THE RECURRENCE COEFFICIENTS OF THE OSCILLATORY GEGENBAUER WEIGHT $(1-x^2)^{\lambda-1/2}e^{i\zeta x}$: THE CLOSED-FORM CONJECTURE OF LYU AND ZHOU CERTIFIED FOR β_n TO $n=13$ AND α_n TO $n=12$, AND THE DEGREE CONJECTURE OF MILOVANOVIĆ, CVETKOVIĆ AND MARJANOVIĆ FOR $n \le 16$

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ABSTRACT. For $\lambda > -1/2$ rational and ζ a nonzero zero of the Bessel function $J_{\lambda-1}$, the monic orthogonal polynomials for the weight $(1-x^2)^{\lambda-1/2}e^{i\zeta x}$ on $[-1,1]$ satisfy $xP_n = P_{n+1}+i\alpha_n P_n+\beta_n P_{n-1}$. Lyu and Zhou (arXiv:2607.19797) derive a first-order coupled system of difference equations for (α_n,β_n), iterate it to $n \le 9$, and conjecture closed forms in terms of monic auxiliary polynomials $X_j(\zeta^2)$ of degree j; the degree pattern $\lfloor n^2/4 \rfloor$ for the underlying Hankel determinant is a 2008 conjecture of Milovanović, Cvetković and Marjanović, who printed coefficients for $n \le 2$ only. We observe that the system is a rational recursion over $\mathbb{Q}(\lambda,\zeta)$ — the Bessel factor common to all moments cancels, and β_0, the only place it could enter, is never read — so that the conjecture is a finite question at each n. Its four displayed formulas collapse to two, and substituting these into the source's own two compatibility conditions yields two bilinear polynomial identities in the X_j alone. Certifying those identities in exact integer arithmetic with λ symbolic, one of them to $n=15$ and the other to $n=12$, and composing with the source's derivation of its system and with the uniqueness of its solution, we prove that every solution of the source's Theorem 2.2 has α_n for $n \le 12$ and β_n for $1 \le n \le 13$ equal to the conjectured closed forms, with explicit monic X_j of the conjectured degrees — against $n \le 9$ in the source and $n \le 2$ in 2008. Every X_j occurring for $n \le 16$ is certified monic of degree exactly j, which is the 2008 degree conjecture on that range. We also observe that the source's two leading-coefficient conjectures are a single formula $H_n = (-1)^{n(n-1)/2}\gamma_n X_{\lfloor n^2/4 \rfloor}$ (verified for $n \le 12$ at three values of λ, not formalised), and we correct a misprint in the 2008 paper. No closed form in n for the X_j themselves is found, and we say why none should be expected. Every certified statement is verified in Lean 4 against *Mathlib*.

1. INTRODUCTION

1.1. **The objects.** Let $\lambda \in \mathbb{Q}$, $\lambda > -1/2$, and let $\zeta \in \mathbb{R} \setminus \{0\}$ be a zero of the Bessel function of the first kind $J_{\lambda-1}$. The *oscillatory Gegenbauer weight* is

$$w(x) = (1-x^2)^{\lambda-1/2}\exp(i\zeta x), \qquad x \in [-1,1]. \tag{1}$$

Milovanović, Cvetković and Marjanović [2] proved (their Theorem 2.2, p. 51) that monic orthogonal polynomials P_n for the (complex-valued, non-Hermitian) functional $\int_{-1}^{1} p(x)w(x)\,dx$ exist for these parameters; their earlier existence result for positive λ is [3]. The polynomials satisfy the three-term recurrence

$$xP_n(x) = P_{n+1}(x) + i\alpha_n P_n(x) + \beta_n P_{n-1}(x), \qquad n \ge 0, \tag{2}$$

with $P_{-1} := 0$, and the question of the two sources is the same: *what are α_n and β_n as functions of λ and ζ?*

The moments are Bessel combinations. By [2] Theorem 2.1 (p. 50),

$$\mu_k = \int_{-1}^{1} x^k w(x)\,dx = \frac{A}{(i\zeta)^k}\big(P_k^\lambda(\zeta)J_\lambda(\zeta) + Q_k^\lambda(\zeta)J_{\lambda-1}(\zeta)\big), \qquad A = (2/\zeta)^\lambda\sqrt{\pi}\,\Gamma(\lambda+1/2), \tag{3}$$

with P_k^λ, Q_k^λ polynomials in ζ with coefficients in $\mathbb{Q}(\lambda)$, generated by the four-term recurrence $y_{k+2} = -(k+2\lambda+1)y_{k+1} - \zeta^2 y_k - k\zeta^2 y_{k-1}$ from $P_0^\lambda = 1$, $P_1^\lambda = -2\lambda$, $P_2^\lambda = 2\lambda(2\lambda+1) - \zeta^2$. At a zero of $J_{\lambda-1}$

the second term dies, and [2] works with the Hankel determinant of the rational part,

$$H_n = \det\big(P^{\lambda}_{i+j}(\zeta)\big)_{0\le i,j\le n-1}, \qquad \beta_n = \frac{1}{(i\zeta)^2}\,\frac{H_{n+1}H_{n-1}}{H_n^2} \quad \text{([2] (2.5), p. 52).} \tag{4}$$

Its authors report (p. 53) that they could obtain the coefficients in symbolic form "for some reasonable small values of n (for $n \leqslant 2$ see Table 1)", and that "Increasing n, the complexity of expressions for α_n and β_n increases quite rapidly." From that data they conjecture, verbatim: "According to symbolic calculations we can conjecture that H_n is a polynomial in ζ^2 of degree m, … where $m = n^2/4$ for even n and $m = (n^2-1)/4$ for odd n", i.e. $\deg_{\zeta^2} H_n = \lfloor n^2/4 \rfloor$; writing $H_n = S_m(\zeta^2)$ they also conjecture a numerator $V_{n(n+1)/2}(\zeta^2)$ for α_n, so that (p. 53)

$$\beta_{2k} = -\frac{S_{k(k+1)}S_{k(k-1)}}{\zeta^2 S_{k^2}^2}, \quad \beta_{2k+1} = -\frac{S_{(k+1)^2}S_{k^2}}{\zeta^2 S_{k(k+1)}^2}, \quad \alpha_{2k} = \frac{V_{k(2k+1)}}{\zeta S_{k^2}S_{k(k+1)}^2}, \quad \alpha_{2k+1} = \frac{V_{(k+1)(2k+1)}}{\zeta S_{k(k+1)}S_{(k+1)^2}} \tag{5}$$

(all arguments ζ^2). The square in the third formula is a misprint; see Proposition 6.2.

Lyu and Zhou [1] take a different route. Using the ladder operators of the Chen–Ismail framework [4, 5] they derive (their Lemma 2.1, p. 4) the two auxiliary functions $A_n(z) = (i\zeta z - \zeta\alpha_n + 2(n+\lambda))/(1-z^2)$ and $B_n(z) = (nz + i\zeta\beta_n - \mathrm{p}(n))/(1-z^2)$, where $\mathrm{p}(n)$ is the sub-leading coefficient of P_n and $i\alpha_n = \mathrm{p}(n) - \mathrm{p}(n+1)$ (their (1.7)), insert them into the compatibility conditions (S_1) and (S_2') (their p. 3), and obtain a *first-order* coupled system.

Theorem ([1], Theorem 2.2, p. 4)**.** *For $n \ge 1$,*

$$\alpha_{n+1} = \frac{3(2n+1+2\lambda)}{2\zeta} - \alpha_n + \frac{2n+1+2\lambda}{2\zeta\beta_{n+1}}\Big[\beta_n - 1 + \alpha_n\Big(\frac{2n-1+2\lambda}{\zeta} - \alpha_n\Big)\Big], \tag{LZ 2.3}$$

$$\beta_{n+2} = \beta_n + \alpha_{n+1}^2 - \alpha_n^2 + \frac{\alpha_n(2n-1+2\lambda) - \alpha_{n+1}(2n+3+2\lambda)}{\zeta}, \tag{LZ 2.4}$$

with the initial conditions

$$\alpha_0 = \frac{2\lambda}{\zeta}, \quad \alpha_1 = \frac{2(\lambda+1)}{\zeta} - \frac{\zeta(2\lambda+1)}{\zeta^2 - 2\lambda}, \quad \beta_1 = 1 - \frac{2\lambda}{\zeta^2}, \quad \beta_2 = -\frac{2(2\lambda+1)\big(\zeta^4 - \zeta^2\lambda(2\lambda+5) + 4\lambda^2\big)}{\zeta^2(\zeta^2-2\lambda)^2}. \tag{LZ 2.5}$$

The three displays are numbered as in the source, with the prefix LZ, and are cited so throughout; the source's other equations are cited by their numbers in the text. Immediately after the theorem the source says, verbatim: "Note that β_0 can be arbitrary." and (Remark 2.3, p. 5) "Consequently, given $\{\alpha_0, \alpha_1, \beta_1, \beta_2\}$ as in (2.5), α_n and β_n for any n can be computed using (2.3)-(2.4), which are independent of the value of β_0." It then iterates (p. 5): "Iterating the difference equations (2.3) and (2.4) for $1 \le n \le 7$ yields α_i for $2 \le i \le 8$ and β_i for $3 \le i \le 9$." The results are its Table 1 (p. 6), written in terms of polynomials $X_j(\zeta^2)$ with the note that "$X_i(\cdot)$ denot[es] a monic polynomial of degree i"; the explicit $X_2, X_3, X_4, X_5, X_6, X_8, X_9$ are its Appendix C (p. 19). From Table 1 it proposes:

Conjecture ([1], Conjecture 2.5, p. 6)**.** *The recurrence coefficients α_n with $n \ge 4$ have the symbolic forms*

$$\alpha_{2k} = \frac{2(\lambda+2k)}{\zeta} + k\zeta\Big((2\lambda+2k-1)\frac{X_{k^2-1}}{X_{k^2}} - (2\lambda+2k+1)\frac{X_{k(k+1)-1}}{2X_{k(k+1)}}\Big),$$

$$\alpha_{2k+1} = \frac{2(\lambda+2k+1)}{\zeta} + (2\lambda+2k+1)\zeta\Big(k\frac{X_{k(k+1)-1}}{2X_{k(k+1)}} - (k+1)\frac{X_{k(k+2)}}{X_{(k+1)^2}}\Big),$$

for $k \ge 2$; and for $n \ge 1$,

$$\beta_{2k} = -2k(2\lambda+2k-1)\frac{X_{k(k-1)}X_{k(k+1)}}{\zeta^2 X_{k^2}^2}, \qquad \beta_{2k+1} = \frac{X_{k^2}X_{(k+1)^2}}{\zeta^2 X_{k(k+1)}^2},$$

which hold for $k \ge 0$ except for $k = 0$ in β_{2k}. Here $X_j(\cdot)$ is a monic polynomial of degree j for $j \ge 0$, with $X_0(\zeta^2) := 1$ and $X_1(\zeta^2) := \zeta^2 - 2\lambda$.

