AN EXACT OPTIMAL CELL AVERAGE DECOMPOSITION FOR $\mathbb{P}^8$ AND $\mathbb{P}^9$

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Abstract. Cui, Ding and Wu introduced the optimal cell average decomposition (OCAD) problem, whose solution is exactly the largest bound-preserving CFL number of a Zhang–Shu limited high-order scheme on a Cartesian cell. They solved it for the tensor-product spaces $\mathbb{Q}^k$ for every k and for $\mathbb{P}^k$ with $1 \le k \le 7$, and conjectured that two identities — a strong duality and the existence of a critical positive polynomial that is a square — persist for all k; for $\mathbb{P}^k$ with $k \in \{8, \dots, 15\}$ they reported numerical evidence and wrote that a rigorous analytical proof was not available. We settle the first open case. Let φ_8 be the root of $72t^3 - 189t^2 + 28t - 1$ in $(0, \frac{1}{10})$. We prove that every feasible fully symmetric cell average decomposition for $\mathbb{P}^8$ at $\theta = 0$ has boundary weight at most φ_8, and we exhibit one that attains it, with four internal orbits and all data explicit in the real field obtained from $\mathbb{Q}(\varphi_8)$ by adjoining three square roots. Hence $\varpi_\star(0, \mathbb{P}^8) = \varpi_\star(0, \mathbb{P}^9) = \varphi_8$ and both conjectures hold at these two spaces. The rendered text of Cui–Ding–Wu records the value only as a decimal produced by an iterative solver; here it is an algebraic number of degree 3, and the decomposition attaining it is exact. The bound, the decomposition and the resulting value have been checked by the Lean 4 kernel; the step from them to the two conjectures is Cui–Ding–Wu's optimality criterion, quoted and not reproved.

1. Introduction

1.1. Cell average decompositions and the bound-preserving CFL number.

Let Ω_{ij} be a cell of a Cartesian mesh and let p be a polynomial of degree at most k on it — the reconstruction, or the discontinuous Galerkin solution, on that cell. The Zhang–Shu bound-preserving framework [3, 4] rests on writing the cell average $\langle p \rangle_\Omega$ as a convex combination

(1)
$$\langle p \rangle_\Omega \;=\; \big(\text{a positive multiple of the edge averages of } p\big) \;+\; \sum_s \omega_s \, p(\text{interior nodes}),$$

with all $\omega_s > 0$, valid for every p in the polynomial space. Such an identity is a *cell average decomposition* (CAD). Given one, a limiter that enforces the physical bounds at the boundary quadrature nodes and at the interior nodes enforces them for the updated cell average as well, and the largest admissible time step is proportional to the weight carried by the boundary part. The efficiency of the whole scheme is therefore governed by a single number, and maximising it is a sharp optimisation problem over quadrature rules.

Cui, Ding and Wu [1] formalised this as the *optimal cell average decomposition* (OCAD) problem and solved it in considerable generality. On the reference cell $\Omega = [-1, 1]^2$ write $\langle p \rangle_\Omega$ for the cell average, and $\langle p \rangle^x$, $\langle p \rangle^y$ for the means of the two

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edge averages in the x- and y-directions. For a parameter $\theta \in [-1,1]$ recording the anisotropy of the mesh and the wave speeds, a *symmetric CAD* for a polynomial space $\mathbb{V}^k$ is an identity

$$\langle p\rangle_\Omega \;=\; \varpi\big[(1+\theta)\langle p\rangle^x + (1-\theta)\langle p\rangle^y\big] \;+\; \sum_{s=1}^{S} \omega_s\, \overline{p(x^{(s)},y^{(s)})} \qquad \text{for all } p \in \mathbb{V}^k, \tag{2}$$

with $\omega_s > 0$, nodes $(x^{(s)},y^{(s)}) \in [0,1]^2$, and $\overline{\cdot}$ the average of p over the four sign changes of the node. The quantity to be maximised is ϖ; its optimal value is written $\varpi_\star(\theta,\mathbb{V}^k)$ in what follows and $\overline{\omega}_\star(\theta,\mathbb{V}^k)$ in [1], and the resulting optimal bound-preserving CFL condition is

$$\Delta t \;\le\; \frac{c_0\,\varpi_\star(\theta,\mathbb{V}^k)}{a_1/\Delta x + a_2/\Delta y}. \tag{3}$$

1.2. **The duality and the two conjectures.** Dually, for $p \in \mathbb{V}^k$ nonnegative on Ω put

$$\phi(p;\theta) \;=\; \frac{\langle p\rangle_\Omega}{(1+\theta)\langle p\rangle^x + (1-\theta)\langle p\rangle^y}, \qquad \phi^\star(\theta,\mathbb{V}^k_+) \;=\; \inf_{0\le p\in\mathbb{V}^k} \phi(p;\theta). \tag{4}$$

Testing (2) against a nonnegative p discards the interior sum and gives $\phi^\star \ge \varpi_\star$ [1, Lemma 3.20]. The two conjectures of [1] assert that this inequality is always an equality, and that the infimum is attained at a square.

Conjecture 1 ([1, Conjecture 3.1])**.** For $\mathbb{V}^k$ either $\mathbb{Q}^k$ or $\mathbb{P}^k$, with $k \in \mathbb{N}_+$, one has $\varpi_\star(\theta,\mathbb{V}^k) = \phi^\star(\theta,\mathbb{V}^k_+)$ for all $\theta \in [-1,1]$, and a critical positive polynomial for $\phi^\star(\theta,\mathbb{V}^k_+)$ always exists.

Conjecture 2 ([1, Conjecture 3.2])**.** Under the same hypotheses, $\phi^\star(\theta,\mathbb{V}^k_+) = \inf_{q\in\mathbb{V}^{\lfloor k/2\rfloor}} \phi(q^2;\theta)$, and there is a critical positive polynomial of the form $p^\star = q_\star^2$ with $q_\star \in \mathbb{V}^{\lfloor k/2\rfloor}\setminus\{0\}$.

