THE ATTAINABLE NUMBERS OF TILES IN O_N-TILE DECOMPOSITIONS

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Abstract. A *tile* of a box $\mathcal{C} = \mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_N}$ is a product $R_1 \times \cdots \times R_N$ of nonempty, not necessarily consecutive, subsets $R_i \subseteq \mathbb{Z}_{d_i}$. Han, Zhang, Shi and Zhang call a partition $\mathcal{C} = \bigsqcup_{j=1}^{s} t_j$ into tiles with $s \geq 3$ an O_N*-tile decomposition* when no union $\bigsqcup_{j\in J} t_j$ with $1 < |J| < s$ is again a tile, and show that each such decomposition produces an unextendible product basis of size $\prod_i d_i - s + 1$. They ask which values of s occur for a given $\mathcal{C}$. We answer this in full for two boxes and up to one value for a third: the attainable set is $\{5, 6, \ldots, 15\}$ for $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, it is $\{5, \ldots, 10\}$ for $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, and for $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_4$ it is $\{5, \ldots, 20\}$ apart from the single undecided value $s = 21$. The upper ends rest on a counting inequality valid for every box, which comes from the observation that two tiles of a decomposition can never agree in all but one coordinate; it gives $\alpha s \leq \lambda \prod_i d_i + \sum_q \prod_{j\neq q} d_j$ and yields the ceilings 16, 11, 21, 25, 27, 34 at the first six boxes considered here. The small values $s = 3, 4$ and the boundary values left open by the inequality are settled by exhaustive searches whose completeness is proved rather than assumed. We show in addition, by an argument with no finite input at all, that *no* box, of any shape and any number of factors, admits a decomposition with exactly three tiles; and we answer the question in full at three further boxes: the attainable set is $\{5\}$ for $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, it is $\{5, 6, 7\}$ for $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$, and it is $\{5, 7, 8, 9\}$ for the four-qubit box $\mathbb{Z}_2^4$ — the first attainable set met here that is not an interval. Weighting the counting inequality for an all-qubit box gives the ceilings 23, 46 and 93 on $\mathbb{Z}_2^5$, $\mathbb{Z}_2^6$ and $\mathbb{Z}_2^7$, in each case stronger than what the tile-to-UPB theorem yields when composed with the known minimum size of a qubit unextendible product basis. Every theorem below is machine-checked in Lean 4; the few statements that are not are marked where they appear.

1. Introduction

Let $d_1, \ldots, d_N \geq 2$ and let $\mathcal{C} = \mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_N}$, a finite set of $\prod_i d_i$ *cells*. A *tile* of $\mathcal{C}$ is a combinatorial box $t = R_1 \times \cdots \times R_N$ with $\emptyset \neq R_i \subseteq \mathbb{Z}_{d_i}$; the sets R_i need not consist of consecutive residues, so $\{0, 2\} \times \{1\} \times \{0\}$ is a tile of $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, and the number of tiles of $\mathcal{C}$ is $\prod_i (2^{d_i} - 1)$. The size of a tile is $|t| = \prod_i |R_i|$.

Han, Zhang, Shi and Zhang [1] single out those partitions of $\mathcal{C}$ into tiles that are irreducible in a strong sense.

Definition 1.1 ([1, Definition 2]). Let $\mathcal{C} = \bigsqcup_{j=1}^{s} t_j$ be a partition of $\mathcal{C}$ into tiles with $s \geq 3$. It is an O_N*-tile decomposition* if for every index set $J \subseteq \{1, 2, \ldots, s\}$ with $1 < |J| < s$ the union $\bigsqcup_{j\in J} t_j$ does not form a tile.

Three features of the definition are used constantly below and are easy to misread. The bound $s \geq 3$ is part of the definition, so a partition into two tiles is excluded outright. The set J indexes tiles, not cells, and the constraint $1 < |J| < s$ excludes only $|J| \in \{0, 1, s\}$. And "forms a tile" is a property of the union *as a set of cells*: the question is whether that set equals $R_1 \times \cdots \times R_N$ for some nonempty R_i, with no adjacency or geometric condition anywhere.

The interest of Definition 1.1 is quantum-informational. An unextendible product basis (UPB) of $\mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_N}$ is a set of orthogonal product vectors whose orthogonal complement contains no product vector. The tile-to-UPB theorem [1, Theorem 1] attaches to every O_N-tile decomposition with s tiles a UPB of cardinality $\prod_i d_i - s + 1$, generalising the bipartite tile constructions of [3] and [4]. Consequently the numbers of tiles that occur control the UPB cardinalities that the construction reaches, and [1] poses this as an open problem.

Problem 1.2 ([1, Problem 2])**.** Given $\mathcal{C} = \mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_N}$, determine the integers s for which $\mathcal{C}$ admits an O_N-tile decomposition with exactly s tiles.

The closing section of [1] returns to the same question, asking for the attainable UPB cardinalities and for "possible obstructions for unattainable sizes". What [1] supplies is a lower half: a satisfiability search produces decompositions, and for $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ eleven of them, one for each $s = 5, 6, \ldots, 15$, are printed in its Appendix B. Nothing there excludes any value of s, and the accompanying archive [2] records only successful searches, over ranges of s that are fixed in advance.

Write

$$\mathcal{A}(\mathcal{C}) \;=\; \big\{\, s \in \mathbb{N} \,:\, \mathcal{C} \text{ admits an } O_N\text{-tile decomposition with exactly } s \text{ tiles} \,\big\}$$

for the set Problem 1.2 asks for. This paper determines $\mathcal{A}(\mathcal{C})$ for the two smallest boxes of the tripartite table of [1] and narrows it to a single undecided value for a third.

Theorem 1.3. *$\mathcal{A}(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3) = \{5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$ and $\mathcal{A}(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3) = \{5, 6, 7, 8, 9, 10\}$. For $\mathcal{C} = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_4$ and every $s \neq 21$, one has $s \in \mathcal{A}(\mathcal{C})$ if and only if $5 \leq s \leq 20$.*

In particular the eleven decompositions printed in [1, Appendix B] already realise every attainable number of tiles for the 3-cube, and the six decompositions its archive holds for $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ realise every attainable number there. Translated through the tile-to-UPB theorem of [1], which we cite and do not reprove, the O_3-tile route to UPBs reaches exactly the cardinalities $13, \ldots, 23$ in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ and exactly $9, \ldots, 14$ in $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$. This is a statement about the construction, not about which UPB cardinalities exist: nothing here excludes a UPB of some other origin.

Two further groups of results are proved below. The first removes the smallest value of s from the reach of any search: it is settled once, at every box simultaneously, by an argument that computes nothing.

Theorem 1.4. *No box $\mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_N}$ — for any N and any $d_1, \ldots, d_N$ — admits an O_N-tile decomposition with exactly three tiles.*

The second group concerns the boxes all of whose factors, or all but one, are $\mathbb{Z}_2$. Writing $\mathbb{Z}_2^N$ for the N-fold product $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, these are the boxes whose decompositions the tile-to-UPB theorem sends to unextendible product bases of the multi-qubit systems $(\mathbb{C}^2)^{\otimes N}$.

Theorem 1.5. *$\mathcal{A}(\mathbb{Z}_2^3) = \{5\}$, $\mathcal{A}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3) = \{5, 6, 7\}$, and $\mathcal{A}(\mathbb{Z}_2^4) = \{5, 7, 8, 9\}$. Moreover every O_N-tile decomposition of $\mathbb{Z}_2^N$ has at most 23, 46, 93 tiles for $N = 5, 6, 7$ respectively.*

The third of these attainable sets is not an interval: $\mathbb{Z}_2^4$ has O_4-tile decompositions with five tiles and with seven, but none with six. We are not aware of another instance of this phenomenon; the values s realised at the eight other boxes treated here run consecutively from 5. The box $\mathbb{Z}_2^4$ is also the only one considered here at which [1] ran no computation at all: the driver of [2] enumerates twenty boxes and $(2, 2, 2, 2)$ is not among them, and none of the 219 recorded searches is over that box, so every value of s there is decided for the first time below. Section 8 states precisely which of these exclusions can, and which cannot, be recovered from the UPB literature through the tile-to-UPB theorem.

Theorem 1.3 is assembled from three kinds of ingredient.

An upper bound valid for every box (Section 3). Two distinct tiles of an O_N-tile decomposition cannot agree in all but one coordinate, since disjointness would then make their union a tile. Hence, for each q, the tiles all of whose factors except the q-th are singletons are pairwise distinguished by that tuple of singletons, and there are only $P_q = \prod_{j \neq q} d_j$ such tuples. Weighing this against $\sum_t |t| = \prod_i d_i$ gives $\alpha s \leq \lambda \prod_i d_i + \sum_q P_q$ for suitable constants, and hence the ceilings in the fourth column of Table 1. No search is involved, and at the top of the range the inequality has so little slack left that it pins the shape of a hypothetical extremal decomposition.