Its Remark 2.6 (pp. 6–7) identifies the S_j of [2] with the X_j for $j = 1, 2, 4, 6, 9, 12, 16, 20, 25$ (that is, H_n for $n \le 10$) and conjectures, for $k \ge 1$ except $k = 1$ in the first line,

(6)
$$S_{k^2} = (-1)^k 2^{k(k-1)} \prod_{j=1}^{k-1} \big(j(2\lambda+2j-1)\big)^{2(k-j)} X_{k^2}, \qquad S_{k(k+1)} = 2^{k^2} \prod_{j=1}^{k} \big(j(2\lambda+2j-1)\big)^{2(k-j)+1} X_{k(k+1)}.$$

Three questions are thus open in [1]: whether Conjecture 2.5 holds beyond $n = 9$; whether the X_j have the conjectured degrees (the 2008 conjecture); and what the leading coefficients (6) are in general.

1.2. **Nothing is transcendental.** The first thing to record is that the problem is one of exact rational arithmetic. At a zero of $J_{\lambda-1}$ every moment is $\mu_k = A(i\zeta)^{-k} P_k^{\lambda}(\zeta) J_\lambda(\zeta)$, so the transcendental factor $J_\lambda(\zeta)$ is common to all moments and cancels from every ratio of Hankel determinants, hence from every α_n and β_n with $n \ge 1$. The source's system makes this invisible: (LZ 2.3)–(LZ 2.5) is a recursion with coefficients in $\mathbb{Q}(\lambda, \zeta)$, and the one place a Bessel value could enter — $\beta_0 = \mu_0$, which [2] takes as $AJ_\lambda(\zeta)$ (p. 53) — is, as the source says, arbitrary and never read. Consequently α_n ($n \ge 0$) and β_n ($n \ge 1$) are rational functions of λ and ζ over $\mathbb{Q}$, computable exactly, and the conjecture is, at each n, a finite question. No Bessel enclosure is used anywhere below.

1.3. **What is proved here.** Write $d_n := \lfloor n^2/4 \rfloor$, so that $d_{2k} = k^2$ and $d_{2k+1} = k(k+1)$ are exactly the indices of the X_j in the denominators of the conjecture, and $d_n - 1$ the indices in the numerators of α_n. With two sequences $E_n := X_{d_n}$, $F_n := X_{d_n-1}$ and two explicit scalar sequences c_n, e_n (Section 2), the conjecture's four formulas are the two formulas

$$\beta_n = \frac{c_n E_{n-1} E_{n+1}}{\zeta^2 E_n^2}, \qquad \alpha_n = \frac{2(\lambda+n)}{\zeta} + \zeta(R_n - R_{n+1}), \qquad R_n := \frac{e_n F_n}{E_n}, \tag{7}$$

and substituting (7) into the source's own two intermediate equations — its (2.7), the consequence of (S_1), and its (2.8), the coefficient of z in (S_2') — gives two *polynomial* identities in the E's and F's, quadratic and quartic respectively (Proposition 2.1). Neither source writes them; they are what makes the conjecture checkable with λ symbolic. The results are then:

(i) (Theorem 4.2) explicit integer tables $\widehat{E}_n$ ($0 \le n \le 16$), $\widehat{F}_n$ ($2 \le n \le 16$) in $\mathbb{Z}[\lambda][\zeta^2]$ are produced, and the two identities are certified for them in exact integer arithmetic — the quadratic one for $1 \le n \le 15$, the quartic one for $1 \le n \le 12$. Composed with the source's own derivation of (LZ 2.3)–(LZ 2.4) and with the uniqueness of the solution of a first-order system, this gives: *every solution of the source's Theorem 2.2 has α_n for $n \le 12$ and β_n for $1 \le n \le 13$ equal to the closed forms* (7) *built from the tables*, λ symbolic; β_0 is not claimed. The source's range is $n \le 9$; that of [2] is $n \le 2$.

(ii) (Theorem 5.1) every $\widehat{E}_n$, $0 \le n \le 16$, has degree exactly d_n in ζ^2 with leading coefficient the constant ε_n, and every $\widehat{F}_n$, $2 \le n \le 16$, degree exactly $d_n - 1$ with leading coefficient φ_n; equivalently every X_j of (12) is monic of degree exactly j: the 2008 degree conjecture, on the range $n \le 16$, for the polynomials the conjecture is about.

(iii) (Observation 6.1, outside the formal development) the two lines of (6) are one formula, $H_n = (-1)^{n(n-1)/2} \gamma_n X_{d_n}$ with $\gamma_n = \prod_{m=1}^{n-1} c_m^{n-m}$, verified for $1 \le n \le 12$ at three values of λ directly against the determinant (4).

(iv) (Proposition 6.2) the exponent 2 in the α_{2k} formula of [2], p. 53, is wrong; the denominator is $\zeta S_{k^2} S_{k(k+1)}$.

(v) (Propositions 4.1, 4.5) an independent implementation of the source's system reproduces its Table 1, all seven polynomials of its Appendix C with their printed normalisations, and Table 1 of [2]; and the conjecture is certified on the source's own range $n \le 9$ from the source's own equations.

Item (i) is the main result. Two negative controls (Proposition 5.3) show the certificates are not vacuous.

1.4. **What is not claimed.** No proof of Conjecture 2.5, or of the degree conjecture, for all n is offered; the results are finite verifications, exact and with λ symbolic, on the stated ranges. The Hankel relation of Observation 6.1 is derived *from* the conjectured form of β_n and is therefore conditional in that direction; unconditionally it is only the finite verification recorded there, which is not machine-checked. And no closed form in n for the X_j themselves is found. None should be expected: $\deg_{\zeta^2} X_{d_n} = d_n \sim n^2/4$, and the degree in λ of $\widehat{E}_n$ grows like $3n^2/8$ (92 at $n = 16$, 117 at $n = 18$; Table 1), so any expression uniform in n must be a determinant of size growing with n — which is exactly what H_n in (4) already is. What is found is the *system* the X_j satisfy (Proposition 2.1), first-order in (E_n, F_n) and free of α, β and divisions.

1.5. **Where this was looked for.** The two sources were read in full: the arXiv PDF of [1] (version 1 of 22 July 2026, the only version on 2026-09-07) together with its e-print compiled locally, and the journal PDF of [2]. Statement, equation, table and page numbers quoted here are those of the arXiv PDF, which agree with the locally compiled e-print. In a local full-text arXiv collection of 890 739 e-prints, the string `oscillatory[- ]Gegenbauer` matches three times, all inside [1], and the weight (1) in its literal LaTeX form occurs in that paper only. OpenAlex (queried 2026-09-07) lists no work citing [1], and exactly three works citing [2]: a 2011 paper on quadrature for a modified Jacobi weight [6], a 2014 survey chapter [8], and a 2024 paper on an oscillatory Laguerre weight [7]. The first and third are [1]'s [25] and [17]; the chapter is not in its 25-item list. None of the three concerns the conjecture. The chapter does treat the weight (1), in its section on Gaussian rules for complex oscillatory weights: it reproduces the moment representation (3) and the Hankel expressions for α_n, β_n, and states one conjecture about them — the asymptotics $\alpha_n \to 0$, $\beta_n \to \frac{1}{4}$, attributed to [3] — while its only reference to [2] is the sentence "Some properties of the corresponding orthogonal polynomials were given in [42]." (The chapter was read in the authors' preprint version.) We read the negative as "not found", not as "new".

2. Preliminaries and the normal form of the conjecture

2.1. **Notation.** Throughout, K is a field of characteristic zero, $\lambda, \zeta \in K$ with $\zeta \neq 0$, and $t := \zeta^2$. For $n \geq 0$ put $d_n := \lfloor n^2/4 \rfloor$, and

$$c_n := \begin{cases} 1, & n \text{ odd}, \\ -n(2\lambda + n - 1), & n \text{ even}, \end{cases} \qquad e_n := \begin{cases} \frac{n}{2}(2\lambda + n - 1), & n \text{ even}, \\ \frac{n-1}{4}(2\lambda + n), & n \text{ odd}, \end{cases} \tag{8}$$

so that $e_0 = e_1 = 0$. Given two sequences E_n $(n \geq 0)$ and F_n $(n \geq 2)$ of elements of K with $E_n \neq 0$, set $F_0 = F_1 := 0$, $R_n := e_n F_n / E_n$ (so $R_0 = R_1 = 0$), and define the *closed forms*

$$\beta_n^c := \frac{c_n E_{n-1} E_{n+1}}{t E_n^2} \ (n \geq 1), \qquad \alpha_n^c := \frac{2(\lambda + n)}{\zeta} + \zeta(R_n - R_{n+1}), \qquad \sigma_n^c := \frac{n(2\lambda + n - 1)}{\zeta} - \zeta R_n. \tag{9}$$

For any sequence (α_n) write $\sigma_n := \sum_{j<n} \alpha_j$; the source's $\mathrm{p}(n)$ is $-i\sigma_n$ (its (1.10), p. 3). In this notation the two intermediate equations of the proof of the source's Theorem 2.2 — its (2.7), the consequence of (S_1) after eliminating $\mathrm{p}(n+1)$, and its (2.8), the coefficient of z in (S_2') — read

$$2\sigma_n + \zeta(\beta_{n+1} + \beta_n - 1) + \alpha_n\big((2n + 1 + 2\lambda) - \zeta\alpha_n\big) = 0, \tag{A_n}$$

$$(2n - 1 + 2\lambda)(\zeta\beta_n - \sigma_n) = \zeta^2 \beta_n(\alpha_{n-1} + \alpha_n). \tag{B_n}$$

The source derives (LZ 2.4) at n from (A_n) at n and $n+1$, and (LZ 2.3) at n from these and (B_n) at $n+1$ (its p. 5); Lemma 3.1 records the two implications in the form used here.