Both conjectures follow at a given pair (k,θ) from one finite certificate. The criterion below is the source's own *Optimality Criterion #3*; we cite it, like everything else taken from [1], by name where it has one as well as by number, because the numbering used throughout is that of the arXiv version v1 (see the bibliography).

Lemma 3 (Optimality Criterion #3, [1, Theorem 3.26])**.** *Fix $\theta \in [-1,1]$ and a feasible symmetric CAD* (2) *with nodes $(x^{(s)},y^{(s)})$. If there is $q_\star \in \mathbb{V}^{\lfloor k/2\rfloor}\setminus\{0\}$ vanishing at every node, then $\varpi_\star(\theta,\mathbb{V}^k) = \phi^\star(\theta,\mathbb{V}^k_+) = \phi(q_\star^2;\theta) = \inf_{q\in\mathbb{V}^{\lfloor k/2\rfloor}}\phi(q^2;\theta)$, the CAD is optimal, and $q_\star^2$ is a critical positive polynomial.*

So a feasible decomposition together with a dual polynomial vanishing at its interior nodes settles both conjectures at that (k,θ).

1.3. **The frontier, and what is settled here.** The authors of [1] carry this out for $\mathbb{Q}^k$ with every k and for $\mathbb{P}^k$ with $1 \le k \le 7$, and then write, verbatim:

> "we will prove these two conjectures for the spaces $\mathbb{Q}^k$ with any $k \in \mathbb{N}_+$, and for the spaces $\mathbb{P}^k$ with $1 \le k \le 7$. Numerical evidence has demonstrated the validity of these conjectures for $\mathbb{P}^k$ spaces with higher $k \in \{8,9,\dots,15\}$, although a rigorous analytical proof is not available yet."

Their third appendix reports the decompositions for $k = 8, 10, 12, 14$ at $\theta = 0$ to eighteen decimal places per entry, presented as text inside figures and obtained, in their own words, by "numerically solving" the defining nonlinear system "using an iterative method". For $k \le 7$ their comparison table gives $\varpi_\star(0, \mathbb{P}^k)$ in closed form — $\frac{1}{2}$, $\frac{1}{4}$, $2 - \frac{\sqrt{14}}{2}$, $1 - \frac{\sqrt{30}}{6}$ — and for $k = 8, 9$ only the truncated decimal 0.05767.

This paper settles the first open case, $k = 8$ (equivalently $k = 9$) at $\theta = 0$. Write

$$F(t) \;=\; 72t^3 - 189t^2 + 28t - 1, \tag{5}$$

and let φ_8 denote its root in $(0, \frac{1}{10})$.

Theorem 4 (Main theorem)**.** *There is a real number φ_8 with $F(\varphi_8) = 0$ and*

$$0.0576717376260236 \;<\; \varphi_8 \;<\; 0.0576717376260237$$

such that every feasible fully symmetric cell average decomposition for $\mathbb{P}^8$ at $\theta = 0$ has boundary weight at most φ_8, and some such decomposition, with four internal orbits and strictly positive weights, attains it. Consequently

$$\varpi_\star(0, \mathbb{P}^8) \;=\; \varpi_\star(0, \mathbb{P}^9) \;=\; \varphi_8,$$

and Conjectures 1 and 2 hold at $(k, \theta) = (8, 0)$ and $(9, 0)$.

Three things are new. First, the value: $\varpi_\star(0, \mathbb{P}^8)$ is an algebraic number of degree three, the root of the explicit cubic (5), whereas the rendered text of [1] records it only as a decimal expansion — but see Remark 11, which withdraws any priority claim for the cubic itself. Second, the upper bound: no rigorous bound on $\varpi_\star(0, \mathbb{P}^8)$ — hence none on the bound-preserving CFL number (3) for $\mathbb{P}^8$-based schemes — appears to have been available, since the numerical evidence of [1] certifies neither direction. Third, the decomposition itself: the four internal orbits and their weights are given exactly, and the dual polynomial is exhibited vanishing on all of them, which is precisely the hypothesis of Lemma 3.

The gain over the classic Zhang–Shu decomposition is quantified along the way: the classic boundary weight at $k = 8$ is $\omega_1^{\mathtt{GL}} = 1/30$, and Theorem 10 (i), (iii) and (iv) together give $1/30 < \varphi_8$ unconditionally. That the classic decomposition is not optimal in more than one dimension is the subject of the same authors' companion paper [2], which [1] cites for the cases $\mathbb{P}^2$ and $\mathbb{P}^3$.

Remark 5. Numerically $30\,\varphi_8 = 1.73015\ldots$, so the optimal decomposition permits a time step more than 1.73 times the classic one at $k = 8$. This ratio is ordinary arithmetic on the two numbers and is not part of the formally verified development; only the strict inequality $1/30 < \varphi_8$ is.

1.4. **Method.** The two halves of Theorem 4 are proved by different means and neither uses analysis beyond the intermediate value theorem.

The upper bound (Section 4) is Lemma 3.20 of [1] instantiated on one explicit test polynomial. The point is that the test polynomial is a *square*: its orbit averages at the unknown interior nodes are nonnegative for free, so no information about those nodes is needed and the bound holds for every feasible decomposition at once, with any number of orbits. Concretely, $w_s q_\star^2$ expands as a fixed linear combination of the nine feasibility functionals, and summing that expansion over s turns the nine identities into a single scalar inequality.

The attainment half (Section 5) is an exact certificate. It was found, not guessed, by the route sketched in Section 5.3; but once written down it is a finite list of polynomial identities over $\mathbb{Q}$ in four indeterminates subject to four defining relations,

and the verification is pure algebra. In particular no step of the proof rests on a floating-point computation, on an exhaustive search whose outcome is taken on trust, or on a stored certificate file.

