

theorem normSum_eight : normSum 8 = 82804531397275762929/53876069761024000000

theorem normSum_strictMono_le_seven :
    normSum 2 < normSum 3 ∧ normSum 3 < normSum 4 ∧ normSum 4 < normSum 5 ∧
    normSum 5 < normSum 6 ∧ normSum 6 < normSum 7

theorem normSum_seven_lt_eight : normSum 7 < normSum 8

theorem normSum_lt_sqrt_exp_one :
    ((normSum 2 : ℚ) : ℝ) < Real.sqrt (Real.exp 1) ∧
    ((normSum 3 : ℚ) : ℝ) < Real.sqrt (Real.exp 1) ∧
    ((normSum 4 : ℚ) : ℝ) < Real.sqrt (Real.exp 1) ∧
    ((normSum 5 : ℚ) : ℝ) < Real.sqrt (Real.exp 1) ∧
    ((normSum 6 : ℚ) : ℝ) < Real.sqrt (Real.exp 1) ∧
    ((normSum 7 : ℚ) : ℝ) < Real.sqrt (Real.exp 1)
```

```
theorem normSum_eight_lt_sqrt_exp_one :
    ((normSum 8 : ℚ) : ℝ) < Real.sqrt (Real.exp 1)
```

The limit (Theorem 5.2, Corollary 5.3).

```
theorem tendsto_of_le_of_le_liminf {a : ℕ → ℝ} {L m : ℝ} (hle : ∀ n, a n ≤ L)
    (hm : ∀ n, m ≤ a n) (hliminf : L ≤ liminf a atTop) :
    Tendsto a atTop (nhds L)

theorem normSum_tendsto_sqrt_exp_one_of_le
    (hle : ∀ n, ((normSum n : ℚ) : ℝ) ≤ Real.sqrt (Real.exp 1))
    (hliminf : Real.sqrt (Real.exp 1) ≤
      liminf (fun n => ((normSum n : ℚ) : ℝ)) atTop) :
    Tendsto (fun n => ((normSum n : ℚ) : ℝ)) atTop (nhds (Real.sqrt (Real.exp 1)))

theorem tendsto_of_monotone_of_le_liminf {a : ℕ → ℝ} {L : ℝ} (hmono : Monotone a)
    (hle : ∀ n, a n ≤ L) (hliminf : L ≤ liminf a atTop) :
    Tendsto a atTop (nhds L)

theorem normSum_tendsto_sqrt_exp_one_of_monotone
    (hmono : Monotone (fun n => ((normSum n : ℚ) : ℝ)))
    (hle : ∀ n, ((normSum n : ℚ) : ℝ) ≤ Real.sqrt (Real.exp 1))
    (hliminf : Real.sqrt (Real.exp 1) ≤
      Filter.liminf (fun n => ((normSum n : ℚ) : ℝ)) Filter.atTop) :
    Filter.Tendsto (fun n => ((normSum n : ℚ) : ℝ)) Filter.atTop
      (nhds (Real.sqrt (Real.exp 1)))
```

Controls (Section 7).

```
theorem control_seven_bracket : ¬ (netSum 7 = 194) ∧ ¬ (netSum 7 = 193) ∧
    193 < netSum 7 ∧ netSum 7 < 194
theorem control_printed_not_exact : netSum 7 ≠ 1935913/10000
theorem control_moments_not_vacuous :
    ¬ (∀ d : ℕ, d < 3 → ∑ k ∈ range 3, (fun _ : ℕ => (1/3 : ℚ)) k * (k : ℚ) ^ d
        = ((d : ℚ) + 1)⁻¹)
theorem control_networks_three : (networks 3).card = 8 ∧
    Fintype.card (Fin 3 → Fin 3 → Bool) = 512
theorem control_tourSum_three : tourSum 3 = 5/4
```

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that MacMahon expanded $\prod_{i \neq k}(\alpha_i + \alpha_k)$ and read off the coefficient of each monomial as the number of tournaments with the corresponding score vector. Stockmeyer abbreviates the journal "Quart. J. Math."; volume, year and pages agree.

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