2.2. **The normal form and the two bilinear identities.** For $n \geq 1$ define the two polynomials in $E_{n-1}, E_n, E_{n+1}, E_{n+2}, F_{n-1}, F_n, F_{n+1}$ (with $F_0 = F_1 = 0$), $P_n := E_n E_{n+1}$ and $\Delta_n := e_n F_n E_{n+1} -$

$e_{n+1}F_{n+1}E_n$:

$$\begin{aligned}\mathcal{B}_n &:= (2\lambda+2n-1)\big[t\,e_nF_nE_n - c_nE_{n-1}E_{n+1} - n(2\lambda+n-1)E_n^2\big]\\ &\quad - t\,c_n\big[e_{n-1}F_{n-1}E_{n+1} - e_{n+1}F_{n+1}E_{n-1}\big], \qquad (10)\\ \mathcal{A}_n &:= 2n(2\lambda+n-1)P_n^2 - 2t\,e_nF_nE_nE_{n+1}^2 - tP_n^2\\ &\quad + c_{n+1}E_n^3E_{n+2} + c_nE_{n-1}E_{n+1}^3 + \big(2(\lambda+n)P_n + t\Delta_n\big)\big(P_n - t\Delta_n\big). \qquad (11)\end{aligned}$$

Proposition 2.1. (a) *With $E_n = X_{d_n}$ and $F_n = X_{d_n-1}$, the four formulas of Conjecture 2.5 are, index by index, the two formulas $\beta_n = \beta_n^c$ ($n \ge 1$) and $\alpha_n = \alpha_n^c$ ($n \ge 4$) of* (9)*; and α_n^c telescopes, $\sum_{j<n}\alpha_j^c = \sigma_n^c$.*

(b) *If $E_n, E_{n+1}, E_{n+2} \ne 0$ then, for the closed forms,*

$$(2n+1+2\lambda)(\zeta\beta_{n+1}^c - \sigma_{n+1}^c) - \zeta^2\beta_{n+1}^c(\alpha_n^c + \alpha_{n+1}^c) = \frac{\mathcal{B}_{n+1}}{\zeta E_{n+1}^2},$$

and if moreover $E_{n+3} \ne 0$,

$$2\sigma_{n+1}^c + \zeta(\beta_{n+2}^c + \beta_{n+1}^c - 1) + \alpha_{n+1}^c\big((2n+3+2\lambda) - \zeta\alpha_{n+1}^c\big) = \frac{\mathcal{A}_{n+1}}{\zeta(E_{n+1}E_{n+2})^2}.$$

Hence $\mathcal{B}_{n+1} = 0$ is exactly (B_n) *at $n+1$, and $\mathcal{A}_{n+1} = 0$ exactly* (A_n) *at $n+1$, for the closed forms.*

Proof. (a) is a re-indexing: at $n = 2k$, $c_n = -2k(2\lambda+2k-1)$, $E_{n\pm1} = X_{k(k\mp1)}, X_{k(k+1)}$, $E_n = X_{k^2}$, $e_n = k(2\lambda+2k-1)$, $F_n = X_{k^2-1}$, $e_{n+1} = \frac{k}{2}(2\lambda+2k+1)$, $F_{n+1} = X_{k(k+1)-1}$; at $n = 2k+1$, $c_n = 1$, $e_{n+1} = (k+1)(2\lambda+2k+1)$, $F_{n+1} = X_{(k+1)^2-1} = X_{k(k+2)}$. Substituting into (9) gives the four displayed formulas. The telescoping sum is $\sum_{j<n} 2(\lambda+j)/\zeta = n(2\lambda+n-1)/\zeta$ and $\zeta(R_0 - R_n) = -\zeta R_n$. (b) is a direct computation after clearing the denominators tE_n^2, E_n, E_{n+1}; both identities are verified in the formal development as the key step of the two implications of Lemma 3.1(e), by `field_simp` and `ring`. □

Remark 2.2. (1) The two E-indices and the F-indices are disjoint for $n \ge 4$ ($d_n - 1 \notin \{d_m\}$), so the naming X_{d_n-1} is unambiguous there. At $n = 3$ it is not: the slot which the $k = 1$ instance of the α_{2k+1} formula would call X_1 is filled in the source's Table 1 (row α_3) by $\zeta^2 - 4\lambda^2 - 4\lambda$, not by $X_1 = \zeta^2 - 2\lambda$, which is presumably why the source states the α part for $n \ge 4$ only. We therefore keep E_n and F_n as two separate sequences and never use the index identification; $F_2 = 1$, $F_3 = \zeta^2 - 4\lambda^2 - 4\lambda$ make (9) reproduce Table 1 at $n = 2, 3$ as well. (2) $\mathcal{B}_n$ is quadratic in the E, F's with t-degree $2d_n$ (+1 for odd n); $\mathcal{A}_n$ is quartic, of t-degree about n^2. $\mathcal{A}_n = 0$ is linear in E_{n+2} with coefficient $c_{n+1}E_n^3$, and $\mathcal{B}_{n+1} = 0$ is linear in F_{n+2} with coefficient $t\,c_{n+1}e_{n+2}E_n$; so from $E_0 = E_1 = 1$, $E_2 = t - 2\lambda$, $F_2 = 1$ the two identities determine the whole sequence over $\mathbb{Q}(\lambda, t)$ — the conjecture, if true, is a first-order recursion in (E_n, F_n).

3. The data-free layer

Everything in this section is independent of the tables. Write, for the right-hand sides of (LZ 2.3) and (LZ 2.4),

$$\begin{aligned}\mathrm{F}_n(x,y,w) &:= \frac{3(2n+1+2\lambda)}{2\zeta} - x + \frac{2n+1+2\lambda}{2\zeta w}\Big(y - 1 + x\Big(\frac{2n-1+2\lambda}{\zeta} - x\Big)\Big),\\ \mathrm{G}_n(x,x',y) &:= y + x'^2 - x^2 + \frac{x(2n-1+2\lambda) - x'(2n+3+2\lambda)}{\zeta},\end{aligned}$$

so that (LZ 2.3) is $\alpha_{n+1} = \mathrm{F}_n(\alpha_n, \beta_n, \beta_{n+1})$ and (LZ 2.4) is $\beta_{n+2} = \mathrm{G}_n(\alpha_n, \alpha_{n+1}, \beta_n)$.

Lemma 3.1. *Let K be a field of characteristic zero and $\zeta \ne 0$.*

(a) *For the closed forms* (9)*, $\sigma_{n+1}^c = \sigma_n^c + \alpha_n^c$ for every $n \ge 0$.*

(b) *Let $N \in K$, $M = N+1$, and $A_0, A_1, B_0, B_1, B_2, S_0, S_1 \in K$ with $S_1 = S_0 + A_0$. If $2S_0 + \zeta(B_1 + B_0 - 1) + A_0((2N+1+2\lambda) - \zeta A_0) = 0$ and $2S_1 + \zeta(B_2 + B_1 - 1) + A_1((2M+1+2\lambda) - \zeta A_1) = 0$, then $B_2 = B_0 + A_1^2 - A_0^2 + (A_0(2N-1+2\lambda) - A_1(2N+3+2\lambda))/\zeta$. ($(\mathrm{A}_n)$ at n and $n+1$ give* (LZ 2.4) *at n.)*

(c) *Let $N \in K$ and $A_0, A_1, B_0, B_1, S_0, S_1 \in K$ with $B_1 \neq 0$ and $S_1 = S_0 + A_0$. If $2S_0 + \zeta(B_1 + B_0 - 1) + A_0((2N+1+2\lambda) - \zeta A_0) = 0$ and $(2N+1+2\lambda)(\zeta B_1 - S_1) = \zeta^2 B_1(A_0 + A_1)$, then $A_1 = \frac{3(2N+1+2\lambda)}{2\zeta} - A_0 + \frac{2N+1+2\lambda}{2\zeta B_1}\big(B_0 - 1 + A_0(\frac{2N-1+2\lambda}{\zeta} - A_0)\big)$. ($(\mathrm{A}_n)$ at n and (B_n) at $n+1$ give* (LZ 2.3) *at n.)*

(d) *(Uniqueness.) Let $N \in \mathbb{N}$, let $a, b, a', b' \colon \mathbb{N} \to K$, and let $F, G \colon \mathbb{N} \to K^3 \to K$ be arbitrary. If $a_0 = a'_0$, $a_1 = a'_1$, $b_1 = b'_1$, $b_2 = b'_2$, and both pairs satisfy $a_{n+1} = F_n(a_n, b_n, b_{n+1})$ and $b_{n+2} = G_n(a_n, a_{n+1}, b_n)$ for $1 \le n \le N$, then $a_n = a'_n$ for $n \le N+1$ and $b_n = b'_n$ for $1 \le n \le N+2$. Nothing is claimed about b_0 and b'_0.*

(e) *For the closed forms, if $E_n, E_{n+1}, E_{n+2} \neq 0$ and $\mathcal{B}_{n+1} = 0$, then (B_n) holds at $n+1$; if $E_n, \dots, E_{n+3} \neq 0$ and $\mathcal{A}_{n+1} = 0$, then (A_n) holds at $n+1$.*

Proof. (a) is the telescoping of Proposition 2.1(a). (b): subtract the two hypotheses and use $S_1 = S_0 + A_0$. (c): solve the second hypothesis for A_1, substitute $S_1 = S_0 + A_0$ and eliminate S_0 by the first; the constant $3/2$ of (LZ 2.3) is $1 + \frac{1}{2}$. (d) is induction on $k \le N$ for the four-tuple $(a_k, a_{k+1}, b_{k+1}, b_{k+2})$. (e) is Proposition 2.1(b). In the formal development (b), (c) are closed by `field_simp` and a `linear_combination` of the hypotheses — $h_{A_1} - h_{A_0} - 2h_S$, respectively $-2h_{B_1} - (2N+1+2\lambda)h_{A_0} - 2(2N+1+2\lambda)h_S$ —, (d) by induction, and none of (a)–(e) uses compiled evaluation. □

Note that in (b) and (c) the index N is an arbitrary element of K, and in (d) the two right-hand sides are arbitrary functions: the lemma is about the *shape* of the source's system, not its content. Consequently, whenever the closed forms satisfy (A_n) for $1 \le n \le N+1$ and (B_n) for $2 \le n \le N+1$, they satisfy (LZ 2.3) and (LZ 2.4) for $1 \le n \le N$, and if in addition they take the initial values (LZ 2.5), (d) identifies them with *every* solution of the source's Theorem 2.2 on the corresponding range. The rest of the paper supplies the hypotheses.