2. Preliminaries

2.1. **The three functionals.** On $\Omega = [-1,1]^2$ set

$$\langle p\rangle_\Omega = \tfrac{1}{4}\iint_\Omega p, \qquad \langle p\rangle^x = \tfrac{1}{2}\big(\langle p\rangle^{-x} + \langle p\rangle^{+x}\big), \qquad \langle p\rangle^y = \tfrac{1}{2}\big(\langle p\rangle^{-y} + \langle p\rangle^{+y}\big), \tag{6}$$

where $\langle p\rangle^{\pm x}$ is the average of p over the edge $x = \pm 1$ and likewise for y. On monomials,

$$\langle x^a y^b\rangle_\Omega = \frac{1}{(a+1)(b+1)}, \qquad \langle x^a y^b\rangle^x = \frac{1}{b+1}, \qquad \langle x^a y^b\rangle^y = \frac{1}{a+1} \tag{7}$$

when a and b are both even, and all three vanish otherwise. These are elementary. In the formal development the three functionals are *defined* by (7) extended by linearity, so that everything in this paper is algebra over $\mathbb{Q}$; deriving (7) from (6) is a routine integration and is not carried out there.

2.2. **Full symmetry and the reduction to nine identities.** Let $\mathscr{G}_f$ be the eight-element dihedral group of the square generated by the two sign changes and the swap $(x,y) \mapsto (y,x)$, and write $\overline{p(x_s,y_s)}$ for the average of p over the $\mathscr{G}_f$-orbit of (x_s,y_s). A CAD whose interior nodes form a $\mathscr{G}_f$-invariant set is called *fully symmetric* [1, §5.2]; at $\theta = 0$ it takes the form

$$\langle p\rangle_\Omega \;=\; \varpi\big[\langle p\rangle^x + \langle p\rangle^y\big] \;+\; \sum_{s=1}^{n} w_s\,\overline{p(x_s,y_s)} \qquad \text{for all } p \in \mathbb{P}^8. \tag{8}$$

Two reductions make (8) finite. First, by [1, Lemma 5.5] a fully symmetric CAD that is feasible on the $\mathscr{G}_f$-invariant subspace $\mathbb{P}^8(\mathscr{G}_f)$ is feasible on all of $\mathbb{P}^8$. Second, $\mathbb{P}^8(\mathscr{G}_f)$ is spanned by the nine symmetrised monomials $x^{2i}y^{2j}+x^{2j}y^{2i}$ with $i \ge j \ge 0$ and $i+j \le 4$. Writing $X_s = x_s^2$ and $Y_s = y_s^2$, the orbit average of $x^{2i}y^{2j}$ at (x_s,y_s) is $\tfrac{1}{2}(X_s^iY_s^j + X_s^jY_s^i)$, and (7) turns (8) into exactly nine scalar identities.

Definition 6. A *feasible datum* is a tuple $(\varpi, n, (w_s), (X_s), (Y_s))_{s=1}^n$ of real numbers with $w_s \ge 0$ and $X_s, Y_s \in [0,1]$ for every s, satisfying

$$\frac{1}{(2i+1)(2j+1)} \;=\; \varpi\left(\frac{1}{2j+1} + \frac{1}{2i+1}\right) \;+\; \sum_{s=1}^{n} w_s\,\frac{X_s^iY_s^j + X_s^jY_s^i}{2} \tag{9}$$

for each of the nine pairs (i,j) with $i \ge j \ge 0$ and $i+j \le 4$.

By the two reductions above, a feasible datum with all $w_s > 0$ and all nodes in $[0,1]^2$ is the same thing as a feasible fully symmetric CAD (8) for $\mathbb{P}^8$ at $\theta = 0$ with n interior orbits. The supremum of ϖ over such data is $\varpi_\star(0,\mathbb{P}^8)$: it is at most $\varpi_\star(0,\mathbb{P}^8)$ because a fully symmetric CAD is in particular a symmetric one, and at least $\varpi_\star(0,\mathbb{P}^8)$ because at $\theta = 0$ the average of a symmetric OCAD with its image under $x \leftrightarrow y$ is a fully symmetric CAD with the same boundary weight — this is the construction of [1, §5.2], their equation (5.7). Definition 6 relaxes $w_s > 0$ to $w_s \ge 0$; this only strengthens the upper bound of Section 4, and the datum constructed in Section 5 has strictly positive weights, so nothing is lost in either direction. Finally, by [1, Lemma 3.15] a symmetric OCAD for $\mathbb{V}^{2k}$ is one for $\mathbb{V}^{2k+1}$ and $\varpi_\star(\theta,\mathbb{V}^{2k+1}) = \varpi_\star(\theta,\mathbb{V}^{2k})$, which is why $\mathbb{P}^8$ and $\mathbb{P}^9$ are settled together.

2.3. **The dual polynomial.** For a real parameter φ define $Q_\varphi \in \mathbb{Q}(\varphi)[X,Y]$ by

$$Q_\varphi(X,Y) \;=\; C_0 + C_1(X+Y) + C_2(X^2+Y^2) + XY, \tag{10}$$

$$C_0 = \tfrac{7}{10} - \tfrac{177}{25}\varphi + \tfrac{72}{25}\varphi^2, \quad C_1 = -\tfrac{137}{30} + 48\varphi - \tfrac{96}{5}\varphi^2, \quad C_2 = \tfrac{1141}{180} - \tfrac{707}{10}\varphi + 28\varphi^2, \tag{11}$$

and put $q_\varphi(x,y) = Q_\varphi(x^2,y^2)$, a fully symmetric element of $\mathbb{P}^4$. This is the critical polynomial of Conjecture 2 for $k=8$: by [1, Theorem 3.29], $\inf_{q\in\mathbb{P}^4}\phi(q^2;\theta)$ is the smallest real root of $\det(\mathbf{M}_\Omega - \phi\,\mathbf{M}_\theta)$, where $\mathbf{M}_\Omega$ and $\mathbf{M}_\theta$ are the Gram matrices of a basis of $\mathbb{P}^4$ under $\langle\,\cdot\,\rangle_\Omega$ and $(1+\theta)\langle\cdot\rangle^x + (1-\theta)\langle\cdot\rangle^y$, and $q_\star$ is read off the corresponding eigenvector. Both Gram matrices are rational, so the value is algebraic and the eigenvector can be normalised to lie in $\mathbb{Q}(\varphi)$; (11) is that normalisation.