Exhaustive searches (Sections 4, 5 and 6). The values $s = 3$ and $s = 4$, and the boundary values $s = 16$ for the 3-cube and $s = 11$ for $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, are excluded by finite searches. For $s = 3, 4$ the search runs over all ordered tuples of tiles and is therefore exhaustive by construction. For the boundary values that is far out of reach, and the search is a depth-first exact cover over a small alphabet of tiles supplied by the counting inequality; there its completeness is a theorem, proved alongside, so that an unsound restriction of the search would be a failure to prove rather than a wrong conclusion.

Known decompositions, re-certified (Sections 5 and 6). The attainability halves are the authors' own objects: the eleven decompositions of [1, Appendix B] and the 54 further decompositions we read out of the archive [2]. We claim no novelty for them, only that each has been checked against Definition 1.1.

Theorems 1.4 and 1.5 add a fourth ingredient, and re-use the first three. Theorem 1.4 (Section 7) is a proof in the ordinary sense — a two-case argument about the product structure, with no finite input anywhere — and it makes six of the box-by-box computations above redundant. The three attainable sets of Theorem 1.5 (Section 8) combine it with the counting inequality run at weights adapted to an all-qubit box, with searches of the kind of Section 4, and with decompositions that are in one case the archive's and in the others found here.

Table 1 summarises what is known after this paper for the nine boxes considered.

TABLE 1. The state of Problem 1.2 for the boxes treated here. "Attained" lists values realised by a decomposition checked here; "excluded" lists values proved unattainable; the last column lists what remains open. The ceiling is the bound of Theorem 3.2.

box $\mathcal{C}$	cells	tiles	ceiling	attained	excluded
$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$	27	343	16	5–15	all others
$\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$	18	147	11	5–10	all others
$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_4$	36	735	21	5–20	$3, 4$ and ≥ 22
$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$	45	1519	25	5–23	3 and ≥ 26
$\mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_4$	48	1575	27	5–24	3 and ≥ 28
$\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$	54	1029	34	5–30	3 and ≥ 35
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	8	27	5	5	all others
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$	12	63	8	5–7	all others
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	16	81	11	$5, 7, 8, 9$	all others

Everything below has been formally verified; Section 10 says exactly what that means, which statements are the marked exceptions, and what is delegated to compiled evaluation.

2. PRELIMINARIES

2.1. **Boxes, tiles, decompositions.** Throughout, $\mathcal{C} = \mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_N}$ with $N \geq 3$ and $d_i \geq 2$, and $\mathcal{T}(\mathcal{C})$ denotes the set of tiles of $\mathcal{C}$, of which there are $\prod_i (2^{d_i} - 1)$. For the 3-cube $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ this is $7^3 = 343$. A *tile partition* is a list $t_1, \ldots, t_s$ of tiles that are pairwise disjoint and cover $\mathcal{C}$; an O_N-tile decomposition is a tile partition with $s \geq 3$ satisfying the sub-union condition of Definition 1.1. We say a value s is *attainable* for $\mathcal{C}$ when $s \in \mathcal{A}(\mathcal{C})$. A tile with one cell is a *single cell*; a tile with two cells is a *domino*. (Section 7 drops the standing assumption $N \geq 3$; nothing before it does.)

We shall use one remark of [1] without reproving it. Call a tile $t = R_1 \times \cdots \times R_N$ *admissible* if at least two of the R_i are proper subsets of the corresponding $\mathbb{Z}_{d_i}$. If a tile of a decomposition is not admissible then its complement is again a tile, namely the union of the other $s - 1 \geq 2$ tiles, which Definition 1.1 forbids; so every tile of an O_N-tile decomposition is admissible [1, §3.2]. We use this only as a pruning rule in enumerations carried out outside the formal development.

2.2. **Two elementary facts.** The first is the criterion by which a set of cells is recognised as a tile. For $S \subseteq \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ write $\pi_i(S)$ for the projection of S to the i-th factor.

Lemma 2.1. *A subset S of $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ is a tile if and only if $S \neq \emptyset$ and $S = \pi_1(S) \times \pi_2(S) \times \pi_3(S)$; that is, if and only if there exist nonempty $R_1, R_2, R_3 \subseteq \mathbb{Z}_3$ with $S = R_1 \times R_2 \times R_3$.*

Proof sketch. If $S = R_1 \times R_2 \times R_3$ with all R_i nonempty then $\pi_i(S) = R_i$, and conversely the displayed equation exhibits S as a product of nonempty sets. The content of the lemma is that the projection criterion is decidable by inspection of S alone, so that "the union $\bigsqcup_{j \in J} t_j$ forms a tile" can be tested on the union without producing the factors; this is what makes the sub-union condition of Definition 1.1 a finite test. In the formal development the criterion is the definition of "forms a tile" and the lemma is the proof that it agrees with Definition 1.1; it is checked by inspecting the 512 subsets of $\mathbb{Z}_3$ triplewise, together with the observation that a tile of the cube is one of the 343 products of nonempty factors. □

The second fact records that nothing is lost by treating the 3-cube either on its own or as an instance of a general box; we use both descriptions below, the first because its arithmetic is cheaper and the second because it applies to every $\mathcal{C}$ at once.

Proposition 2.2. *Let $\mathcal{C} = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. For every $S \subseteq \mathcal{C}$, S is a tile of the 3-cube if and only if it is a tile of $\mathcal{C}$ read as the box with $(d_1, d_2, d_3) = (3, 3, 3)$; and for every finite list of tiles, it is an O_3-tile decomposition in one reading if and only if it is one in the other. In particular $\mathcal{A}(\mathcal{C})$ is the same set computed either way.*

Proof sketch. The two descriptions of a tile agree on every subset of the cube by Lemma 2.1: one side asks for nonempty factors, the other for membership in the list of the 343 products of nonempty factors. The remaining clauses of Definition 1.1 — disjointness, coverage, $s \geq 3$, and the quantifier over index sets — are transcribed identically in the two settings, so they agree list by list. □

3. A counting bound for an arbitrary box

The whole upper half of Problem 1.2, at every box treated here, comes from one observation about pairs of tiles. For a tile $t = R_1 \times \cdots \times R_N$ and $1 \leq q \leq N$ put

$$\iota_q(t) = \begin{cases} 1, & |R_j| = 1 \text{ for every } j \neq q, \\ 0, & \text{otherwise,} \end{cases} \qquad\qquad P_q = \prod_{j \neq q} d_j.$$

Lemma 3.1. *Let $t_1, \ldots, t_s$ be an O_N-tile decomposition of $\mathcal{C}$ and let $t \neq t'$ be two of its tiles, say $t = R_1 \times \cdots \times R_N$ and $t' = R'_1 \times \cdots \times R'_N$. Then there are at least two indices j with $R_j \neq R'_j$. Consequently, for each q, the map $t \mapsto (R_j)_{j \neq q}$ is injective on the decomposition, and*

$$\#\{\, t \in \{t_1, \ldots, t_s\} : \iota_q(t) = 1 \,\} \;\leq\; P_q.$$

Proof sketch. Suppose $R_j = R'_j$ for every $j \neq q$. The two tiles are disjoint and their factors agree off q, so $R_q \cap R'_q = \emptyset$, and then

$$t \sqcup t' = R_1 \times \cdots \times (R_q \cup R'_q) \times \cdots \times R_N$$

is again a tile. Its index set J has $|J| = 2$, and $2 < s$ because $s \geq 3$ is part of Definition 1.1; so $1 < |J| < s$ and the definition is violated. This is exactly the $|J| = 2$ instance of the sub-union condition, and it is the only instance the bound of this section uses. For the last assertion, a tile with $\iota_q(t) = 1$ is labelled by the tuple $(R_j)_{j \neq q}$ of singletons, of which there are P_q, and by injectivity distinct tiles carry distinct labels. In the formal development the displayed union identity is checked once over the alphabet of tiles — for every tile t, every coordinate q and every subset m of $\mathbb{Z}_{d_q}$, the cell set of t with R_q replaced by $R_q \cup m$ is the union of the cell sets of t and of t with R_q replaced by m — and the pigeonhole step is carried out by splitting the count over the P_q labels. Proposition 7.2 below removes that check, by proving the identity at every box at once. □

Theorem 3.2. *Let $\lambda \geq 1$ and $\alpha = \min(\lambda + N,\ 2\lambda + 1,\ 4\lambda)$. Every O_N-tile decomposition of $\mathcal{C}$ into s tiles satisfies*

$$\alpha\, s \ \leq\ \lambda \prod_{i=1}^{N} d_i \ +\ \sum_{q=1}^{N} P_q.$$

More generally, if $w\colon \mathcal{T}(\mathcal{C}) \to \mathbb{N}$ satisfies $\alpha + w(t) \leq \lambda|t| + \sum_q \iota_q(t)$ for every tile t, then $\alpha s + \sum_j w(t_j) \leq \lambda \prod_i d_i + \sum_q P_q$.