4. The certified tables and the main theorem

4.1. **The tables.** The computation of Section 7 produces, for $0 \le n \le 16$, an integer polynomial $\widehat{E}_n \in \mathbb{Z}[\lambda][t]$ and a positive integer ε_n, and for $2 \le n \le 16$ an integer polynomial $\widehat{F}_n$ and a positive integer φ_n, such that

$$X_{d_n} := \widehat{E}_n/\varepsilon_n, \qquad X_{d_n-1} := \widehat{F}_n/\varphi_n \tag{12}$$

are the polynomials of the conjecture; Table 1 lists their sizes. Every statement below is about *these* polynomials, evaluated at $(\lambda, t) = (\lambda, \zeta^2)$ in K; we keep writing E_n, F_n for the evaluated values and α_n^c, β_n^c, σ_n^c for the closed forms (9) built from them. The first entries are $\widehat{E}_0 = \widehat{E}_1 = 1$, $\widehat{E}_2 = t - 2\lambda$, $\widehat{E}_3 = t^2 - t\lambda(2\lambda+5) + 4\lambda^2$, $\widehat{F}_2 = 1$, $\widehat{F}_3 = t - 4\lambda - 4\lambda^2$, with $\varepsilon_n = \varphi_n = 1$ there; so X_1 and X_2 agree with the source's $X_1 := \zeta^2 - 2\lambda$ and with its Appendix C.

4.2. **The initial values, and the sources' printed data.**

Proposition 4.1. (a) *For $\zeta \neq 0$ and $\zeta^2 \neq 2\lambda$, the closed forms built from the tables take the source's initial values* (LZ 2.5)*: $\alpha_0^c = 2\lambda/\zeta$, $\beta_1^c = 1 - 2\lambda/\zeta^2$, $\alpha_1^c = 2(\lambda+1)/\zeta - \zeta(2\lambda+1)/(\zeta^2 - 2\lambda)$ and $\beta_2^c = -2(2\lambda+1)(\zeta^4 - \zeta^2\lambda(2\lambda+5) + 4\lambda^2)/(\zeta^2(\zeta^2 - 2\lambda)^2)$.*

(b) *(Outside the formal development.) An implementation of* (LZ 2.3)–(LZ 2.5) *written independently from the e-print, in exact rational-function arithmetic, reproduces coefficient by coefficient the source's Table 1 ($\alpha_0, \dots, \alpha_8$ and $\beta_1, \dots, \beta_9$, in its X-form), every polynomial of its Appendix C — X_2, X_3, X_4, X_5, X_6, X_8, X_9, including the printed normalising factors $2X_3$, $4X_4$, $8X_5$, $8X_6$, $16X_8$, $16X_9$ — and Table 1 of* [2] *(α_0, α_1, α_2, β_1, β_2). The tables of Section 4.1 are obtained from that implementation.*

n	$d_n = \deg_t X_{d_n}$	$\deg_\lambda \widehat{E}_n$	ε_n	$\deg_t X_{d_n-1}$	φ_n
0	0	0	1	–	–
1	0	0	1	–	–
2	1	1	1	0	1
3	2	2	1	1	1
4	4	5	4	3	2
5	6	8	8	5	8
6	9	12	16	8	16
7	12	16	64	11	96
8	16	22	256	15	256
9	20	28	2048	19	4096
10	25	35	16384	24	16384
11	30	42	262144	29	655360
12	36	51	4194304	35	2097152
13	42	60	67108864	41	100663296
14	49	70	1073741824	48	1073741824
15	56	80	17179869184	55	120259084288
16	64	92	1099511627776	63	549755813888

TABLE 1. The certified tables: t-degree (the conjectured d_n, Theorem 5.1), λ-degree of $\widehat{E}_n$ (as computed; 117 at $n = 18$), and the normalisers ε_n, φ_n of (12), which are the leading t-coefficients of $\widehat{E}_n$, $\widehat{F}_n$. At $n = 13$ the table $\widehat{E}_{13}$ has 1597 integer coefficients.

Proof. (a) is read off $\widehat{E}_0, \dots, \widehat{E}_3$, $\widehat{F}_2$ and the definitions (9): $\alpha_0^c = 2\lambda/\zeta$ since $R_0 = R_1 = 0$; $\beta_1^c = c_1E_0E_2/(tE_1^2) = (t - 2\lambda)/t$; $\alpha_1^c = 2(\lambda + 1)/\zeta - \zeta R_2$ with $R_2 = e_2F_2/E_2 = (2\lambda + 1)/(t - 2\lambda)$; $\beta_2^c = c_2E_1E_3/(tE_2^2)$ with $c_2 = -2(2\lambda + 1)$. (b) is the compute-first check recorded in Section 7; its evidence is the exact agreement of some thousand integer coefficients with the printed ones. □

4.3. **The main theorem.**

Theorem 4.2. *Let K be a field of characteristic zero, $\lambda, \zeta \in K$, $\zeta \neq 0$, and let E_n, F_n be the tables of Section 4.1 evaluated at (λ, ζ^2).*

(a) *$\mathcal{B}_n = 0$ for $1 \le n \le 15$ and $\mathcal{A}_n = 0$ for $1 \le n \le 12$, identically in λ and ζ (these are polynomial identities in $\mathbb{Z}[\lambda][t]$).*

(b) *If $E_0, \dots, E_{16} \neq 0$, the closed forms satisfy (B_n) for $1 \le n \le 15$ and (A_n) for $1 \le n \le 12$; if moreover $\beta_2^c, \dots, \beta_{13}^c \neq 0$, they satisfy (LZ 2.3) for $1 \le n \le 12$ and (LZ 2.4) for $1 \le n \le 11$.*

(c) *Suppose $\zeta^2 \neq 2\lambda$, $E_n \neq 0$ for $0 \le n \le 14$, and $\beta_j^c \neq 0$ for $1 \le j \le 13$. Let $a, b\colon \mathbb{N} \to K$ be any two sequences with*

$$a_0 = \frac{2\lambda}{\zeta}, \quad a_1 = \frac{2(\lambda+1)}{\zeta} - \frac{\zeta(2\lambda+1)}{\zeta^2 - 2\lambda}, \quad b_1 = 1 - \frac{2\lambda}{\zeta^2}, \quad b_2 = -\frac{2(2\lambda+1)(\zeta^4 - \zeta^2\lambda(2\lambda+5) + 4\lambda^2)}{\zeta^2(\zeta^2 - 2\lambda)^2},$$

satisfying $a_{n+1} = \mathrm{F}_n(a_n, b_n, b_{n+1})$ and $b_{n+2} = \mathrm{G}_n(a_n, a_{n+1}, b_n)$ for $1 \le n \le 11$ — that is, the source's Theorem 2.2 on that range. Then

$$a_n = \alpha_n^c \quad (0 \le n \le 12) \qquad \text{and} \qquad b_n = \beta_n^c \quad (1 \le n \le 13).$$

Nothing is claimed about b_0.

In particular the recurrence coefficients of the weight (1) are given by Conjecture 2.5 for $n \le 12$ (α_n) and $1 \le n \le 13$ (β_n), with the explicit monic polynomials X_j of (12), whenever the nonvanishing hypotheses of (c) hold at (λ, ζ).

Proof. (a) Each identity, multiplied by the normalisers ε, φ of the tables involved and by one further positive integer L, is an equality of two explicit integer polynomials in two variables built from the tables by ring operations; those are evaluated exactly, as nested coefficient lists, and compared entry by entry (Section 8). The multipliers L grow from 4 at $n = 1$ to about $3.3 \cdot 10^{22}$ at $n = 15$ for $\mathcal{B}$

and from 16 to about $2.9 \cdot 10^{30}$ at $n = 12$ for $\mathcal{A}$; since $L \neq 0$ in characteristic zero, the identity itself follows. This is the only step of the paper that rests on exhaustive computation, and it is exactly 27 list identities: $\mathcal{B}_1, \dots, \mathcal{B}_{15}$ and $\mathcal{A}_1, \dots, \mathcal{A}_{12}$. (b) The first half is Lemma 3.1(e) applied to (a); the second half is Lemma 3.1(b) with (A_n) at $n, n+1$ and (c) with (A_n) at n and (B_n) at $n+1$, together with the telescoping (a) of the lemma. The range of (LZ 2.3) is limited by $\mathcal{A}_{12}$, that of (LZ 2.4) by $\mathcal{A}_{12}$ at $n+1 = 12$. (c) By Proposition 4.1(a) the closed forms take the same four initial values as a, b; by (b) they satisfy the same system for $1 \leq n \leq 11$; Lemma 3.1(d) with $N = 11$ gives the conclusion. □

Remark 4.3 (The hypotheses at the parameters of the weight). The nonvanishing conditions are those of a statement in pure algebra; at the actual parameters ($\lambda \in \mathbb{Q}$, $\lambda > -1/2$, ζ a nonzero real zero of $J_{\lambda-1}$) they hold. Each E_n is a nonzero polynomial in ζ^2 with coefficients in $\mathbb{Q}(\lambda) = \mathbb{Q}$ (monic, by Theorem 5.1), and $\zeta^2 - 2\lambda$ is another; since ζ is transcendental, none of them vanishes at ζ. Then $\beta_j^c = c_j E_{j-1} E_{j+1} / (\zeta^2 E_j^2) \neq 0$ as well, since $c_j \neq 0$ for $\lambda > -1/2$. And the coefficients of the weight satisfy the hypotheses on a, b by the source's Theorem 2.2.

That ζ is transcendental is the fact on which the existence proof of [2], Theorem 2.2, rests. [2] states it (p. 51) as: "non-trivial zeros of the Bessel functions J_λ, $\lambda \in \mathbb{Q}$, $-\lambda \notin \mathbb{N}$, are transcendental and cannot be the zeros of the polynomials with rational coefficients unless polynomials are identically equal to zero", and attributes it to "[13, p. 220], based on the results by [11], [12] (see also [6])". In [2]'s numbering, [13] is Shidlovskii's monograph [10]; [11] is Siegel's 1929 memoir [9], the primary source; [12] is Siegel's 1949 monograph and [6] the Fel'dman–Shidlovskii survey. The order occurring here is $\lambda - 1$, so the hypothesis reads $\lambda - 1 \in \mathbb{Q}$ and $1 - \lambda \notin \mathbb{N}$, which for $\lambda > -1/2$ excludes only $\lambda = 0$; at $\lambda = 0$ one has $J_{-1} = -J_1$, and the statement applies at the order 1 instead. This remark is not part of the formal development.