Two elements of $\mathbb{Q}(\varphi)$ record the relevant moments of q_φ^2:

$$\alpha(\varphi) = \tfrac{139472}{455625} - \tfrac{160448}{50625}\varphi + \tfrac{512}{405}\varphi^2, \qquad \beta(\varphi) = \tfrac{1230592}{455625} - \tfrac{1432456}{50625}\varphi + \tfrac{557888}{50625}\varphi^2, \tag{12}$$

namely $\alpha(\varphi) = \langle q_\varphi^2\rangle_\Omega$ and $\beta(\varphi) = \langle q_\varphi^2\rangle^x = \langle q_\varphi^2\rangle^y$ whenever $F(\varphi) = 0$ — as polynomials in the parameter φ these moments have degree four, and (12) is their reduction modulo the monic form of F — the two edge moments being equal (for every φ) because q_φ is symmetric in x and y.

3. The number φ_8

Theorem 7. *The cubic $F(t) = 72t^3 - 189t^2 + 28t - 1$ has a root φ_8 with*

$$\frac{576717376260236}{10^{16}} \;<\; \varphi_8 \;<\; \frac{576717376260237}{10^{16}}.$$

Proof sketch. F is continuous, F is negative at the left endpoint and positive at the right, and the intermediate value theorem on that closed interval produces a zero; the two endpoint evaluations are exact rational arithmetic, and strictness at each endpoint follows because F is nonzero there. □

The enclosure of Theorem 7 pins the first sixteen decimal places of φ_8, and they are those of the value 0.057671737626023616 printed in the third appendix of [1]. That printed value carries seventeen significant figures, and its last one is not correct: exact rational arithmetic outside the formal development gives

$$\varphi_8 \;=\; 0.05767173762602361298\ldots,$$

so the printed decimal is high by about 3×10^{-18}. The discrepancy is the accuracy of the iterative solve behind that table, not of the value above. The enclosure of Theorem 7 is deliberately no tighter than the endpoint evaluations require.

Proposition 8. *If $F(\varphi) = 0$ then $\alpha(\varphi) = 2\,\beta(\varphi)\,\varphi$. Consequently, if in addition $\beta(\varphi) \neq 0$,*

$$\phi(q_\varphi^2;0) \;=\; \frac{\langle q_\varphi^2\rangle_\Omega}{\langle q_\varphi^2\rangle^x + \langle q_\varphi^2\rangle^y} \;=\; \frac{\alpha(\varphi)}{2\beta(\varphi)} \;=\; \varphi.$$

In particular $\phi(q_{\varphi_8}^2;0) = \varphi_8$.

Proof sketch. Both α and β are quadratics in φ with rational coefficients, so $\alpha(\varphi) - 2\beta(\varphi)\varphi$ is a cubic in φ over $\mathbb{Q}$; it is the rational multiple $-\frac{1115776}{50625}$ of the monic form $\varphi^3 - \frac{21}{8}\varphi^2 + \frac{7}{18}\varphi - \frac{1}{72}$ of F, hence vanishes. Positivity of β on the relevant range is Lemma 9. □

Lemma 9. *If $\frac{1}{20} < \varphi < \frac{3}{50}$ then $\beta(\varphi) > 0$.*

The next theorem collects the guards on the development: that the enclosure of Theorem 7 could not have been placed lower, that the constant 2 in Proposition 8 is forced, that φ_8 is pinned rather than merely bracketed, and that the classic decomposition is genuinely suboptimal at $k = 8$.

Theorem 10. (i) *$F(t) < 0$ for all $t \in (0, \frac{1}{20}]$; in particular F has no root below $\frac{1}{20}$.*
(ii) *If $F(\varphi) = 0$ and $\frac{1}{20} < \varphi < \frac{3}{50}$ then $\alpha(\varphi) \neq 3\,\beta(\varphi)\,\varphi$.*
(iii) *If $F(a) = F(b) = 0$ and $a, b \in (\frac{1}{20}, \frac{3}{50})$ then $a = b$.*
(iv) *There is a root φ of F with $\frac{1}{30} < \varphi < \frac{3}{50}$.*

Part (iv) is the kernel-checked half of the comparison with the classic decomposition: the root it produces lies above $\frac{1}{20}$ by (i) and is therefore φ_8 by (iii), so $1/30 < \varphi_8$. That the classic Zhang–Shu boundary weight at $k = 8$ equals $\omega_1^{\mathrm{GL}} = 1/30$ is the assertion of [1, Table 1] and is cited here, not reproved. Parts (i)–(iii) are proved by `nlinarith`-style real arithmetic from the defining cubic.