Proof sketch. Fix a tile t and let k be the number of its factors that are singletons. If $k = N$ then $|t| = 1$ and $\iota_q(t) = 1$ for every q, so $\lambda|t| + \sum_q \iota_q(t) = \lambda + N$. If $k = N-1$ then $|t| \geq 2$ and exactly one ι_q is 1, giving at least $2\lambda+1$. If $k \leq N-2$ then at least two factors have two or more elements, so $|t| \geq 4$ and no ι_q is 1, giving at least 4λ. Hence $\alpha \leq \lambda|t| + \sum_q \iota_q(t)$ for every tile. Summing over the s tiles of the decomposition and using $\sum_j |t_j| = \prod_i d_i$, which holds because the tiles partition $\mathcal{C}$, together with Lemma 3.1 in the form $\sum_q \#\{j : \iota_q(t_j) = 1\} \leq \sum_q P_q$, yields the inequality. The weighted form is the same computation with the sharper local inequality. ☐

Minimising the resulting ceiling $\lfloor(\lambda \prod_i d_i + \sum_q P_q)/\alpha\rfloor$ over λ gives, for every tripartite box below, $\lambda = 2$ and $\alpha = 5$, and for the quadripartite box $\lambda = 3$ and $\alpha = 7$. Each of these ceilings coincides with the optimum of the integer programme

$$\max\ a + \sum_q b_q + c \quad \text{subject to} \quad a + 2\sum_q b_q + 4c \leq \prod_i d_i, \qquad a + b_q \leq P_q,$$

in which a counts single cells, b_q the tiles whose only non-singleton factor is the q-th, and c the rest; so no further weighting of these statistics improves the bound. Section 9 shows that the statistics themselves cannot be pushed further. At the all-qubit boxes $\mathbb{Z}_2^N$ of Section 8 the optimal choice is $\lambda = N-1$, which for $N \geq 4$ is not 2, and the difference is the content of Theorem 8.9.

Theorem 3.3. *Every O_3-tile decomposition of $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ has at most 16 tiles; in particular no such decomposition has 17 or more tiles. Moreover, one with exactly 16 tiles would have every tile of at most two cells, hence would consist of five single cells and eleven dominoes.*

Proof sketch. Theorem 3.2 with $\lambda = 2$, $\alpha = 5$, $N = 3$ and $P_1 = P_2 = P_3 = 9$ gives $5s \leq 2\cdot 27 + 27 = 81$, so $s \leq 16$. For the shape, take $w(t) = 2$ when $|t| \geq 3$ and $w(t) = 0$ otherwise; the local inequality $5 + 2\,[\,|t| \geq 3\,] \leq 2|t| + \sum_q \iota_q(t)$ holds for every tile of the cube, as one checks on the three classes exactly as in the proof of Theorem 3.2. At $s = 16$ the weighted inequality reads $80 + \sum_j w(t_j) \leq 81$, so w vanishes on every tile of the decomposition and every tile has at most two cells. If a tiles have one cell and b have two, then $a + b = 16$ and $a + 2b = 27$, so $a = 5$ and $b = 11$. In the formal development the two local inequalities are verified by inspecting all 343 tiles of the cube, and the pigeonhole bound $\#\{j : \iota_q(t_j) = 1\} \leq 9$ is obtained by splitting that count over its nine labels. ☐

Theorem 3.4. *An O_3-tile decomposition of $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_4$ has at most 21 tiles; one of $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$ has at most 25; one of $\mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_4$ has at most 27; and an O_4-tile decomposition of $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ has at most 34 tiles.*

Proof sketch. Theorem 3.2. For $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_4$, $\lambda = 2$ and $\alpha = 5$ give $5s \leq 2\cdot 36 + (12+12+9) = 105$; for $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$, $5s \leq 2\cdot 45 + (15+15+9) = 129$; for $\mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_4$, $5s \leq 2\cdot 48 + (16+12+12) = 136$. For the quadripartite box $N = 4$, and $\lambda = 3$, $\alpha = 7$ give $7s \leq 3 \cdot 54 + (27+18+18+18) = 243$. Definition 1.1 is stated for general N in [1], so the quadripartite case is an instance of it and not an extension. The only finite input in each case is the union identity of Lemma 3.1, checked over the alphabet of the box; the largest of them has 1575 tiles. ☐

4. Exhaustive searches, and why they are exhaustive

Two of the values left open by Section 3 — $s = 16$ for the 3-cube and $s = 11$ for $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ — and the small values $s = 3, 4$ need a search. For $s = 3$ and $s = 4$ the search can be exhaustive by construction: one runs over *all* ordered tuples of tiles, with the last tile forced to be the complement of the union of the others, so there is no ordering convention to justify. For $s = 16$ and $s = 11$ that is hopeless, and a depth-first exact cover is needed; then exhaustiveness is a statement to be proved. This section records the two tools that make such a search both affordable and provably complete.

The first replaces the 2^s index sets of Definition 1.1 by a scan over the tile alphabet. In a partition an index set is determined by its union: if $\bigsqcup_{j\in J} t_j$ is a tile T, then the tiles inside T are precisely the t_j with $j \in J$, since the tiles are disjoint and nonempty.

Proposition 4.1. *Let $t_1, \dots, t_s$ be a tile partition of $\mathcal{C}$ with every t_j nonempty. Suppose that for every tile T of $\mathcal{C}$ other than $\mathcal{C}$ itself, either the t_j contained in T fail to cover T, or at most one of them is contained in T. Then no union $\bigsqcup_{j\in J} t_j$ with $1 < |J| < s$ forms a tile, so the partition is an O_N-tile decomposition as soon as $s \geq 3$.*

Proof sketch. Let J with $1 < |J| < s$ have $T = \bigsqcup_{j\in J} t_j$ a tile; $T \neq \mathcal{C}$ because $|J| < s$ and the tiles are nonempty. Every t_j, $j \in J$, is contained in T, so the tiles contained in T cover T and there are at least $|J| \geq 2$ of them, contradicting the hypothesis. The two halves of the count — that every $j \in J$ contributes a tile inside T, and that nothing else lies inside T — are where disjointness and nonemptiness enter. □

The cost of the test of Proposition 4.1 is proportional to $|\mathcal{T}(\mathcal{C})|\cdot s$ rather than $2^s(2s+|\mathcal{T}(\mathcal{C})|)$, which is what brings decompositions with s up to 30 within reach at all. Outside the formal development the two tests were compared on every tile partition of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ and $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ — 1 823 452 partitions with $s \geq 3$ — with no disagreement.

The second tool is the search itself. It scans the cells of $\mathcal{C}$ in a fixed order, carrying the set of covered cells; at the first uncovered cell it branches over the tiles of a candidate alphabet containing that cell, and a branch is abandoned when a budget on the number of tiles of each size is exceeded, or when some proper tile T of $\mathcal{C}$ is exactly the union of at least two already placed tiles.

Proposition 4.2. *The two restrictions above are sound: an O_N-tile decomposition with the prescribed profile never has a proper tile of $\mathcal{C}$ as the union of two or more of its tiles, and any sub-collection of it respects each budget. Consequently, if the search abandons every branch, no O_N-tile decomposition of the prescribed profile exists.*

Proof sketch. For the first restriction, suppose a proper tile $T \neq \mathcal{C}$ is the union of the tiles of the decomposition contained in it and that at least two are. Those tiles form an index set J with $1 < |J|$, and $|J| < s$ because $T \neq \mathcal{C}$; so Definition 1.1 is violated. For the second, a branch knows only a sub-collection of a hypothetical decomposition, and what transfers to a sub-collection is the inclusion of the cells covered: if all members of a class have k cells, then k times the number of members of that class in the sub-collection is the number of cells that class covers, which is at most the corresponding number for the whole decomposition. Completeness is then an induction on the cell index: the scanned cell is uncovered, the decomposition covers it, so one of its tiles contains that cell; that tile lies in the candidate alphabet, it is disjoint from everything already placed, and neither restriction can have fired — so the branch that places it is one the search explores. □

The point of Proposition 4.2 is that the restrictions are discharged inside the completeness proof. A mistaken restriction therefore blocks the proof rather than producing a false conclusion; the only thing the searches below contribute is the single Boolean value "every branch was abandoned".

5. The cube $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$

5.1. **The published decompositions.** Appendix B of [1] prints one O_3-tile decomposition of the 3-cube for each $s = 5, 6, \dots, 15$.