Remark 4.4. What the theorem does *not* do is prove that the polynomials of the conjecture are uniquely determined: it exhibits explicit X_j and proves that the closed forms built from them are the recurrence coefficients. (Uniqueness of the X_{d_n} given the β_n and the normalisation is elementary from (9), but is not formalised.)

4.4. **The source's own range.** The source iterated its system for $1 \leq n \leq 7$; the following is that range, certified from the source's own equations rather than recomputed.

Proposition 4.5. *$\mathcal{B}_n = 0$ and $\mathcal{A}_n = 0$ for $1 \leq n \leq 9$; the closed forms satisfy (B_n) and (A_n) for $1 \leq n \leq 9$ (given $E_0, \dots, E_{11} \neq 0$), and (LZ 2.3), (LZ 2.4) for $1 \leq n \leq 7$ (given also $\beta_2^c, \dots, \beta_8^c \neq 0$). Hence α_n for $n \leq 8$ and β_n for $1 \leq n \leq 9$ — the source's Table 1 — are the closed forms.*

Proof. The instances $n \leq 9$ of Theorem 4.2(a),(b); the composition is as in Theorem 4.2(c) with $N = 7$. □

5. The degree conjecture, and the controls

Theorem 5.1. *For every $0 \leq n \leq 16$ the table $\widehat{E}_n$ has exactly $d_n + 1$ coefficients in t and its leading t-coefficient is the constant ε_n; for every $2 \leq n \leq 16$ the table $\widehat{F}_n$ has exactly d_n coefficients in t and its leading t-coefficient is the constant φ_n. Equivalently: X_{d_n} is monic of degree exactly $d_n = \lfloor n^2/4 \rfloor$ in ζ^2 for $0 \leq n \leq 16$, and $X_{d_n - 1}$ is monic of degree exactly $d_n - 1$ for $2 \leq n \leq 16$.*

Proof. Thirty-two decidable facts about the stored lists (length, and last entry equal to the integer listed in Table 1), each checked by compiled evaluation; with (12) they are the statement. The list of j covered is $\{0, 1, 2, 3, 4, 5, 6, 8, 9, 11, 12, 15, 16, 19, 20, 24, 25, 29, 30, 35, 36, 41, 42, 48, 49, 55, 56, 63, 64\}$, twenty-nine polynomials. □

Remark 5.2 (Relation to the 2008 conjecture). The 2008 conjecture is about H_n, not about the X_j; the two are related by (4). For the range of Theorem 4.2, (4) together with $\beta_n = \beta_n^c$ gives $H_{n+1} H_{n-1} / H_n^2 = -c_n E_{n-1} E_{n+1} / E_n^2$ for $1 \leq n \leq 13$, so H_n / E_n satisfies a second-order multiplicative recursion with constant (in ζ) coefficients and H_n has the degree of E_n up to $n = 14$; this is the content of Observation 6.1 below, and it is not machine-checked. The naive bound from the degrees of the

entries ($\deg_{\zeta^2} P^\lambda_{2k} = \deg_{\zeta^2} P^\lambda_{2k+1} = k$, [2] Lemma 2.1, p. 51) is $\deg H_n \le n(n-1)/2$, far above $\lfloor n^2/4 \rfloor$; the content of the conjecture is that cancellation, and a proof of it for all n is not attempted here. Theorem 5.1 therefore states the degree conjecture for the X_j only. The transfer to H_n is the argument just given: it is not machine-checked, and it reaches $n \le 14$, not $n \le 16$, because it consumes $\beta_n = \beta_n^c$ for $1 \le n \le 13$.

Proposition 5.3 (Controls)**.** (a) *None of $\widehat{E}_0, \dots, \widehat{E}_{16}$ is the zero polynomial.*

(b) *Replacing $\widehat{E}_6$ by $\widehat{E}_6 + 1$ (one unit added to its constant coefficient) in the certificate of $\mathcal{B}_5$ makes the two sides of the cleared identity differ.*

(c) *The cleared identity is index-sensitive: the left side of the certificate of $\mathcal{B}_5$ differs from the right side of that of $\mathcal{B}_4$.*

(d) *Adding one unit to the cleared polynomial certifying $\mathcal{A}_5 = 0$ makes it nonzero.*

(e) *$\widehat{E}_6$ has neither d_6 nor $d_6 + 2$ coefficients in t (it has $d_6 + 1 = 10$).*

Proof. Twenty-one compiled evaluations — seventeen for (a), one each for (b)–(e) — on the stored and the perturbed lists. (b)–(d) show that the certificates of Theorem 4.2(a) are not identities of a degenerate kind (an all-zero expression, or an equality holding for any data), and (a), (e) that Theorem 5.1 is not vacuous. □

6. The Hankel determinant

6.1. **One formula for the leading coefficients.** Put $\gamma_n := \prod_{m=1}^{n-1} c_m^{n-m}$ ($\gamma_0 = \gamma_1 = 1$), a polynomial in λ.

Observation 6.1 (Verified, not formalised)**.** With H_n the determinant (4) of [2] and X_{d_n} the polynomials of the conjecture,

$$H_n = (-1)^{n(n-1)/2}\, \gamma_n\, X_{d_n}(\zeta^2). \tag{13}$$

Expanding γ_n at $n = 2k$ and $n = 2k+1$ gives exactly the two products of (6), sign included; (13) is thus the source's two-case conjecture in one line. The relation was verified directly, in exact arithmetic, for $1 \le n \le 12$ at $\lambda = \frac{1}{3}, \frac{7}{2}, 5$: P_k^λ from the four-term recurrence of [2] Theorem 2.1, $H_n = \det(P^\lambda_{i+j})$, and X_{d_n} from the source's system (LZ 2.3)–(LZ 2.5), two independent routes to the same polynomials. The source lists S_j only for $j = 1, 2, 4, 6, 9, 12, 16, 20, 25$ ($n \le 10$).

Derivation. Assume $\beta_n = c_n E_{n-1} E_{n+1}/(\zeta^2 E_n^2)$ (the conjecture) and (4), with $H_0 := 1$ and $H_1 = P_0^\lambda = 1$ (the source's Remark 2.6: "$S_0 = H_1 = 1$"). Then $r_n := H_n/E_n$ satisfies $r_{n+1} r_{n-1}/r_n^2 = -c_n$ and $r_0 = r_1 = 1$, whose unique solution is $r_n = (-1)^{n(n-1)/2}\gamma_n$: indeed $\gamma_{n+1}\gamma_{n-1}/\gamma_n^2 = c_n$ and the sign factor contributes -1 at every step. The derivation is conditional on the β half of the conjecture; by Theorem 4.2 that half holds for $1 \le n \le 13$, but the composition is not part of the formal development, and (13) is recorded only as the finite verification above. □

The same computation is a cross-check of the whole paper: the Hankel route (moments, determinants, $\beta_n = H_{n+1}H_{n-1}/((i\zeta)^2 H_n^2)$) reproduces the β_n obtained from the difference equations for $1 \le n \le 10$ at $\lambda = \frac{1}{3}$ — two derivations with nothing in common, the same rational functions.

6.2. **A misprint in the 2008 paper.** On p. 53, [2] prints $\alpha_{2k} = V_{k(2k+1)}(\zeta^2)/\big(\zeta S_{k^2}(\zeta^2) S_{k(k+1)}(\zeta^2)^2\big)$, as in (5). The degrees forbid it: $\deg V_{k(2k+1)} = 2k^2 + k$ and $\deg(S_{k^2} S_{k(k+1)}) = 2k^2 + k$, so $\zeta\alpha_{2k}$ is a ratio of equal degrees (as it must be, $\alpha_n = 2(\lambda + n)/\zeta + O(\zeta^{-1})$ by (9)), whereas the printed denominator has degree $3k^2 + 2k$. The source [1] restates the formula in its Remark 2.6 with a single power, without comment. The case $k = 1$ is settled from the 2008 paper's own data.

Proposition 6.2. *Let $X_2 = z^4 - z^2\lambda(2\lambda + 5) + 4\lambda^2$, $S_1 = -(z^2 - 2\lambda)$, $S_2 = 2(2\lambda + 1)X_2$ and $V_3 = -(2\lambda+1)\big((6\lambda+7)z^6 - 4\lambda(\lambda+4)(2\lambda+3)z^4 + 32\lambda^2(2\lambda+3)z^2 - 32\lambda^3(\lambda+2)\big)$, the S_1, S_2, V_3 of* [2] *as reproduced in Remark 2.6 of* [1]*, and let $\alpha_2 = A/D$ with $A = (7+6\lambda)z^6 - 4\lambda(4+\lambda)(3+2\lambda)z^4 + 32\lambda^2(3+$*

$2\lambda)z^2 - 32\lambda^3(2+\lambda)$ *and* $D = 2z(z^2-2\lambda)X_2$*, the entry of Table 1 of* [2]*. Then, in any commutative ring, identically in* λ, z*,*

$$V_3\,D = z\,S_1\,S_2\,A \qquad \textit{and} \qquad z\,S_1\,S_2^2\,A = (V_3D)\,S_2,$$

while at $(\lambda, z) = (1,3)$ *in* $\mathbb{Q}$*,* $V_3D \neq z\,S_1\,S_2^2\,A$*. That is:* $\alpha_2 = V_3/(\zeta S_1 S_2)$ *holds identically,* $\alpha_2 = V_3/(\zeta S_1 S_2^2)$ *does not, and the discrepancy is exactly the factor* S_2*.*

Proof. The two identities are polynomial identities, closed by `ring`; the inequality is a numerical evaluation ($S_2(1,3) \neq 1$), closed by `norm_num`. No compiled evaluation is involved. □

Numerically, at $(\lambda, \zeta) = (1,3)$: the printed formula $V_3/(\zeta S_1 S_2^2)$ gives $907/40656$, while α_2 — from Table 1 of [2], and equally from the source's system (LZ 2.3)–(LZ 2.5) — is $907/308 = V_3/(\zeta S_1 S_2)$; the ratio is $S_2(1,3) = 132$.

7. The computations outside the formal development

All exact computations were done in PARI/GP [11] and in Python with exact rational arithmetic, under low process priority; the tables were then frozen into the formal development. The steps, with their recorded costs, are the following.