Remark 11. The rendered text of [1] characterises $\varpi_\star(0, \mathbb{P}^8)$ only by its decimal expansion, and gives closed forms only for $k \leq 7$. Its arXiv v1 LaTeX source shows, however, that an algebraic characterisation was known to the authors. A commented-out block of that source — a variant of their comparison table, at line 4410 of `main.tex` — carries the entry

$$\varpi_\star(0, \mathbb{P}^8) \;=\; \frac{7}{8} + \frac{\sqrt{3297}}{36}\,\cos\!\left(\frac{2\pi}{3} + \frac{1}{3}\arccos\frac{21015\sqrt{3297}}{1207801}\right).$$

This is exactly the trigonometric form of the root of F. Substituting $t = u + \frac{7}{8}$ in $\frac{1}{72}F(t)$ gives $u^3 - \frac{1099}{576}u - \frac{2335}{2304}$, whose three real roots are

$$2\sqrt{\tfrac{1099}{1728}}\,\cos\!\big(\tfrac{1}{3}\arccos c - \tfrac{2\pi m}{3}\big), \qquad m = 0, 1, 2, \qquad c = \frac{21015\sqrt{3297}}{1207801}.$$

Since $2\sqrt{1099/1728} = \sqrt{3297}/36$, adding $\frac{7}{8}$ back turns these into the three roots of F, and the branch displayed above is the one in $(0, \frac{1}{10})$. Neither this expression nor any equivalent one occurs in arXiv v1 as rendered — the string `3297` does not occur anywhere in the text of the document compiled from that source — and the published version could not be consulted (see the bibliography). We therefore make no priority claim for the cubic (5) itself and record it only as the form in which the value enters the proofs below.

4. The upper bound

Theorem 12. *Let $(\varpi, n, (w_s), (X_s), (Y_s))$ be a feasible datum in the sense of Definition 6, and let φ satisfy $F(\varphi) = 0$ and $\frac{1}{20} < \varphi < \frac{3}{50}$. Then $\varpi \leq \varphi$.*

Proof sketch. The argument is [1, Lemma 3.20] made explicit at $(k, \theta) = (8, 0)$ with the single test polynomial q_φ^2, and it is entirely algebraic.

Step 1 (a pointwise expansion). There are nine elements $\lambda_{ij} \in \mathbb{Q}[\varphi]$, each of degree at most 2, such that for every s

$$w_s\, Q_\varphi(X_s, Y_s)^2 \;=\; \sum_{(i,j)} \lambda_{ij}(\varphi)\; w_s\, \frac{X_s^i Y_s^j + X_s^j Y_s^i}{2}, \tag{13}$$

the sum running over the same nine pairs as in (9). This is an identity of polynomials in φ, X_s, Y_s modulo the relation $F(\varphi) = 0$: expanding the square of (10) produces monomials $X^a Y^b$ with $a, b \le 4$ and $a + b \le 4$ after the reduction, and matching coefficients determines the λ_{ij}. Verification is one cofactor multiple of the monic cubic. Note that the orbit averages appearing on the right of (13) are exactly the ones appearing in (9), which is the whole point of the choice of q_φ.

Step 2 (summation). Summing (13) over s and substituting the nine identities (9) — each used once, with coefficient $-\lambda_{ij}(\varphi)$ — collapses the interior data entirely:

$$\sum_{s=1}^{n} w_s \, Q_\varphi(X_s, Y_s)^2 \;=\; \alpha(\varphi) - \varpi \cdot 2\beta(\varphi). \tag{14}$$

Step 3 (positivity). The left side of (14) is a sum of products of a nonnegative weight with a square, hence nonnegative. Here is where the choice of a *square* test polynomial pays: no information at all about the nodes (X_s, Y_s) is used, so the conclusion holds for every feasible datum simultaneously, whatever n is. Thus $\varpi \cdot 2\beta(\varphi) \le \alpha(\varphi)$, and by Proposition 8 $\alpha(\varphi) = 2\beta(\varphi)\varphi$, so $\varpi \cdot 2\beta(\varphi) \le 2\beta(\varphi)\varphi$. Dividing by $2\beta(\varphi) > 0$ (Lemma 9) gives $\varpi \le \varphi$. □

Theorem 12 with $\varphi = \varphi_8$ appears to be the first rigorous upper bound on $\varpi_\star(0, \mathbb{P}^8)$, and hence, by (3), on the optimal bound-preserving CFL number of a $\mathbb{P}^8$-based Zhang–Shu scheme on a Cartesian mesh; we have not found one in the literature, but the negative is of course a search result and not a theorem.

5. The exact decomposition

5.1. **The data.** Fix a root φ of F and work in $K = \mathbb{Q}(\varphi)$. Put

$$d_1 = \tfrac{32}{5}\varphi^2 - \tfrac{1954}{125}\varphi + \tfrac{6799}{4500}, \qquad d_2 = \tfrac{384}{5}\varphi^2 - \tfrac{23568}{125}\varphi + \tfrac{6604}{375}, \tag{15}$$

and let $s_1, s_2 > 0$ satisfy $s_1^2 = d_1$, $s_2^2 = d_2$; write $L = K(s_1, s_2)$. The first three interior orbits are

$$X_1 = -\tfrac{144}{7}\varphi^2 s_1 + \tfrac{192}{35}\varphi^2 + 52\varphi s_1 - \tfrac{482}{35}\varphi - \tfrac{22}{7}s_1 + \tfrac{41}{35}, \qquad Y_1 = 0, \tag{16}$$

$$X_2 = \tfrac{152208}{13129}\varphi^2 s_2 + \tfrac{115632}{65645}\varphi^2 - \tfrac{384678}{13129}\varphi s_2 - \tfrac{287022}{65645}\varphi + \tfrac{20529}{13129}s_2 + \tfrac{37581}{65645}, \quad Y_2 = X_2, \tag{17}$$

$$X_3 = \text{the image of } X_2 \text{ under } s_2 \mapsto -s_2, \qquad Y_3 = X_3. \tag{18}$$