Proposition 5.1. *Each of the eleven decompositions of* [1, Appendix B] *is a tile partition of* $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ *satisfying Definition 1.1, with* $s = 5, 6, \ldots, 15$ *respectively. Hence* $\{5, 6, \ldots, 15\} \subseteq \mathcal{A}(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)$*.*

Proof. Each is checked against Definition 1.1 directly: disjointness, coverage, $s \geq 3$, and the full quantifier over all 2^s index sets, the largest instance being 2^{15} subsets. The decompositions are the authors' own; nothing is claimed here beyond the check. $\square$

5.2. **Small numbers of tiles.**

Theorem 5.2. $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ *admits no* O_3*-tile decomposition with exactly three tiles, and none with exactly four.*

Proof sketch. Both statements are established by exhaustive computation, verified formally. For $s = 3$ the computation runs over all ordered pairs (t_1, t_2) of the 343 tiles of the cube; in a partition the cell set of t_3 is forced to be the complement of the union of the first two, so the third tile is not a free choice but a test: the complement must itself be a tile and the resulting triple must satisfy Definition 1.1. For $s = 4$ the same reduction leaves all ordered triples of tiles, about 4×10^7 of them, with early abandonment as soon as the first two chosen tiles overlap. There is no ordering convention and no completeness lemma to prove: the enumeration is over all ordered tuples of the whole alphabet, and every tile of the cube is in that alphabet. The two searches together take 82 seconds. Outside the formal development the same two facts were reproduced by an independent program using a different search order: the cube has 111 partitions into three tiles, none of them with all tiles admissible, and 1143 partitions into four tiles, of which 297 have all tiles admissible and every one of those contains two tiles whose union is a tile. (The case $s = 3$ is also a special case of Theorem 7.3, which needs no computation.) $\square$

5.3. **Sixteen tiles.** By Theorem 3.3 the only value not yet decided for the cube is $s = 16$, and the same theorem says what such a decomposition would look like: five single cells and eleven dominoes. That is a strong enough restriction to search.

Theorem 5.3. $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ *admits no* O_3*-tile decomposition with exactly sixteen tiles.*

Proof sketch. By Theorem 3.3 such a decomposition uses only tiles of at most two cells, of which the cube has 108: its 27 single cells and 81 dominoes. Moreover its single cells cover exactly five cells and its dominoes exactly twenty-two. The search of Proposition 4.2 is run over that alphabet: the 27 cells are scanned in a fixed order, and at the first uncovered cell the seven small tiles containing it — the cell itself and the six dominoes on it — are tried in turn. A branch is abandoned when the five-cell or twenty-two-cell budget is exceeded, or when some proper tile of at most nine cells containing the cell just filled is exactly the union of at least two placed tiles, which Proposition 4.2 forbids. Every branch is abandoned; the search performs 575 077 placements and takes 24 seconds. Its completeness is proved, so the conclusion is that no sixteen-tile decomposition exists. An independently written program in C performs the same search and reports the same number of placements. $\square$

Theorem 5.4. *For every* s*, the cube* $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ *admits an* O_3*-tile decomposition with exactly* s *tiles if and only if* $5 \leq s \leq 15$*; that is,* $\mathcal{A}(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3) = \{5, 6, \ldots, 15\}$*.*

Proof. Values $s \leq 2$ are excluded by Definition 1.1 itself, $s = 3$ and $s = 4$ by Theorem 5.2, $s \geq 17$ by Theorem 3.3, and $s = 16$ by Theorem 5.3. The values $5, \ldots, 15$ are attained by Proposition 5.1. $\square$

Corollary 5.5. *Through the tile-to-UPB theorem of* [1]*, which is cited and not reproved here, the* O_3*-tile construction produces unextendible product bases of* $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ *of cardinality* $28 - s$ *for exactly the values* $s \in \{5, \ldots, 15\}$*, that is, of cardinalities* $13, 14, \ldots, 23$ *and no others.*

Remark 5.6. Corollary 5.5 constrains the construction, not the objects: it does not exclude unextendible product bases of $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ of cardinality 24 or 25 obtained by other means, and indeed [1, Table 2] lists 7 as an attained cardinality, which no tile decomposition of the cube produces.

Remark 5.7. Outside the formal development, a complete enumeration of the O_3-tile decompositions of the cube — a depth-first exact cover over the 343 tiles, pruned by admissibility and by the $|J| = 2$ instances of Definition 1.1, with the full condition applied at every complete cover — finds 5 064 336 of them in 58 seconds over 54 352 072 search nodes, namely

$$270,\ 1944,\ 12960,\ 46980,\ 267786,\ 804492,\ 1420956,\ 1444608,\ 865728,\ 193752,\ 4860$$

for $s = 5, 6, \dots, 15$ and none for $s \geq 16$. These counts are computations, not theorems of the present development; only the statement that the count vanishes for $s \notin \{5, \dots, 15\}$ is, by Theorem 5.4.

6. Other boxes

The argument of Section 3 never uses $d_i = 3$, and the search of Section 4 is stated for an arbitrary box, so both apply verbatim to the other boxes for which [1] reports computations. The attainability halves come from the authors' archive [2], which stores one decomposition per box and per value of s in a range fixed in advance; the paper itself prints none of these, its Appendix B being the 3-cube only.

6.1. $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$.

Theorem 6.1. *$\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ admits no O_3-tile decomposition with three tiles and none with four; it admits none with twelve or more tiles; and one with exactly eleven tiles would have every tile of at most three cells.*

Proof sketch. The two small values are excluded exactly as in Theorem 5.2, by an enumeration over all ordered tuples of the 147 tiles of this box with the last tile forced to be the complement of the others. The ceiling is Theorem 3.2 with $\lambda = 2$, $\alpha = 5$ and $(P_1, P_2, P_3) = (9, 6, 6)$: $5s \leq 2{\cdot}18 + 21 = 57$, so $s \leq 11$. For the shape at $s = 11$ the weighted form leaves a budget of $57 - 55 = 2$, and a tile with four or more cells already consumes three units of it. $\square$

Proposition 6.2. *Each of the six decompositions of $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ recorded in* [2]*, with $s = 5, \dots, 10$, satisfies Definition 1.1; so do the recorded decompositions of $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$, $\mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ with $s = 5, \dots, 16$. In particular $\{5, \dots, 10\} \subseteq \mathcal{A}(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3)$ and $\{5, \dots, 16\}$ is attainable for each of the other four boxes.*

Proof. Fifty-four decompositions in total, read from [2] and checked against Definition 1.1 with its quantifier over all 2^s index sets. All 87 recorded decompositions of these five boxes were in addition checked outside the formal development by two independent implementations of Definition 1.1. The decompositions are the authors' own. $\square$

Theorem 6.3. *$\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ admits no O_3-tile decomposition with eleven tiles. Hence, for every s, it admits one with exactly s tiles if and only if $5 \leq s \leq 10$; that is, $\mathcal{A}(\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3) = \{5, 6, \dots, 10\}$.*

Proof sketch. Write n_k for the number of tiles with k cells in a hypothetical eleven-tile decomposition. By Theorem 6.1 every tile has at most three cells, so $n_1 + n_2 + n_3 = 11$, and counting cells, $n_1 + 2n_2 + 3n_3 = 18$; hence $n_2 + 2n_3 = 7$. A three-cell tile of this box is a full $\mathbb{Z}_3$-row, whose slack in the weighted inequality of Theorem 3.2 is $2 \cdot 3 + 1 - 5 = 2$, which is the entire budget $57 - 55 = 2$; so $n_3 \leq 1$ and $(n_1, n_2, n_3) \in \{(4, 7, 0), (5, 5, 1)\}$. Each profile fills the eighteen cells exactly, so any complete cover found under either budget has exactly eleven tiles, and it suffices to run the search of Proposition 4.2 once per profile. Both close: 2 681 and 4 751 placements, with every branch abandoned. The attainability half is Proposition 6.2, and the remaining values are excluded by Theorem 6.1 and by the clause $s \geq 3$ of Definition 1.1. $\square$

By the tile-to-UPB theorem of [1] the corresponding unextendible product bases of $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$ have cardinalities exactly $19 - s = 9, 10, \dots, 14$, which is the range [1, Table 2] lists for that system apart from the isolated entry 6.

Remark 6.4. An independent enumeration outside the formal development counts

$$90, \quad 432, \quad 1512, \quad 3204, \quad 1614, \quad 72$$

decompositions of this box for $s = 5, \dots, 10$ respectively, and none for $s = 11$. As in Remark 5.7 these counts are computations, and only the vanishing at $s = 11$ is a theorem here.