The tables. Iterating (LZ 2.3)–(LZ 2.5) with both λ and ζ symbolic is impractical (the bivariate rational-function gcds explode: 1 ms, 24 ms, 1.7 s, more than 100 s at $n = 1, 2, 3, 4$), but at a fixed rational λ the same iteration reaches $n = 16$ in 0.7 s. So the system was iterated in $\mathbb{Q}(\zeta)$ at 130 integer values of λ, E_n and F_n were extracted from β_n and α_n by (9) ($E_{n+1} = \zeta^2\beta_n E_n^2/(c_n E_{n-1})$, then F_n from R_n), and the λ-dependence was recovered by exact Lagrange interpolation. The interpolation is exact provided the number of samples exceeds the λ-degree, which was *checked*, not assumed: the recovered degrees of $\widehat{E}_0, \ldots, \widehat{E}_{18}$ are $0, 0, 1, 2, 5, 8, 12, 16, 22, 28, 35, 42, 51, 60, 70, 80, 92, 104, 117$, all below 129, and the fit was confirmed at a value of λ outside the sample set. (An earlier run with 90 samples silently returned a wrong $\widehat{E}_{16}$, with recovered degree pinned at 89 — the tell.) Cost: 6 processor-minutes, 5.8 MB of tables to $n = 18$; the formal development carries $n \leq 16$.

The compute-first check (Proposition 4.1(b)) compared these tables and the resulting α_n, β_n with the source's Table 1, its Appendix C, and Table 1 of [2]. For the source's Table 2 ($\lambda = 0$, $n \leq 13$, polynomials Y_j) the re-indexing $X_{d_n}(\zeta^2)|_{\lambda=0} = \zeta^{2n-2}Y_{d_n-n+1}(\zeta^2)$ was derived and confirmed (ζ-valuation $2n-2$ for $1 \leq n \leq 28$, and Y_2, Y_4 matched against Appendix C); $d_n - n + 1$ is $(k-1)^2$ at $n = 2k$ and $k(k-1)$ at $n = 2k+1$, exactly the indices the source prints.

Structure checks. With ζ symbolic, for every $0 \leq n \leq 20$ at $\lambda \in \{0, -\frac{1}{3}, \frac{1}{3}, \frac{1}{2}, 1, \frac{7}{2}, 5\}$ and for every $0 \leq n \leq 30$ at $\lambda \in \{0, \frac{1}{3}\}$, E_n is a polynomial in ζ^2, monic of degree exactly d_n, and F_n monic of degree $d_n - 1$; under 4 processor-minutes. Before any formalisation, $\mathcal{B}_n = 0$ was confirmed over $\mathbb{Z}[\lambda][t]$ for $1 \leq n \leq 14$ and both identities at $\lambda = \frac{1}{3}, 5, \frac{7}{2}$ for $1 \leq n \leq 16$ and at $\lambda = 0$ for $1 \leq n \leq 12$. These are supporting evidence, not results.

The Hankel route (Observation 6.1): under one processor-minute.

The certificates. For each n, the identity $\mathcal{B}_n = 0$ (resp. $\mathcal{A}_n = 0$) was multiplied by the normalisers of the tables it involves and by the least positive integer L clearing the rational scalars of (8); the resulting integer weights (five per n for $\mathcal{B}$, eight for $\mathcal{A}$) were validated at random points before being written into the development.

Cost of the formal check. Parsing the 1.4 MB of tables: 5 s; the fifteen $\mathcal{B}$ certificates: 104 s; the twelve $\mathcal{A}$ certificates: 479 s at 7.0 GB peak resident set; transporting the 27 list identities to the field: 59 s; the degree facts and controls: 23 s (the 7.0 GB is the peak resident set of the `CertA` module alone). A first attempt at $\mathcal{A}_1, \ldots, \mathcal{A}_{14}$ in one file was stopped after 29.5 processor-minutes without finishing; the identity is quartic in the tables, so its cost grows like n^4 (about $1.5 \cdot 10^7$ big-integer coefficient products at $n = 10$, $1.9 \cdot 10^8$ at $n = 14$), and $\mathcal{A}_{13}$, $\mathcal{A}_{14}$ are left uncertified. The steps recorded above sum to about 22 processor-minutes; with the stopped run, about 50.

8. Verification

Propositions 2.1, 4.1(a), 4.5, 5.3, 6.2, Lemma 3.1 and Theorems 4.2, 5.1 are verified in Lean 4 against *Mathlib* [12, 13]; the statements are reproduced verbatim in Appendix A. The development is twelve modules, 1 854 lines (1.4 MB of which are the tables), 206 theorems and 139 definitions: bivariate integer polynomials as nested coefficient lists with a Horner evaluation `beval` into any commutative ring, proved to be a ring homomorphism (`Poly`, 20 lemmas); the tables and the integer weights (`Data`); the 27 cleared identities as list equalities, checked by compiled evaluation (`Cert`, `CertA`); the two implications of Lemma 3.1(b),(c) (`Algebra`) and the uniqueness (d) (`Unique`); the closed forms, $\mathcal{B}_n$, $\mathcal{A}_n$, and the transport of each list identity to an identity in the field (`Core`); Lemma 3.1(a),(e) and their 27 instances (`Bridge`); the thirty-two degree facts, the seventeen non-vanishing facts and the four controls (`Controls`); the misprint (`Mcm`); and the initial values, the 23 instances of (LZ 2.3), (LZ 2.4) for the closed forms, and the composed `closed_form` of Theorem 4.2(c) (`Final`). Compiled evaluation (`native_decide`) is confined to finite integer-list arithmetic on the tables: the 27 certificates, the small table look-ups, and the decidable facts of `Controls`. The axiom print of `closed_form` is `propext`, `Classical.choice`, `Quot.sound` and 322 `native_decide` axioms (the union over the 27 certificates and the look-ups); that of the reasoning layer — `d1_of`, `d2_of`, `unique_of`, `beval_bmul`, `beval_eq_of_sub`, and the three theorems of `Mcm` — contains no `native_decide` axiom at all. There is no `sorry` and no axiom declared by hand. A thirteenth file holding the two uncertified instances $\mathcal{A}_{13}$, $\mathcal{A}_{14}$ exists but is not part of the development and is not cited. The docstrings of the development cite the source's difference system as its "Theorem 2.1, (2.9)–(2.11)" and its closed-form conjecture as "Conjecture 2.1"; in the arXiv PDF and in the compiled e-print these are Theorem 2.2, (2.3)–(2.5) and Conjecture 2.5, the numbers used in this paper. Only the docstrings are affected: the statements themselves name no equation numbers. Outside the formal development, and labelled as such where they occur, are Proposition 4.1(b), Observation 6.1, Remark 4.3, and everything in Section 7. The development and the programs are currently in a private repository: reference to be added at submission.

On the reference list. Statement, equation, table and page numbers attributed to [1] are those of the arXiv PDF of version 1 (25 pages), which agree with the e-print compiled locally with `tectonic` (its `.aux` resolves the labels `diff2`, `diff1`, `al0bt1` to (2.3), (2.4), (2.5) and `S1`, `S1-1`, `S2'-1` to (2.6)–(2.8)): Lemma 2.1, Theorem 2.2, (2.3)–(2.5) and "Note that β_0 can be arbitrary" (p. 4); the proof with (2.6)–(2.8), Remark 2.3 and the iteration sentence (p. 5); Table 1, Conjecture 2.5 and Remark 2.6 (p. 6, the remark continuing on p. 7); Table 2 for $\lambda = 0$ (p. 7); references (pp. 15–16); Appendix C (p. 19). The initial values are printed as (2.5a) and (2.5b) and referred to collectively as (2.5) by the source itself (its Remark 2.3). Numbers attributed to [2] are those of the journal PDF (pp. 49–60): Theorem 2.1 (p. 50); Lemma 2.1, Theorem 2.2 and the definition of H_n (p. 51); (2.3)–(2.5) (p. 52); Table 1, the degree conjecture and the α_{2k} formula (p. 53). Bibliographic data of [3, 4, 5, 6, 7] agree with the reference list of [1] (its [16], [2], [12], [25], [17]) and were checked against Crossref, which supplied the DOIs; [8], which is not in that list, is from Crossref alone. [9] and [10] are as printed in the reference list of [2] (its [11] and [13]).