The fourth is generic. Its elementary symmetric functions $S_4 = X_4 + Y_4$ and $T_4 = X_4 Y_4$ lie in $K(s_1)$:

$$\begin{aligned} S_4 &= \tfrac{31794816}{2129813}\varphi^2 s_1 + \tfrac{94001568}{10649065}\varphi^2 - \tfrac{80322176}{2129813}\varphi s_1 - \tfrac{235760388}{10649065}\varphi + \tfrac{4397408}{2129813}s_1 + \tfrac{21746274}{10649065}, \\ T_4 &= \tfrac{553589832576}{3240073867835}\varphi^2 s_1 + \tfrac{15746600668896}{16200369339175}\varphi^2 - \tfrac{196306344128}{462867695405}\varphi s_1 - \tfrac{39240375901516}{16200369339175}\varphi \\ &\quad - \tfrac{22176534112}{3240073867835}s_1 + \tfrac{3580885823023}{16200369339175}, \end{aligned} \tag{19, 20}$$

and one further square root splits the orbit into its two coordinates: with $s_3 > 0$ satisfying $s_3^2 = S_4^2 - 4T_4$,

$$X_4 = \tfrac{1}{2}(S_4 + s_3), \qquad Y_4 = \tfrac{1}{2}(S_4 - s_3). \tag{21}$$

The four weights $w_1, \dots, w_4$ are the solution of the linear system that (9) becomes once the nodes are fixed; they are elements of L, with w_1 and w_4 in $K(s_1)$ and w_2, w_3 conjugate over $K(s_1)$ under $s_2 \mapsto -s_2$. Their exact expressions are long — w_2, w_3 and w_4 have coefficients with numerators of up to twenty-five digits — and are recorded in the formal development rather than reproduced here. Numerically, at the root $\varphi = \varphi_8$ of Theorem 7:

s	X_s	Y_s	w_s
1	0.2268189911432481	0	0.1762036490189027
2	0.0946045837770378	$= X_2$	0.2077276517268407
3	0.5577691825578472	$= X_3$	0.1620943710533683
4	0.6545592897338122	0.0919248669299542	0.3386308529488411

Since $X_s = x_s^2$, the corresponding nodes are $x_1 = 0.4762551744004974$, $x_2 = y_2 = 0.3075785814666518$, $x_3 = y_3 = 0.7468394623731711$, $x_4 = 0.8090483852859557$, $y_4 = 0.3031911392668892$. These agree with the table of the third appendix of [1] to at least thirteen significant digits — worst case the weight w_2, whose printed value is 0.207727651726835549 against the exact 0.20772765172684069 . . .. The gap is the accuracy of the iterative solve behind that table, not of the values above, which are the exact algebraic numbers rounded to sixteen decimal places.

5.2. **The two halves of the certificate.**

Theorem 13. *Let φ, s_1, s_2, s_3 be real numbers satisfying*

$$\varphi^3 - \tfrac{21}{8}\varphi^2 + \tfrac{7}{18}\varphi - \tfrac{1}{72} = 0, \qquad s_1^2 = d_1, \qquad s_2^2 = d_2, \qquad s_3^2 = S_4^2 - 4T_4,$$

and define $X_1, \dots, X_4$, $Y_1, \dots, Y_4$ and $w_1, \dots, w_4$ by (16)–(21) *and the linear solve above. Then all nine identities* (9) *hold with $\varpi = \varphi$ and $n = 4$.*

Theorem 14. *Under the same hypotheses, the dual polynomial vanishes at every interior orbit:*

$$Q_\varphi(X_1, Y_1) = Q_\varphi(X_2, Y_2) = Q_\varphi(X_3, Y_3) = Q_\varphi(X_4, Y_4) = 0.$$

Proof sketch for Theorems 13 and 14. Each of the thirteen statements is a polynomial identity in the four indeterminates φ, s_1, s_2, s_3 over $\mathbb{Q}$, to be proved modulo the ideal generated by the four displayed relations. Since those four relations are monic in $\varphi^3, s_1^2, s_2^2, s_3^2$ respectively, they form a triangular system and reduction is division with remainder in that order: each identity is a single explicit combination

$$A \cdot h_0 + B \cdot h_1 + C \cdot h_2 + E \cdot h_3 \;=\; 0,$$

with $h_0, \dots, h_3$ the four relations and $A, B, C, E \in \mathbb{Q}[\varphi, s_1, s_2, s_3]$ the cofactors produced by that reduction. No Gröbner basis is needed. The cofactors were computed by exact rational arithmetic; the largest has 48 monomials with coefficients of up to 89 digits, and they occupy some 80 kilobytes in total. They are not trusted: each identity is re-derived from its cofactor by ring normalisation in the verification of Section 7, so a wrong cofactor would fail rather than mislead. □

Theorem 15. *Let φ be as in Theorem 7 and $s_1, s_2, s_3 > 0$ the positive square roots above. Then*

$$w_s > 0 \quad (1 \le s \le 4), \qquad X_s \in (0,1) \quad (1 \le s \le 4), \qquad Y_4 \in (0,1),$$

so that the data of Theorems 13 and 14 form a feasible datum with four interior orbits and $\varpi = \varphi$ — a feasible fully symmetric cell average decomposition for $\mathbb{P}^8$ at $\theta = 0$ with boundary weight φ, in the sense of (8).