6.2. **The four larger boxes.**

Theorem 6.5. *None of $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$, $\mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ admits a tile decomposition of the kind of Definition 1.1 with exactly three tiles, and $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_4$ admits none with exactly four.*

Proof sketch. As in Theorem 5.2: the enumeration runs over all ordered tuples of the box's tiles with the last tile forced to be the complement of the union of the others, so it is exhaustive by construction, and the step from the finite computation to the statement is proved once for an arbitrary box. The four three-tile enumerations, over 735^2, 1519^2, 1575^2 and 1029^2 ordered pairs respectively, together take 17.9 seconds. The four-tile enumeration for $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_4$ examines 81 264 300 disjoint ordered triples and takes 280 seconds. Independent enumerations in C reproduce the same conclusion at every box, finding no tile partition into at most four admissible tiles that even covers the box. The four three-tile statements are also instances of Theorem 7.3. □

Proposition 6.6. *Every decomposition recorded in* [2] *for $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_4$, $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$, $\mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ — with s running up to* 20, 23, 24 *and* 30 *respectively — satisfies Definition 1.1. Hence $\{5, \dots, 20\}$, $\{5, \dots, 23\}$, $\{5, \dots, 24\}$ and $\{5, \dots, 30\}$ are attainable for these four boxes.*

Proof. Thirty-three further decompositions, read from [2] and certified through the test of Proposition 4.1, which is what makes the values $s = 17, \dots, 30$ affordable at all: the direct quantifier over 2^{30} index sets is not. The conversion from the archive's format was calibrated against three decompositions transcribed independently, which it reproduces exactly, and all 219 decompositions the archive records, over 19 boxes, were checked outside the formal development in two independent ways. □

Theorem 6.7. *For $\mathcal{C} = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_4$ and every $s \neq 21$: $\mathcal{C}$ admits an O_3-tile decomposition with exactly s tiles if and only if $5 \leq s \leq 20$. For the remaining three boxes, every attainable s satisfies $s = 4$ or $5 \leq s$, and $s \leq 25$, $s \leq 27$, $s \leq 34$ respectively.*

Proof. For $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_4$: $s \leq 2$ is excluded by Definition 1.1, $s = 3$ and $s = 4$ by Theorem 6.5, $s \geq 22$ by Theorem 3.4, and $s = 5, \dots, 20$ are attained by Proposition 6.6. Only $s = 21$ is left. For the other three boxes the same three ingredients give the stated brackets, with $s = 4$ not excluded there. □

The value $s = 21$ for $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_4$ is the natural next target: the weighted form of Theorem 3.2 leaves budget $2 \cdot 36 + (12 + 12 + 9) - 5 \cdot 21 = 0$, so such a decomposition would consist of six single cells and fifteen dominoes, with six dominoes in each of the first two directions and three in the third, and the corresponding search visits about 3.6×10^6 nodes. The values $s = 4$ on the three boxes other than $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_4$, and the top few values on each of them, are open for the same reason: they are searches whose cost was measured and not paid.

7. Three tiles are never enough

Sections 5 and 6 excluded $s = 3$ box by box, each time by an enumeration over all ordered pairs of that box's tiles: 343^2, 147^2, 735^2, 1519^2, 1575^2 and 1029^2. For $s = 3$ that enumeration can be dispensed with altogether. The statement holds at every box, and the proof is a two-case argument about the product structure with no finite input anywhere.

In this section $N \geq 1$ and $d_1, \dots, d_N$ are arbitrary — in particular bipartite boxes $\mathbb{Z}_m \times \mathbb{Z}_n$ are allowed, which the standing convention of Section 2 excluded and at which neither [1] nor its archive computes. Nothing is assumed beyond Definition 1.1, which [1] states for general N. We first record the two facts about the encoding of cells that the proof uses; both are proved with the box left free, so no box-by-box computation enters below.

Lemma 7.1. *Let $t = R_1 \times \cdots \times R_N$ and $t' = R'_1 \times \cdots \times R'_N$ be tiles of $\mathcal{C}$. Then $t \cap t' = \emptyset$ if and only if $R_q \cap R'_q = \emptyset$ for some coordinate q.*

Proof sketch. $t \cap t' = \prod_i (R_i \cap R'_i)$, and a product of sets is empty exactly when one of its factors is. In the formal development a cell of $\mathcal{C}$ is a bit position of a natural number, in the mixed-radix order, and the cell set of a tile is assembled coordinate by coordinate out of the bit masks of its factors. What must be proved there is that this assembly carries the coordinatewise intersection of two tile codes to the intersection of the corresponding cell masks, and that the cell mask of a tile is nonzero; both follow from one description of the bit at position n of an assembled mask, and both hold with N and the d_i free. □

Proposition 7.2. *The union identity used in the proof of Lemma 3.1 — that replacing the q-th factor R_q of a tile by $R_q \cup m$ produces the union of the corresponding cell sets — holds at every box, with no finite verification.*

Proof sketch. It is the linearity of the coordinate-by-coordinate assembly in a single coordinate, and it follows from the same bit-level description as Lemma 7.1. Previously the identity was verified separately at each box, by inspecting that box's tile alphabet against every subset of every coordinate; the general proof removes that verification everywhere, including at boxes whose alphabet is too large to inspect. Consequently Lemma 3.1, and with it every ceiling derived from Theorem 3.2, is available at an arbitrary box without computation. □

Theorem 7.3. *No box $\mathcal{C} = \mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_N}$ admits an O_N-tile decomposition with exactly three tiles.*

Proof. Suppose $\mathcal{C} = t_1 \sqcup t_2 \sqcup t_3$ is one, with $t_a = R^a_1 \times \cdots \times R^a_N$. The three tiles are pairwise disjoint, so by Lemma 7.1 we may choose coordinates p, q, r with

$$R^1_p \cap R^2_p = \emptyset, \qquad R^1_q \cap R^3_q = \emptyset, \qquad R^2_r \cap R^3_r = \emptyset.$$

Case 1: two of p, q, r coincide. Then one coordinate separates one of the three tiles from both of the others; after renaming the tiles, say $p = q$, so that R^1_p is disjoint from R^2_p and from R^3_p. Put $u = R^1_p \times \prod_{i \neq p} \mathbb{Z}_{d_i}$. Then u is disjoint from t_2 and from t_3, so $u \subseteq t_1$ because the three tiles cover $\mathcal{C}$; and $t_1 \subseteq u$ because $R^1_i \subseteq \mathbb{Z}_{d_i}$ for every $i \neq p$. Hence $t_1 = u$ is a slab, and

$$t_2 \sqcup t_3 \;=\; \mathcal{C} \setminus u \;=\; (\mathbb{Z}_{d_p} \setminus R^1_p) \times \prod_{i \neq p} \mathbb{Z}_{d_i}$$

is again a tile: its p-th factor is nonempty because R^1_p misses the nonempty set R^2_p. Taking $J = \{2, 3\}$, which satisfies $1 < |J| = 2 < 3 = s$, contradicts Definition 1.1.

Case 2: p, q, r are pairwise distinct. Let e be the tile whose p-th factor is R^1_p, whose q-th factor is R^3_q, whose r-th factor is R^2_r, and whose remaining factors are all of $\mathbb{Z}_{d_i}$. Every factor of e is nonempty, so e is a tile of $\mathcal{C}$ and $e \neq \emptyset$. But $e \cap t_2 = \emptyset$ because their p-th factors are disjoint, $e \cap t_1 = \emptyset$ because their q-th factors are, and $e \cap t_3 = \emptyset$ because their r-th factors are. So e is disjoint from $t_1 \sqcup t_2 \sqcup t_3 = \mathcal{C}$ while $\emptyset \neq e \subseteq \mathcal{C}$ — a contradiction. It is here that the three coordinates must be distinct: the three requirements on e then constrain three different factors and can be met simultaneously. □

Corollary 7.4. $3 \notin \mathcal{A}(\mathcal{C})$ *for every box $\mathcal{C}$. In particular the $s = 3$ halves of Theorems 5.2, 6.1 and 6.5 hold with no computation, and the same conclusion holds at each of the* 19 *boxes over which* [2] *records searches, at every bipartite grid $\mathbb{Z}_m \times \mathbb{Z}_n$, and at boxes far beyond any enumeration, such as $\mathbb{Z}_2^{100}$ and $\mathbb{Z}_{10^6} \times \mathbb{Z}_{10^6}$.*

Remark 7.5. The following statement is proved here in prose and is *not* part of the formal development. At every bipartite box $\mathbb{Z}_m \times \mathbb{Z}_n$ the value $s = 4$ is unattainable as well; with Theorem 7.3 and the clause $s \geq 3$ of Definition 1.1 it follows that the least number of tiles of an O_2-tile decomposition of a grid is exactly 5, a value attained at $\mathbb{Z}_3 \times \mathbb{Z}_3$. The argument classifies, for a hypothetical $t_i = R_i \times S_i$ $(i = 1, \ldots, 4)$, the *classes* $I(x) = \{i : x \in R_i\}$ of the rows $x \in \mathbb{Z}_m$. The tiles containing x cut its row into the S_i with $i \in I(x)$, so those S_i partition $\mathbb{Z}_n$; no class is a singleton and no tile lies in every class, since either would make some tile a slab, whose complement is a tile and violates the condition at $|J| = 3$; two classes are never $\subseteq$-comparable, and never differ by exchanging one index for another, since either forces two tiles to share a factor and hence to merge at $|J| = 2$. With four tiles these constraints leave five possible class families, and each of them either leaves a tile in no class or puts two tiles in exactly the same classes, both impossible. Outside the formal development the statement was confirmed at the ten grids $\mathbb{Z}_m \times \mathbb{Z}_n$ with $2 \leq m \leq n \leq 5$: no grid has a four-tile decomposition, while $s = 5$ is attained at $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_3 \times \mathbb{Z}_4$, $\mathbb{Z}_4 \times \mathbb{Z}_4$ and $\mathbb{Z}_3 \times \mathbb{Z}_5$ in 18, 108, 648 and 450 ways respectively.