Appendix A. The formal statements

The Lean statements corresponding to the results above, verbatim (statements only; proofs, docstrings and attributes omitted; declarations that differ only in the index are shown for the first and last index). `UP` and `BP` are univariate and bivariate integer polynomials as coefficient lists (outer index the power of $t = \zeta^2$, inner the power of λ); `beval p l u` evaluates `p` at $\lambda =$ `l`, $t =$ `u`; `Ed n`, `Fd n` are $\widehat{E}_n$, $\widehat{F}_n$ (parsed from string tables `eTab`, `fTab`) and `edenN n`, `fdenN n` are ε_n, φ_n; `wB n i`, `wA n i` are the integer weights of the certificates. In `Core`, `Bridge`, `Algebra` and `Final` the context is `{K : Type*} [Field K] [CharZero K]`; in `Unique` it is `{K : Type*}`; in `Mcm` it is `[CommRing K]`; in the evaluation section of `Poly` it is `{R : Type*} [CommRing R]`. `Xm l z n`, `Ym l z n` are E_n, F_n evaluated at $(\lambda, \zeta^2) =$ (`l`, `z`^{2}); `cc`, `ee`, `Rc`, `Bc`, `Ac`, `Sc` are c_n, e_n, R_n, β_n^c, α_n^c, σ_n^c; `BilBd`, `BilAd` are $\mathcal{B}_n$, $\mathcal{A}_n$ and `BilB`, `BilA` the propositions $\mathcal{B}_n = 0$, $\mathcal{A}_n = 0$; `Frhs`, `Grhs` are F_n, G_n.

```
-- Poly.lean
```

```
abbrev UP := List ℤ
abbrev BP := List UP
def beval : BP → R → R → R
  | [], _, _ => 0
  | c :: p, l, u => peval c l + u * beval p l u
theorem beval_bmul (a b : BP) (l u : R) :
    beval (bmul a b) l u = beval a l u * beval b l u
theorem beval_eq_of_sub {p q : BP} (h : bIsZero (bsub p q) = true) (l u : R) :
    beval p l u = beval q l u
def bIsZero (p : BP) : Bool := p.all (fun c => c.all (fun a => a == 0))
def tdeg (p : BP) : ℕ := p.length - 1
def tlead (p : BP) : UP := p.getLastD []

-- Data.lean
def NMAX : Nat := 16
def edenN (n : Nat) : Nat := edenL.getD n 1
def fdenN (n : Nat) : Nat := fdenL.getD n 1
def wB (n i : Nat) : Nat := (wBL.getD (n-1) []).getD i 0
def wA (n i : Nat) : Nat := (wAL.getD (n-1) []).getD i 0
def Ed (n : Nat) : BP := eTab.getD n []
def Fd (n : Nat) : BP := fTab.getD n []

-- Cert.lean
def MMp (n : Nat) : BP := [[2*(n:ℤ)-1, 2]]
def NNp (n : Nat) : BP := [[(n:ℤ)*((n:ℤ)-1), 2*(n:ℤ)]]
def CCp (n : Nat) : BP := if n % 2 = 1 then [[1]] else [[-((n:ℤ)*((n:ℤ)-1)), -2*(n:ℤ)]]
def GGp (n : Nat) : BP :=
  if n % 2 = 0 then [[2*(n:ℤ)*((n:ℤ)-1), 4*(n:ℤ)]] else [[((n:ℤ)-1)*(n:ℤ), 2*((n:ℤ)-1)]]
def Tp : BP := [[], [1]]
def Zc (k : Nat) : BP := [[(k:ℤ)]]
def Lnp (n : Nat) : BP := [[2*(n:ℤ), 2]]
def certBL (n : Nat) : BP :=
  bmul (MMp n)
    (bsub (bsub (bmul (bmul Tp (bmul (GGp n) (Zc (wB n 0)))) (bmul (Fd n) (Ed n)))
                (bmul (bmul (CCp n) (Zc (wB n 1))) (bmul (Ed (n-1)) (Ed (n+1)))))
          (bmul (bmul (NNp n) (Zc (wB n 2))) (bmul (Ed n) (Ed n))))
def certBR (n : Nat) : BP :=
  bmul Tp (bmul (CCp n)
    (bsub (bmul (bmul (GGp (n-1)) (Zc (wB n 3))) (bmul (Fd (n-1)) (Ed (n+1))))
          (bmul (bmul (GGp (n+1)) (Zc (wB n 4))) (bmul (Fd (n+1)) (Ed (n-1))))))
def certAP (n : Nat) : BP :=
  let T2 := bmul Tp Tp
  let g0 := GGp n
  let g1 := GGp (n+1)
  bsum
    [ bmul (bmul (Zc (wA n 0)) (bsub (badd (bmul (Zc 2) (NNp n)) (Lnp n)) Tp))
           (bmul (bmul (Ed n) (Ed n)) (bmul (Ed (n+1)) (Ed (n+1)))),
      bneg (bmul (bmul (Zc (wA n 1)) (bmul Tp (bmul g0 [[2*(n:ℤ)+1, 2]])))
                 (bmul (Fd n) (bmul (Ed n) (bmul (Ed (n+1)) (Ed (n+1)))))),
      bmul (bmul (Zc (wA n 2)) (CCp (n+1)))
           (bmul (bmul (Ed n) (bmul (Ed n) (Ed n))) (Ed (n+2))),
      bmul (bmul (Zc (wA n 3)) (CCp n))
           (bmul (Ed (n-1)) (bmul (Ed (n+1)) (bmul (Ed (n+1)) (Ed (n+1))))),
      bmul (bmul (Zc (wA n 4)) (bmul (MMp n) (bmul Tp g1)))
           (bmul (Fd (n+1)) (bmul (Ed n) (bmul (Ed n) (Ed (n+1))))),
      bneg (bmul (bmul (Zc (wA n 5)) (bmul T2 (bmul g0 g0)))
                 (bmul (Fd n) (bmul (Fd n) (bmul (Ed (n+1)) (Ed (n+1)))))),
      bmul (bmul (Zc (2 * wA n 6)) (bmul T2 (bmul g0 g1)))
           (bmul (Fd n) (bmul (Fd (n+1)) (bmul (Ed n) (Ed (n+1))))),
      bneg (bmul (bmul (Zc (wA n 7)) (bmul T2 (bmul g1 g1)))
                 (bmul (Fd (n+1)) (bmul (Fd (n+1)) (bmul (Ed n) (Ed n))))) ]
theorem certB1 : bIsZero (bsub (certBL 1) (certBR 1)) = true
```

```
theorem certB15 : bIsZero (bsub (certBL 15) (certBR 15)) = true

-- CertA.lean
theorem certA1 : bIsZero (certAP 1) = true
theorem certA12 : bIsZero (certAP 12) = true

-- Core.lean
def Xm (l z : K) (n : Nat) : K := beval (Ed n) l (z^2) / (edenN n : K)
def Ym (l z : K) (n : Nat) : K := beval (Fd n) l (z^2) / (fdenN n : K)
def cc (l : K) (n : Nat) : K := if n % 2 = 1 then 1 else -(n:K)*(2*l+(n:K)-1)
def ee (l : K) (n : Nat) : K :=
  if n % 2 = 0 then (n:K)/2*(2*l+(n:K)-1) else ((n:K)-1)/4*(2*l+(n:K))
def Rc (l z : K) (n : Nat) : K := ee l n * Ym l z n / Xm l z n
def Bc (l z : K) (n : Nat) : K :=
  cc l n * Xm l z (n-1) * Xm l z (n+1) / (z^2 * (Xm l z n)^2)
def Ac (l z : K) (n : Nat) : K := 2*(l+(n:K))/z + z*(Rc l z n - Rc l z (n+1))
def Sc (l z : K) (n : Nat) : K := (n:K)*(2*l+(n:K)-1)/z - z * Rc l z n
def BilBd (l z : K) (n : Nat) : K :=
  (2*l+2*(n:K)-1) * (z^2 * ee l n * Ym l z n * Xm l z n
      - cc l n * Xm l z (n-1) * Xm l z (n+1) - (n:K)*(2*l+(n:K)-1) * (Xm l z n)^2)
    - z^2 * cc l n * (ee l (n-1) * Ym l z (n-1) * Xm l z (n+1)
      - ee l (n+1) * Ym l z (n+1) * Xm l z (n-1))
def BilB (l z : K) (n : Nat) : Prop := BilBd l z n = 0
def BilAd (l z : K) (n : Nat) : K :=
  2*(n:K)*(2*l+(n:K)-1) * (Xm l z n * Xm l z (n+1))^2
  - 2*z^2*(ee l n * Ym l z n) * Xm l z n * (Xm l z (n+1))^2
  + cc l (n+1) * (Xm l z n)^3 * Xm l z (n+2)
  + cc l n * Xm l z (n-1) * (Xm l z (n+1))^3
  - z^2 * (Xm l z n * Xm l z (n+1))^2
  + (2*(l+(n:K))*(Xm l z n * Xm l z (n+1))
      + z^2*(ee l n * Ym l z n * Xm l z (n+1) - ee l (n+1) * Ym l z (n+1) * Xm l z n))
    * (Xm l z n * Xm l z (n+1)
      - z^2*(ee l n * Ym l z n * Xm l z (n+1) - ee l (n+1) * Ym l z (n+1) * Xm l z n))
def BilA (l z : K) (n : Nat) : Prop := BilAd l z n = 0
theorem bilB1 (l z : K) : BilB l z 1
theorem bilB15 (l z : K) : BilB l z 15
theorem bilA1 (l z : K) : BilA l z 1
theorem bilA12 (l z : K) : BilA l z 12

-- Algebra.lean
theorem d1_of {l z N M A0 A1 B0 B1 B2 S0 S1 : K} (hz : z ≠ 0) (hM : M = N + 1)
    (hS : S1 = S0 + A0)
    (hA0 : 2*S0 + z*(B1 + B0 - 1) + A0*((2*N+1+2*l) - z*A0) = 0)
    (hA1 : 2*S1 + z*(B2 + B1 - 1) + A1*((2*M+1+2*l) - z*A1) = 0) :
    B2 = B0 + A1^2 - A0^2 + (A0*(2*N-1+2*l) - A1*(2*N+3+2*l))/z
theorem d2_of {l z N A0 A1 B0 B1 S0 S1 : K} (hz : z ≠ 0) (hB : B1 ≠ 0)
    (hS : S1 = S0 + A0)
    (hA0 : 2*S0 + z*(B1 + B0 - 1) + A0*((2*N+1+2*l) - z*A0) = 0)
    (hB1 : (2*N+1+2*l)*(z*B1 - S1) = z^2*B1*(A0 + A1)) :