Proof sketch. Every quantity above is a $\mathbb{Q}$-linear combination of the thirteen monomials

$$1,\ \varphi,\ \varphi^2,\ s_1,\ \varphi s_1,\ \varphi^2 s_1,\ s_2,\ \varphi s_2,\ \varphi^2 s_2,\ s_1 s_2,\ \varphi s_1 s_2,\ \varphi^2 s_1 s_2,\ s_3$$

(with s_3 entering only through X_4 and Y_4). Rational two-sided enclosures of φ — the one of Theorem 7, of width 10^{-16} — and of s_1, s_2, s_3, of width below 10^{-7}, are propagated to enclosures of the eleven nonconstant monomials other than φ by interval multiplication, and the fourteen strict inequalities above then follow by linear arithmetic on those enclosures. The enclosures of s_1, s_2, s_3 are obtained from the enclosure of φ in Theorem 7: d_1, d_2 are quadratics in φ with rational coefficients, and $S_4^2 - 4T_4$ is a polynomial in φ and s_1, so each is enclosed in turn and its positive square root bracketed by squaring. Positivity of the radicands, which is what makes the square roots real, comes out of the same enclosures. □

5.3. **How the data was found.** The proofs above are independent of how the certificate was discovered, but the route is short enough to record, and it explains why the field is as small as it is.

By [1, Theorem 3.29], $\phi^\star(0, (\mathbb{P}^4)^2)$ is the smallest real root of $\det(\mathbf{M}_\Omega - \phi\,\mathbf{M}_0)$ with $\mathbf{M}_\Omega, \mathbf{M}_0$ rational Gram matrices of size $\dim \mathbb{P}^4 = 15$. Decomposing $\mathbb{P}^4$ into isotypic components for the dihedral group of the square — the four parity classes, then the symmetric and antisymmetric parts under $x \leftrightarrow y$ inside the two classes that admit them — block-diagonalises both matrices. For $k = 8$ the relevant block is the fully symmetric one, of dimension 4, and its characteristic determinant is the cubic (5) up to a rational factor. Root isolation was done by Sturm sequences in exact rational arithmetic.

The structural observation that makes the nodes computable in closed form is that all four interior orbits lie on the curve $Q_\varphi = 0$, which is Theorem 14 read backwards. Given that: the orbit on the x-axis has $Y = 0$, so its X is a root of the quadratic $C_2X^2 + C_1X + C_0$ over K, of discriminant d_1; the two orbits on the diagonal have $X = Y$, so that common value is a root of $(2C_2 + 1)X^2 + 2C_1X + C_0$ over K, of discriminant d_2; and for the generic orbit, $Q_\varphi = 0$ is linear in $T = XY$ once $S = X + Y$ is fixed, so T is determined by S. Substituting into five of the nine identities and requiring the resulting 5×5 system in the four weights to be consistent gives a determinant that is a quartic in S over $K(s_1)$ whose roots are the three already-known values of S together with the wanted S_4; the first three make two columns of the determinant coincide. Hence S_4 is recovered from the quartic's trace, with no elimination at all, and T_4 follows. The weights are then a linear solve. The whole computation is exact rational and L-arithmetic in a few hundred lines of Python with no computer algebra system, and costs about one CPU-minute and under 300 megabytes.

Remark 16. $[L : \mathbb{Q}] = 12$. Indeed F has no rational root — by the rational root theorem one would be $\pm 1/m$ with $m \mid 72$, and $F(t) < 0$ for every $t \le 0$, so only the twelve values $1/m$ need testing and none is a root — so F is irreducible and $K = \mathbb{Q}(\varphi)$ has degree 3; it is totally real, the roots of F being approximately 0.05767, 0.09751 and 2.46982. Then $[K(s_1, s_2) : K] = 4$ as soon as none of d_1, d_2,

$d_1 d_2$ is a square in K, and none is, because a square in K has square norm over $\mathbb{Q}$ while

$$N_{K/\mathbb{Q}}(d_1) = \frac{1091131}{18750000}, \qquad N_{K/\mathbb{Q}}(d_2) = -\frac{2434048}{390625},$$

neither of which — nor their product — is the square of a rational number. (For d_2 the sign alone suffices: it is negative at the middle real place of K, and a square in a totally real field is nonnegative at every real embedding.) These norms are exact rational arithmetic carried out outside the formal development, and nothing in this paper depends on them: s_1, s_2, s_3 enter every statement above only as real numbers with the stated squares.

6. The main theorem

Theorem 17. *There is a real number φ_8 such that*

(i) $72\varphi_8^3 - 189\varphi_8^2 + 28\varphi_8 - 1 = 0$;
(ii) $0.0576717376260236 < \varphi_8 < 0.0576717376260237$;
(iii) *every feasible datum has $\varpi \le \varphi_8$; and*
(iv) *there is a feasible datum with $\varpi = \varphi_8$.*

Proof sketch. Take φ_8 from Theorem 7, which gives (i) and (ii). Part (iii) is Theorem 12, whose hypotheses $\frac{1}{20} < \varphi_8 < \frac{3}{50}$ follow from (ii). For (iv), the three square roots s_1, s_2, s_3 are constructed inside the proof: the enclosures of Theorem 15 show that the three radicands d_1, d_2 and $S_4^2 - 4T_4$ are positive, so each has a positive real square root, and the defining relations $s_i^2 = \cdots$ hold by construction. Theorems 13 and 15 then supply the datum, with $\varpi = \varphi_8$ by definition. In particular the statement carries no hypotheses. □

Corollary 18. *$\varpi_\star(0, \mathbb{P}^8) = \varpi_\star(0, \mathbb{P}^9) = \varphi_8$, and Conjectures 1 and 2 hold at $(k, \theta) = (8, 0)$ and $(9, 0)$ with critical positive polynomial $p^\star = q_{\varphi_8}^2$, where $q_{\varphi_8} \in \mathbb{P}^4$ is the polynomial* (10)–(11).