Remark 7.6. Read without its quantum-informational motivation, Theorem 7.3 says that a partition of a product of finite sets into three combinatorial boxes always has two parts whose union is a box — indeed one part is always a slab. The nearest literature we found is the work on minimal partitions of a box into boxes [13, 15], where the parts are required to be *proper* — every factor a nonempty proper subset, with every $d_i \geq 2$ — so that a partition has at least 2^N parts, by the theorem of Alon, Bohman, Holzman and Kleitman [14], and three-part partitions therefore do not occur at all for $N \geq 2$; or else the parts are required to be pairwise *dichotomous* in the sense of [15], meaning that in some coordinate one part's factor is the complement of the other's, which is stronger than being disjoint. Tiles in the sense of [1] need be neither, and we found no statement of Theorem 7.3 in that literature.

8. The qubit boxes

Write $\mathbb{Z}_2^N$ for the N-fold product $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$; it has 2^N cells and 3^N tiles. These boxes differ from the tripartite ones treated above in two ways. On the combinatorial side, a tile of $\mathbb{Z}_2^N$ has 2^k cells, where k is the number of factors equal to $\mathbb{Z}_2$, and $\sum_q \iota_q(t)$ equals N, 1 or 0 according as $k = 0$, $k = 1$ or $k \geq 2$; so the choice $\lambda = 2$ that is optimal for a tripartite box ceases to be optimal once $N \geq 4$, and Theorem 8.9 below is what the right choice gives. On the quantum-informational side, the tile-to-UPB theorem sends decompositions of $\mathbb{Z}_2^N$ to unextendible product bases of the multi-qubit systems $(\mathbb{C}^2)^{\otimes N}$, whose possible cardinalities have been studied in their own right; Remarks 8.3, 8.8 and 8.11 record exactly what that comparison does and does not give.

8.1. $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$.

Theorem 8.1. $\mathcal{A}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) = \{5\}$.

Proof sketch. The value $s = 3$ is excluded by Theorem 7.3 and $s \leq 2$ by Definition 1.1. For $s = 4$ the enumeration of Theorem 5.2 is run over all ordered triples of the 27 tiles of this box with the fourth tile forced to be the complement of the union of the others, and is exhaustive by construction. The ceiling is Theorem 3.2 with $\lambda = 2$, $\alpha = 5$ and $P_1 = P_2 = P_3 = 4$: $5s \leq 2 \cdot 8 + 12 = 28$, so $s \leq 5$. Finally $s = 5$ is attained by the one decomposition [2] records at this box, checked against Definition 1.1; its tiles are $\{0\} \times \{0\} \times \{0\}$, $\{0\} \times \{1\} \times \mathbb{Z}_2$, $\{1\} \times \{1\} \times \{1\}$, $\{1\} \times \mathbb{Z}_2 \times \{0\}$ and $\mathbb{Z}_2 \times \{0\} \times \{1\}$. Outside the formal development an independent enumeration finds exactly eight O_3-tile decompositions of this box, all with $s = 5$, forming one orbit under the 48 symmetries of the box. $\square$

By the tile-to-UPB theorem the O_3-tile route in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ therefore produces exactly the cardinality $8 - 5 + 1 = 4$ and no other. That is the only cardinality [1, Table 2] records for this system; the entry is attributed there to [5] and was not produced by the authors' own search, whose driver was run at this box only at $s = 5$.

Theorem 8.2. $\mathcal{A}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3) = \{5, 6, 7\}$.

Proof sketch. As before $s \le 2$ and $s = 3$ are excluded by Definition 1.1 and Theorem 7.3, and $s = 4$ by the enumeration over all ordered triples of the 63 tiles of this box with the fourth forced. The ceiling is Theorem 3.2 with $\lambda = 2$, $\alpha = 5$ and $(P_1, P_2, P_3) = (6, 6, 4)$: $5s \le 2 \cdot 12 + 16 = 40$, so $s \le 8$. The values $s = 5, 6, 7$ are attained by the three decompositions [2] records at this box, checked against Definition 1.1. There remains $s = 8$, where the weighted form of Theorem 3.2 leaves budget $40 - 5 \cdot 8 = 0$: every tile has zero slack, hence at most two cells, and $n_1 + n_2 = 8$ with $n_1 + 2n_2 = 12$ forces four single cells and four dominoes. The search of Proposition 4.2 at that budget abandons every branch. Outside the formal development an independent enumeration counts 24, 72, 24 decompositions at $s = 5, 6, 7$ and none at any other s. □

Here the tile-to-UPB theorem gives the cardinalities $13 - s = 6, 7, 8$ in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$, exactly the row [1, Table 2] prints for that system. The value $s = 8$ had been run nowhere: the driver of [2] is hard-coded to $s \in \{5, 6, 7\}$ at this box, as it is to $s = 5$ at $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Remark 8.3. Parts of Theorems 8.1 and 8.2 can also be reached from the UPB literature through the tile-to-UPB theorem of [1], which we cite and do not reprove. A three- or four-tile decomposition of $\mathbb{Z}_2^3$ would give a three-qubit UPB of cardinality 6 or 5, and no n-qubit UPB has cardinality $2^n - 1$, $2^n - 2$ or $2^n - 3$. That is [7, Proposition 2], where the size $2^n - 1$ is called well known — a pure state with positive partial transpose is separable — and the sizes $2^n - 2$ and $2^n - 3$ are deduced from the theorem of Chen and Đoković [8] that a multipartite state of rank 2 or 3 with positive partial transpose is separable; the neighbouring size $2^n - 5$, which [7] settles for $n = 4$ by computer search and leaves open for $n \ge 5$, was excluded in [11]. A decomposition of $\mathbb{Z}_2^3$ with $s \ge 6$ would give a UPB of cardinality at most 3, below the minimum $f(3) = 4$ of [6, Theorem 1]. Likewise an eight-tile decomposition of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ would give a UPB of $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^3$ of cardinality 5, whereas the minimum cardinality for that system is 6 — a value [9, §2] records and attributes to Feng [10]. What is proved here is the tile-side statement, in the combinatorial category and without the passage through quantum information; but a reader who grants the tile-to-UPB theorem should regard the negative halves of these two theorems as known.

8.2. **$\mathbb{Z}_2^4$: an attainable set with a gap.**

Theorem 8.4. $\mathcal{A}(\mathbb{Z}_2^4) = \{5, 7, 8, 9\}$. *In particular $\mathbb{Z}_2^4$ admits O_4-tile decompositions with five tiles and with seven tiles, but none with six.*

Proof sketch. The ceiling comes from Theorem 3.2 at $N = 4$ with $\lambda = 3$, so that $\alpha = \min(3 + 4, 7, 12) = 7$, and $P_q = 8$ for each of the four coordinates: $7s \le 3 \cdot 16 + 32 = 80$, that is $s \le 11$. The choice $\lambda = 2$ used at the tripartite boxes would give only $s \le 12$; one value of s is removed by the choice of weight alone.

The values $s \le 2$ and $s = 3$ are excluded by Definition 1.1 and Theorem 7.3; $s = 4$ by the enumeration over all ordered triples of the 81 tiles of this box with the fourth forced to be the complement of the union of the others, exhaustive by construction.

For $s = 6, 10, 11$ we use the weighted form of Theorem 3.2 with $w(t) = 3|t| + \sum_q \iota_q(t) - 7$, which is 0 for a single cell and for a domino, 5 for a four-cell tile, 17 for an eight-cell slab and 41 for the whole box; the weighted inequality reads $7s + \sum_j w(t_j) \le 80$. Writing n_k for the number of tiles with k cells, so that $\sum_k n_k = s$ and $\sum_k k\, n_k = 16$, the residual budget $80 - 7s$ pins the profile (n_1, n_2, n_4, n_8) in each case: at $s = 11$ the budget is 3, so every tile is a single cell or a domino and the profile is $(6, 5, 0, 0)$; at $s = 10$ it is 10, so there is no slab and at most two four-cell tiles, leaving $(4, 6, 0, 0)$, $(6, 3, 1, 0)$ and $(8, 0, 2, 0)$; at $s = 6$ it is 38, so the whole box is excluded and there are at most two slabs, leaving $(4, 0, 1, 1)$, $(2, 3, 0, 1)$, $(2, 1, 3, 0)$ and $(0, 4, 2, 0)$. Each of these eight profiles is ruled out by one run of the search of Proposition 4.2 at the corresponding budget; the eight runs perform 9 261 placements in total, and all eight are kernel evaluations rather than compiled ones.