    A1 = 3*(2*N+1+2*l)/(2*z) - A0
        + (2*N+1+2*l)/(2*z*B1)*(B0 - 1 + A0*((2*N-1+2*l)/z - A0))

-- Unique.lean
theorem unique_of {N : ℕ} {a b a' b' : ℕ → K} {F G : ℕ → K → K → K}
    (ha0 : a 0 = a' 0) (ha1 : a 1 = a' 1) (hb1 : b 1 = b' 1) (hb2 : b 2 = b' 2)
    (hA : ∀ n, 1 ≤ n → n ≤ N → a (n+1) = F n (a n) (b n) (b (n+1)))
    (hA' : ∀ n, 1 ≤ n → n ≤ N → a' (n+1) = F n (a' n) (b' n) (b' (n+1)))
    (hB : ∀ n, 1 ≤ n → n ≤ N → b (n+2) = G n (a n) (a (n+1)) (b n))
    (hB' : ∀ n, 1 ≤ n → n ≤ N → b' (n+2) = G n (a' n) (a' (n+1)) (b' n)) :
    (∀ n, n ≤ N+1 → a n = a' n) ∧ (∀ n, 1 ≤ n → n ≤ N+2 → b n = b' n)

-- Bridge.lean
```

```
theorem Sc_succ (l z : K) (n : Nat) (hz : z ≠ 0) :
    Sc l z (n+1) = Sc l z n + Ac l z n
theorem eqB_of (l z : K) (n : Nat) (hz : z ≠ 0)
    (hX0 : Xm l z n ≠ 0) (hX1 : Xm l z (n+1) ≠ 0) (hX2 : Xm l z (n+2) ≠ 0)
    (h : BilB l z (n+1)) :
    (2*((n : Nat):K)+1+2*l) * (z * Bc l z (n+1) - Sc l z (n+1))
      = z^2 * Bc l z (n+1) * (Ac l z n + Ac l z (n+1))
theorem eqA_of (l z : K) (n : Nat) (hz : z ≠ 0)
    (hX0 : Xm l z n ≠ 0) (hX1 : Xm l z (n+1) ≠ 0) (hX2 : Xm l z (n+2) ≠ 0)
    (hX3 : Xm l z (n+3) ≠ 0) (h : BilA l z (n+1)) :
    2*Sc l z (n+1) + z*(Bc l z (n+2) + Bc l z (n+1) - 1)
      + Ac l z (n+1)*((2*(((n+1 : Nat):K))+1+2*l) - z*Ac l z (n+1)) = 0
theorem eqB1 (l z : K) (hz : z ≠ 0) (h0 : Xm l z 0 ≠ 0) (h1 : Xm l z 1 ≠ 0)
    (h2 : Xm l z 2 ≠ 0) :
    (2*((0 : Nat):K)+1+2*l) * (z * Bc l z 1 - Sc l z 1)
      = z^2 * Bc l z 1 * (Ac l z 0 + Ac l z 1)
theorem eqB15 (l z : K) (hz : z ≠ 0) (h0 : Xm l z 14 ≠ 0) (h1 : Xm l z 15 ≠ 0)
    (h2 : Xm l z 16 ≠ 0) :
    (2*((14 : Nat):K)+1+2*l) * (z * Bc l z 15 - Sc l z 15)
      = z^2 * Bc l z 15 * (Ac l z 14 + Ac l z 15)
theorem eqA1 (l z : K) (hz : z ≠ 0) (h0 : Xm l z 0 ≠ 0) (h1 : Xm l z 1 ≠ 0)
    (h2 : Xm l z 2 ≠ 0) (h3 : Xm l z 3 ≠ 0) :
    2*Sc l z 1 + z*(Bc l z 2 + Bc l z 1 - 1)
      + Ac l z 1*((2*((1 : Nat):K)+1+2*l) - z*Ac l z 1) = 0
theorem eqA12 (l z : K) (hz : z ≠ 0) (h0 : Xm l z 11 ≠ 0) (h1 : Xm l z 12 ≠ 0)
    (h2 : Xm l z 13 ≠ 0) (h3 : Xm l z 14 ≠ 0) :
    2*Sc l z 12 + z*(Bc l z 13 + Bc l z 12 - 1)
      + Ac l z 12*((2*((12 : Nat):K)+1+2*l) - z*Ac l z 12) = 0

-- Final.lean
theorem Ac_zero (l z : K) : Ac l z 0 = 2*l/z
theorem Bc_one (l z : K) (hz : z ≠ 0) : Bc l z 1 = 1 - 2*l/z^2
theorem Ac_one (l z : K) (hz : z ≠ 0) (h2 : z^2 - 2*l ≠ 0) :
    Ac l z 1 = 2*(l+1)/z - z*(2*l+1)/(z^2-2*l)
theorem Bc_two (l z : K) (hz : z ≠ 0) (h2 : z^2 - 2*l ≠ 0) :
    Bc l z 2 = -2*(2*l+1)*(z^4 - z^2*l*(2*l+5) + 4*l^2)/(z^2*(z^2-2*l)^2)
def Frhs (l z : K) (n : Nat) (x y w : K) : K :=
  3*(2*((n : Nat):K)+1+2*l)/(2*z) - x
    + (2*((n : Nat):K)+1+2*l)/(2*z*w)*(y - 1 + x*((2*((n : Nat):K)-1+2*l)/z - x))
def Grhs (l z : K) (n : Nat) (x x' y : K) : K :=
  y + x'^2 - x^2 + (x*(2*((n : Nat):K)-1+2*l) - x'*(2*((n : Nat):K)+3+2*l))/z
theorem conj_d2_1 (l z : K) (hz : z ≠ 0) (hb : Bc l z 2 ≠ 0) (x0 : Xm l z 0 ≠ 0) (x1 : Xm l z
      1 ≠ 0) (x2 : Xm l z 2 ≠ 0) (x3 : Xm l z 3 ≠ 0) :
    Ac l z 2 = Frhs l z 1 (Ac l z 1) (Bc l z 1) (Bc l z 2)
theorem conj_d2_12 (l z : K) (hz : z ≠ 0) (hb : Bc l z 13 ≠ 0) (x11 : Xm l z 11 ≠ 0) (x12 :
      Xm l z 12 ≠ 0) (x13 : Xm l z 13 ≠ 0) (x14 : Xm l z 14 ≠ 0) :
    Ac l z 13 = Frhs l z 12 (Ac l z 12) (Bc l z 12) (Bc l z 13)
theorem conj_d1_1 (l z : K) (hz : z ≠ 0) (x0 : Xm l z 0 ≠ 0) (x1 : Xm l z 1 ≠ 0) (x2 : Xm l z
      2 ≠ 0) (x3 : Xm l z 3 ≠ 0) (x4 : Xm l z 4 ≠ 0) :
    Bc l z 3 = Grhs l z 1 (Ac l z 1) (Ac l z 2) (Bc l z 1)
theorem conj_d1_11 (l z : K) (hz : z ≠ 0) (x10 : Xm l z 10 ≠ 0) (x11 : Xm l z 11 ≠ 0) (x12 :
      Xm l z 12 ≠ 0) (x13 : Xm l z 13 ≠ 0) (x14 : Xm l z 14 ≠ 0) :
    Bc l z 13 = Grhs l z 11 (Ac l z 11) (Ac l z 12) (Bc l z 11)
theorem closed_form (l z : K) (hz : z ≠ 0) (h2 : z^2 - 2*l ≠ 0)
    (x0 : Xm l z 0 ≠ 0) (x1 : Xm l z 1 ≠ 0) (x2 : Xm l z 2 ≠ 0) (x3 : Xm l z 3 ≠ 0) (x4 : Xm
      l z 4 ≠ 0) (x5 : Xm l z 5 ≠ 0) (x6 : Xm l z 6 ≠ 0) (x7 : Xm l z 7 ≠ 0) (x8 : Xm l z 8 ≠
      0) (x9 : Xm l z 9 ≠ 0) (x10 : Xm l z 10 ≠ 0) (x11 : Xm l z 11 ≠ 0) (x12 : Xm l z 12 ≠
      0) (x13 : Xm l z 13 ≠ 0) (x14 : Xm l z 14 ≠ 0)
    (hb : ∀ j, 1 ≤ j → j ≤ 13 → Bc l z j ≠ 0)
    (a b : Nat → K)
    (ha0 : a 0 = 2*l/z)
    (ha1 : a 1 = 2*(l+1)/z - z*(2*l+1)/(z^2-2*l))
```

```
    (hb1 : b 1 = 1 - 2*l/z^2)
    (hb2 : b 2 = -2*(2*l+1)*(z^4 - z^2*l*(2*l+5) + 4*l^2)/(z^2*(z^2-2*l)^2))
    (hd2 : ∀ n, 1 ≤ n → n ≤ 11 → a (n+1) = Frhs l z n (a n) (b n) (b (n+1)))
    (hd1 : ∀ n, 1 ≤ n → n ≤ 11 → b (n+2) = Grhs l z n (a n) (a (n+1)) (b n)) :
    (∀ n, n ≤ 12 → a n = Ac l z n) ∧ (∀ n, 1 ≤ n → n ≤ 13 → b n = Bc l z n)

-- Controls.lean
theorem Ed_shape0 : (Ed 0).length = 1 ∧ tlead (Ed 0) = [(1 : ℤ)]
theorem Ed_shape16 : (Ed 16).length = 65 ∧ tlead (Ed 16) = [(1099511627776 : ℤ)]
theorem Fd_shape2 : (Fd 2).length = 1 ∧ tlead (Fd 2) = [(1 : ℤ)]
theorem Fd_shape16 : (Fd 16).length = 64 ∧ tlead (Fd 16) = [(549755813888 : ℤ)]
theorem Ed_ne0 : bIsZero (Ed 0) = false
theorem Ed_ne16 : bIsZero (Ed 16) = false
def EdPert : BP := badd (Ed 6) [[1]]
def certBLpert : BP :=
  bmul (MMp 5)
    (bsub (bsub (bmul (bmul Tp (bmul (GGp 5) (Zc (wB 5 0)))) (bmul (Fd 5) (Ed 5)))
                (bmul (bmul (CCp 5) (Zc (wB 5 1))) (bmul (Ed 4) EdPert)))
          (bmul (bmul (NNp 5) (Zc (wB 5 2))) (bmul (Ed 5) (Ed 5))))
def certBRpert : BP :=
  bmul Tp (bmul (CCp 5)
    (bsub (bmul (bmul (GGp 4) (Zc (wB 5 3))) (bmul (Fd 4) EdPert))
          (bmul (bmul (GGp 6) (Zc (wB 5 4))) (bmul (Fd 6) (Ed 4)))))
theorem certB5_pert_fails : bIsZero (bsub certBLpert certBRpert) = false
theorem certB_index_mismatch : bIsZero (bsub (certBL 5) (certBR 4)) = false
theorem certA5_shift_fails : bIsZero (badd (certAP 5) [[1]]) = false
theorem Ed6_len_not_off : (Ed 6).length ≠ 9 ∧ (Ed 6).length ≠ 11

-- Mcm.lean
def X2 (l z : K) : K := z^4 - z^2*l*(2*l+5) + 4*l^2
def S1 (l z : K) : K := -(z^2 - 2*l)
def S2 (l z : K) : K := 2*(2*l+1) * X2 l z
def V3 (l z : K) : K :=
  -(2*l+1)*((6*l+7)*z^6 - 4*l*(l+4)*(2*l+3)*z^4 + 32*l^2*(2*l+3)*z^2 - 32*l^3*(l+2))
def A2num (l z : K) : K :=
  (7+6*l)*z^6 - 4*l*(4+l)*(3+2*l)*z^4 + 32*l^2*(3+2*l)*z^2 - 32*l^3*(2+l)
def A2den (l z : K) : K := 2*z*(z^2-2*l)*X2 l z
theorem alpha2_single_power (l z : K) :
    V3 l z * A2den l z = z * S1 l z * S2 l z * A2num l z
theorem alpha2_square_power_false :
    V3 (1 : ℚ) 3 * A2den (1 : ℚ) 3 ≠ 3 * S1 (1 : ℚ) 3 * (S2 (1 : ℚ) 3)^2 * A2num (1 : ℚ) 3
theorem alpha2_square_off_by_S2 (l z : K) :
    (z * S1 l z * (S2 l z)^2 * A2num l z) = (V3 l z * A2den l z) * S2 l z
```

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