Proof. The decomposition of Theorem 15 is a feasible fully symmetric CAD for $\mathbb{P}^8$ at $\theta = 0$ with strictly positive weights and interior nodes in the open cell, hence admissible in the Symmetric 2D OCAD Problem [1, Problem 3.13]; so $\varpi_\star(0, \mathbb{P}^8) \ge \varphi_8$. For the reverse inequality, Theorem 17 (iii) bounds the boundary weight of every *fully* symmetric CAD by φ_8, and at $\theta = 0$ that is no loss: by the symmetrisation of [1, §5.2] recalled in Section 2, $\varpi_\star(0, \mathbb{P}^8)$ is attained by a fully symmetric CAD. Finally $\varpi_\star(0, \mathbb{P}^9) = \varpi_\star(0, \mathbb{P}^8)$ by [1, Lemma 3.15]. For the conjectures, Theorem 14 says the nonzero polynomial $q_{\varphi_8} \in \mathbb{P}^4 = \mathbb{P}^{\lfloor 8/2 \rfloor}$ vanishes at every interior node of that decomposition, which is exactly the hypothesis of Lemma 3 at $(k, \theta) = (8, 0)$; the lemma yields both conjectured identities and identifies $q_{\varphi_8}^2$ as a critical positive polynomial. The case $k = 9$ follows since $\lfloor 9/2 \rfloor = 4$ and the same decomposition serves $\mathbb{P}^9$. □

Corollary 19. *For a $\mathbb{P}^8$- or $\mathbb{P}^9$-based Zhang–Shu bound-preserving scheme on a Cartesian mesh with $a_1 \Delta y = a_2 \Delta x$, the optimal bound-preserving CFL number of* (3) *is φ_8 exactly. It exceeds the classic value* 1/30*, by Theorem 10 (i), (iii) and (iv).*

Remark 20 (not part of the verified development). The same exact route — rational Gram matrices, isotypic reduction, Sturm isolation — identifies the remaining values reported numerically in the third appendix of [1] as algebraic numbers. Outside

the formal development, $\varpi_\star(0,\mathbb{P}^{10})$, $\varpi_\star(0,\mathbb{P}^{12})$ and $\varpi_\star(0,\mathbb{P}^{14})$ are the roots near 0.043783667884911184, 0.032438153462135161 and 0.026235904303028568 of

$$6480t^5 - 14643t^4 + 7632t^3 - 1064t^2 + 56t - 1,$$

$$1080t^4 - 2532t^3 + 857t^2 - 56t + 1,$$

$$3628800t^7 - 8175816t^6 + 5134032t^5 - 1210383t^4 + 118432t^3 - 5484t^2 + 120t - 1$$

respectively. In each of the three cases the printed value agrees with the exact root to sixteen significant figures and differs in the seventeenth, exactly as at $k = 8$ (Section 3). These identifications are stated as computations, not as theorems: they have not been machine-checked, and in each case the attainment half — which for $k = 10, 12, 14$ needs 6, 8 and 10 interior orbits over base fields of degree 5, 4 and 7 — has not been carried out. The bound of Section 4 transfers verbatim once the corresponding $q_\star$ is known, with one change: for $k \equiv 2 \pmod 4$ the critical polynomial lies in the x-even, y-odd isotypic block, so $\langle q_\star^2\rangle^x \neq \langle q_\star^2\rangle^y$ and the two edge constants in (12) differ.

7. Verification

Every numbered theorem, proposition and lemma of Sections 3–5, and Theorem 17, has been formalised and checked by the Lean 4 [6] kernel against Mathlib [5]. What has *not* been formalised is: the remarks, all of which record computations made outside the development (Remarks 11, 16 and 20, and the ratio $30\,\varphi_8$ in Section 1.3); the decimal table of Section 5; and Corollaries 18 and 19. The last point is worth spelling out. The kernel-checked statement is Theorem 17: a real number with the stated cubic relation and enclosure bounds the boundary weight of every feasible datum, and is attained by one. Identifying that number with $\varpi_\star(0,\mathbb{P}^8)$, and deducing the two conjectures from it, uses four results of [1] that are quoted here and not reproved — Optimality Criterion #3 (Lemma 3), the symmetrisation of its §5.2, its Lemma 3.15 and its Problem 3.13 — together with the identification of the nine identities (9) with feasibility, which rests on its Lemma 5.5 and on the elementary moment formula (7).

The formalisation defines the feasible data of Definition 6 as a structure carrying the boundary weight, the orbit count, the weights and squared coordinates with their range conditions, and the nine identities (9) as fields; Theorem 12 is then a theorem about an arbitrary inhabitant of that structure, and Theorem 15 constructs one. The proofs use only ring normalisation with explicit cofactors, linear and nonlinear real arithmetic over ordered fields, and one application of the intermediate value theorem; there is no appeal to kernel reduction of a compiled decision procedure, no external certificate file, and no floating-point arithmetic anywhere. Each of the fourteen top-level declarations depends on exactly the three standard axioms of Mathlib's real numbers — propositional extensionality, the axiom of choice and quotient soundness — and on no other axiom; in particular none depends on the axiom introduced by an unfinished proof. The repository is private: repository reference to be added at submission.

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item and equation numbers cited above are those of the arXiv version v1, which at the time of writing is the only arXiv version and whose printed numbering was read off a compilation of its LaTeX source; the published SIAM text is behind a paywall that could not be cleared here and was therefore never compared with the preprint. (Its landing page does show that the published version carries a supplement with sections titled "Auxiliary verification of the feasibility of CAD" and "Sharpness of optimal BP CFL condition" — titles occurring nowhere in v1 — so the two texts certainly differ at least in arrangement.) Two consequences should be kept in mind. The numbering of the published version may differ, which is why every item is also cited by name where it has one. And the frontier quoted in Section 1.3 — that no rigorous proof of the two conjectures is available for $\mathbb{P}^k$ with $k \in \{8, \dots, 15\}$ — is the preprint's; should the published version have closed the case $k = 8$, the results of Sections 3–5 stand unchanged but their novelty does not. The digital object identifier above is the one registered with Crossref for this article; the identifier `10.1137/22M1503732`, which some citing bibliographies print for it, is not a registered DOI at all.

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