For attainability, $s = 5$ is inherited from Theorem 8.1: appending a full coordinate to every tile of an O_N-tile decomposition of $\mathcal{C}$ yields an O_{N+1}-tile decomposition of $\mathcal{C} \times \mathbb{Z}_d$, because $A \times \mathbb{Z}_d$ forms a tile exactly when A does and the sub-union condition is unchanged. Decompositions with $s = 7, 8, 9$ are exhibited; they were found by the enumeration described in Remark 8.6 and are checked against Definition 1.1 with its full quantifier over 2^s index sets. □

Corollary 8.5. *Through the tile-to-UPB theorem of* [1]*, the O_4-tile construction produces unextendible product bases of $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ of cardinality $17 - s$ for exactly the values $s \in \{5, 7, 8, 9\}$, that is, of cardinalities $8, 9, 10, 12$ and no others.*

Remark 8.6. Outside the formal development, an independently written complete enumeration — an exact cover over the 81 tiles, branching on the lowest uncovered cell so that each tile partition is reached once, with the full condition of Definition 1.1 applied at every complete cover — finds 856 O_4-tile decompositions of $\mathbb{Z}_2^4$ among 89 512 tile partitions, namely 32, 576, 200 and 48 at $s = 5, 7, 8, 9$ and none at any other s. Before being run at $\mathbb{Z}_2^4$ this enumerator was required to reproduce the counts already known here for $\mathbb{Z}_2^3$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$ and $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, which it does exactly. As in Remark 5.7 these counts are computations; only their vanishing outside $\{5, 7, 8, 9\}$ is a theorem here, by Theorem 8.4.

Remark 8.7. $\mathbb{Z}_2^4$ is the only box in Table 1 at which [1] reports no computation. Its archive [2] drives a satisfiability search over twenty boxes, listed explicitly in the driver, and $(2, 2, 2, 2)$ is not among them; none of the 219 recorded searches is over that box, and the archive holds no artefact for it. Table 2 of [1] has a single quadripartite entry, for $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$. So every value of s in Theorem 8.4 is decided here for the first time.

Remark 8.8. As in Remark 8.3, part of the negative half of Theorem 8.4 is implied by published results about four-qubit UPBs, through the tile-to-UPB theorem. Johnston [7] classifies the 1446 inequivalent nontrivial four-qubit UPBs — those of fewer than 16 states — and reports there that none has 11 states; a six-tile decomposition of $\mathbb{Z}_2^4$ would give one, so $6 \notin \mathcal{A}$ follows. The exclusions $s \geq 12$ follow from the minimum $f(4) = 6$ of [6], and $s = 4$ from the nonexistence quoted in Remark 8.3. The exclusions $s = 10$ and $s = 11$ do *not* follow from that classification — four-qubit UPBs of 7 and of 6 states exist — but they are consistent with, and were anticipated by, a remark of Sainz et al. [12]: in the scenario of four parties with two measurements each, where a UPB has at most two bases per site, they report that the unextendible product bases obtained in this way "have only 8, 9, 10 or 12 elements", and $\{8, 9, 10, 12\}$ is exactly the set of Corollary 8.5. The coincidence is not accidental: at $d_i = 2$ the local states produced by the tile-to-UPB theorem are $|0\rangle$, $|1\rangle$, $|+\rangle$, $|-\rangle$ and the stopper is $|+\rangle^{\otimes N}$, so a tile-derived qubit UPB uses two bases per site. Their sentence, however, reports the output of a particular search, whereas Theorem 8.4 and Remark 8.6 are an exhaustive statement about the O_4-tile route: it attains each of $8, 9, 10, 12$ and nothing else. We record all of this because a reader who grants the tile-to-UPB theorem may reasonably regard the negative half of Theorem 8.4 as known; what is new is the tile-side statement, the enumeration, and the fact that the gap at $s = 6$ is a phenomenon of the tile problem itself.

8.3. **Ceilings at the larger qubit boxes.**

Theorem 8.9. *Every O_5-tile decomposition of $\mathbb{Z}_2^5$ has at most 23 tiles; every O_6-tile decomposition of $\mathbb{Z}_2^6$ has at most 46; every O_7-tile decomposition of $\mathbb{Z}_2^7$ has at most 93.*

Proof sketch. Theorem 3.2 with $\lambda = N - 1$, so that $\alpha = \min(\lambda + N, 2\lambda + 1, 4\lambda) = 2N - 1$ for $N \geq 2$, and with $P_q = 2^{N-1}$ for each of the N coordinates. The inequality reads

$$(2N - 1)\, s \ \leq\ (N - 1)\, 2^N + N\, 2^{N-1},$$

which for $N = 5, 6, 7$ is $9s \leq 208$, $11s \leq 512$ and $13s \leq 1216$, giving $s \leq 23$, $s \leq 46$ and $s \leq 93$. The only finite input is the per-tile inequality $\alpha \leq \lambda|t| + \sum_q \iota_q(t)$, checked over the 243, 729 and 2187 tiles of the three boxes; the union identity that Lemma 3.1 needs is supplied by Proposition 7.2 and costs nothing. The bound is not vacuous: $\mathbb{Z}_2^5$ has a nine-tile decomposition, obtained from the nine-tile decomposition of $\mathbb{Z}_2^4$ by appending a full fifth coordinate, and $9 \leq 23$. The choice of λ is not free either: at $\mathbb{Z}_2^5$ the tripartite choice $\lambda = 2$, $\alpha = 5$ also satisfies the per-tile inequality and yields the weaker $s \le 28$, while one step past the optimum, $\lambda = 4$ with $\alpha = 10$, fails it — a domino has $4 \cdot 2 + 1 = 9 < 10$ — so $s \le 20$ is not available from this argument. □

Remark 8.10. The following is proved here in prose and is *not* part of the formal development. At every all-qubit box the same computation gives

$$(2N-1)\, s \;\le\; 2^{N-1}(3N-2), \qquad \text{i.e.} \qquad s \;\le\; \left\lfloor \tfrac{2^{N-1}(3N-2)}{2N-1} \right\rfloor,$$

and $\lambda = N-1$ is the optimal choice: since a tile of $\mathbb{Z}_2^N$ has $(|t|, \sum_q \iota_q(t))$ equal to $(1, N)$, to $(2, 1)$, or to $(2^k, 0)$ with $k \ge 2$, the per-tile inequality holds exactly when $\alpha \le \min(\lambda + N, 2\lambda + 1, 4\lambda)$, and the resulting ceiling $(\lambda 2^N + N 2^{N-1})/\alpha$ equals $2^{N-1}\big(1 + (N-1)/(2\lambda+1)\big)$ for $\lambda \le N-1$ and $2^{N-1}(2\lambda + N)/(\lambda + N)$ for $\lambda \ge N-1$, decreasing and then increasing. The values at $N = 3, \ldots, 7$ are $5, 11, 23, 46, 93$, the first two being the ceilings of Theorems 8.1 and 8.4 and the last three those of Theorem 8.9; the optimality of $\lambda = N-1$ was also checked by brute force over $\lambda \le 60$ for $N = 3, \ldots, 10$. Formalising this with N free needs a closed formula for the number of cells of a tile of $\mathbb{Z}_2^N$, which the present development has only box by box.

Remark 8.11. The following comparison, too, lies outside the formal development; it composes two published results and is recorded because it is the natural competitor of Theorem 8.9. An s-tile decomposition of $\mathbb{Z}_2^N$ gives, by the tile-to-UPB theorem of [1], an N-qubit UPB of cardinality $2^N - s + 1$; so $s \le 2^N + 1 - f(N)$, where $f(N)$ is the minimum cardinality of an N-qubit UPB, determined in [6, Theorem 1] to be $N+1$ for odd N, $N+2$ for $N = 4$ and $N \equiv 2 \pmod 4$, $N+3$ for $N = 8$ and $N+4$ otherwise (the values were confirmed independently against sequence A211390 of the On-Line Encyclopedia of Integer Sequences). This gives $s \le 5, 11, 27, 57, 121$ at $N = 3, \ldots, 7$. The comparison with Remark 8.10 is

N	3	4	5	6	7
Theorem 3.2 at $\lambda = 2$	5	12	28	64	140
Theorem 3.2 at $\lambda = N-1$	5	11	**23**	**46**	**93**
$2^N + 1 - f(N)$	5	11	27	57	121

so the bound obtained by citation ties the counting inequality at $N = 3, 4$ and is strictly weaker from $N = 5$ on, the two sides behaving as $\frac{3}{4} \cdot 2^N$ against $2^N - N$. In particular the three ceilings of Theorem 8.9 are not consequences of the UPB literature.

9. Sharpness, and negative controls

Definition 1.1 is a conjunction of four clauses, and an exhaustive search that silently weakened any one of them would still prove theorems — about a different definition. The following statements were verified for that reason. In each case the exhibited object satisfies every clause but one.

Proposition 9.1. *For $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ there exist: a tile partition into two tiles, rejected only by the clause $s \ge 3$; a list of five tiles covering $\mathcal{C}$ but with a repetition, rejected only by disjointness; a list of four pairwise disjoint tiles, rejected only by coverage; a tile partition into six tiles, rejected only by the sub-union clause and already at $|J| = 2$; and a tile partition into sixteen tiles no two of whose tiles have a union forming a tile, which is nevertheless not an O_3-tile decomposition, because five of its tiles are the layer $\{0\} \times \mathbb{Z}_3 \times \mathbb{Z}_3$.*

The last of these deserves a sentence. It shows that the quantifier over all index sets in Definition 1.1 is strictly stronger than its $|J| = 2$ instances, so no sharpening of the counting argument of Section 3 — which uses only those instances — can settle the top of the attainable range. It also refutes the claim $16 \in \mathcal{A}(\mathbb{Z}_3^3)$ at the level of a concrete candidate, independently of Theorem 5.3.

Proposition 9.2. *The same five controls exist for $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. Moreover there are tile partitions of that box into ten and into eleven tiles, no two of whose tiles have a union forming a tile, which* *are not O_3-tile decompositions; for the ten-tile one a violating index set with $|J| = 5$ is exhibited. The eleven-tile one sits at exactly the value that Theorem 6.1 leaves open.*

Remark 9.3. That $|J| = 5$ is the *smallest* violating size for those two partitions was checked outside the formal development; what is verified here is that no pair violates the condition and that the exhibited five-element set does.

Two further families of controls concern the searches rather than the definition. A search can close for the wrong reason — because its restrictions are too aggressive, or because it never explores anything — and the following statements exclude that.

Proposition 9.4. *The search of Theorem 5.3, run with its tile restriction weakened to proper tiles of at most four cells, reaches* 1296 *complete covers rather than none; run with the budget set for fifteen tiles instead of sixteen, it reaches* 4104*. The sixteen-tile partition of Proposition 9.1 is refuted a second time, as an instance of Theorem 5.3 rather than by a direct test of its* 2^{16} *index sets.*

The number 1296 is the count of sixteen-tile tile partitions of the cube in which no two tiles have a union forming a tile, so the weakened search recovers exactly the objects the pairwise condition cannot exclude; Theorem 5.3 says that the full condition excludes all of them.

Proposition 9.5. *The test of Proposition 4.1 and the direct quantifier of Definition 1.1 agree on a genuine ten-tile decomposition of* $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$*, on a list failing coverage, and on the two pairwise-irreducible partitions of Proposition 9.2; in particular the test is not a disguised pairwise test. The two searches of Theorem 6.3, run with the tile restriction weakened to proper tiles of at most four cells, reach* 36 *and* 0 *complete covers; run with no restriction at all they reach* 117 900 *and* 132 732*; and run with the budget set for a ten-tile decomposition the first reaches* 72 *complete covers, which is the number of ten-tile O_3-tile decompositions of that box found by an independent enumeration.*

The results of Sections 7 and 8 carry the corresponding controls.

Proposition 9.6. $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ *has a tile partition into three tiles, namely* $\{0\} \times \mathbb{Z}_2 \times \mathbb{Z}_2$*,* $\{1\} \times \{0\} \times \mathbb{Z}_2$ *and* $\{1\} \times \{1\} \times \mathbb{Z}_2$*; so Theorem 7.3 is not vacuous for want of three-tile partitions. It is rejected by the sub-union clause alone, and the violating index set is the one the first case of the proof predicts:* $J = \{2, 3\}$*, whose union is the slab* $\{1\} \times \mathbb{Z}_2 \times \mathbb{Z}_2$*. The theorem is furthermore instantiated at* $\mathbb{Z}_2^{100}$ *and at the bipartite grid* $\mathbb{Z}_{10^6} \times \mathbb{Z}_{10^6}$*, and the* $s = 3$ *statement of Theorem 6.5 for* $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_4$ *— previously an enumeration over* 735^2 *pairs — is re-derived from it.*

Proposition 9.7. *The search that excludes* $s = 8$ *for* $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$*, run with its sub-union restriction removed, reaches* 1149 *complete covers — the number of ways four single cells and four dominoes tile that box — and run with the budget of a seven-tile decomposition it reaches* 24*, the number of seven-tile decompositions an independent enumeration finds. So that search is emptied by the condition of Definition 1.1 and not by its budget.*

Proposition 9.8. $\mathbb{Z}_2^4$ *has a tile partition into six tiles no two of which have a union forming a tile, so neither Theorem 8.4 is vacuous for want of six-tile partitions at* $s = 6$*, nor can the* $|J| = 2$ *instances of Definition 1.1 — the only ones Theorem 3.2 uses — settle that value. A violating index set with* $|J| = 5$ *is exhibited, its union being the slab* $\{0\} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$*. The search used in Theorem 8.4, run so as to count complete covers instead of closing, reaches* 32*,* 576 *and* 48 *at the budgets of a five-, seven- and nine-tile decomposition — the counts of Remark 8.6 — and* 14 208 *at the eleven-tile budget with the sub-union restriction removed, which is that enumeration's number of tile partitions of shape* $(6, 5, 0, 0)$*; so the search at* $s = 11$ *is emptied by the condition and not by the budget.*

Remark 9.9. That $|J| = 5$ is the smallest violating size for the six-tile partition of Proposition 9.8 was checked outside the formal development, as in Proposition 9.2; what is verified here is that no pair violates the condition and that the exhibited five-element set does.

10. Verification

Every theorem, proposition, lemma and corollary of this paper has been formally verified in Lean 4 (toolchain `v4.32.0`); the development is free of `sorry` and imports no mathematical library, Mathlib included, so its trusted base is the Lean kernel alone. The exceptions, marked as such where they appear, are the enumeration counts that the remarks below describe as computations rather than theorems; the three statements proved here in prose only, in Remarks 7.5, 8.10 and 8.11; and Corollaries 5.5 and 8.5, whose tile-side content is verified but whose passage from tiles to unextendible product bases is the theorem of [1] that we cite and do not reprove. The repository is currently private: repository reference to be added at submission.

The arguments of Sections 3 and 4 — the merge lemma, the counting bound in its plain and weighted forms, the shape statements, the test of Proposition 4.1 and the completeness of the search of Proposition 4.2 — are checked by the kernel and depend on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`; so are the whole of Section 7, including Theorem 7.3, which computes nothing at all, and Proposition 7.2, which replaces a per-box computation by a proof. The eleven decompositions of Proposition 5.1 are likewise checked by the kernel, with no appeal to compiled evaluation.

What is delegated to compiled evaluation, through Lean's `native_decide`, is exactly some of the finite searches and sweeps: the two enumerations of Theorem 5.2 and their analogues in Theorem 6.5; the union identity of Lemma 3.1 at each box, until Proposition 7.2 removed it; the searches of Theorems 5.3 and 6.3; the certification of the archive's decompositions in Propositions 6.2 and 6.6; the $s = 4$ enumerations of Theorems 8.2 and 8.4; the two sweeps behind the case $N = 7$ of Theorem 8.9, whose kernel evaluation was measured at 23 gigabytes of resident memory and rejected on that ground; and the control computations of Section 9, apart from those of Proposition 9.6 and the partition and violating-set statements of Proposition 9.8, which are kernel evaluations. Theorem 8.1 uses no compiled evaluation anywhere, and neither do the eight searches behind Theorem 8.4 or the cases $N = 5, 6$ of Theorem 8.9: they are kernel evaluations, which is affordable because the searches are small.

In each case the mathematics is separated from the computation: the statement proved is that a specific Boolean-valued function evaluates to `true`, and every inference from that value — in particular the exhaustiveness of a search — is a kernel-checked theorem, as Proposition 4.2 describes. Every numerical value quoted was first produced by an independent implementation in C or Python, and the formal proof was then written against it; the node counts of the searches in Theorems 5.3, 6.3 and 8.4 agree exactly with those independent implementations, and the counts of Remarks 5.7 and 8.6 were reproduced by two differently written enumerators.

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