NEW LOWER BOUNDS FOR THE NON-ORIENTABLE FOUR-GENUS: ONE HUNDRED AND EIGHTEEN KNOTS OF AT MOST THIRTEEN CROSSINGS

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Abstract. The *non-orientable four-genus* $\gamma_4(K)$ of a knot $K \subset S^3$ is the least first Betti number of a smooth, compact, properly embedded surface in B^4 with boundary K; $\gamma_4(K) = 1$ says that K bounds a Möbius band. Lee and Sabloff have recently surveyed γ_4 over all knots of at most 14 crossings, implementing the Murakami–Yasuhara linking-form obstruction only under the hypothesis that $\det(K)$ is square-free; their data records 14,319 knots on which that obstruction was in consequence never evaluated at all. We evaluate it on the part of that set for which KnotInfo publishes a diagram and the first homology of the double branched cover is cyclic, and obtain $\gamma_4(K) \geq 2$ for 118 knots whose γ_4 their pipeline left undetermined — five with 11 crossings, six with 12, and one hundred and seven with 13. For 91 of them the upper bound recorded in KnotInfo is 2, so $\gamma_4(K) = 2$ exactly; the remaining 27 have $\gamma_4(K) \in \{2, 3\}$. These 118 values raise the count in Lee–Sabloff's Theorem 1.1 from "at least 22,412" to "at least 22,530". The obstruction argument is not new: for 87 of the 118 the applicable published statement is a corollary of Gilmer and Livingston, for 23 it is a lemma of Fairchild, and for 8 it is Lee–Sabloff's own corollary. What is new is the values. We also record the uniform decidable form of those statements for cyclic first homology, exhibit 322 knots in Lee–Sabloff's own data showing that the arithmetic of their corollary is unsound if applied outside its square-free hypothesis, and re-verify the arithmetic behind their corrections to six KnotInfo entries. Every finite computation reported here is machine-checked in Lean 4.

1. Introduction

1.1. **The invariant.** For a knot $K \subset S^3$, let $\gamma_4(K)$ denote the minimum first Betti number among all smooth, compact, properly embedded surfaces in B^4 with boundary K. Following Lee–Sabloff [8] we call this the *non-orientable four-genus*; a knot with $\gamma_4(K) = 1$ bounds a Möbius band in B^4, and a slice knot has $\gamma_4(K) = 0$. KnotInfo [7] tabulates this invariant in a column named `smooth_4d_crosscap_number`, whose description page defines it as the minimum first Betti number among the *non-orientable* surfaces bounded by K in B^4; the two definitions agree except on slice knots, which KnotInfo assigns the value 1. Every knot we treat below is non-slice, so the distinction does not affect any statement here.

Lower bounds on γ_4 come from a short list of obstructions. Two of them are arithmetic in a strong sense: everything they need can be read off an integer matrix associated with a diagram or a Seifert surface, so that a computer can evaluate them over a whole knot table. The first is due to Yasuhara [14] and Gilmer–Livingston [4] and is stated by Lee–Sabloff as their Theorem 2.1: if

$$\sigma(K) + 4\,\mathrm{Arf}(K) \equiv 4 \pmod 8,$$

then K has no Möbius band filling. (On a knot separately known not to be slice this gives $\gamma_4(K) \geq 2$, which is how it is used in Section 6.) The second is the linking form of the double branched cover, and it is the subject of this paper.

1.2. **The source, and the gap in its pipeline.** Lee and Sabloff [8] search for Möbius band fillings algorithmically across the 59,937 prime knots of at most 14 crossings, and publish their per-knot results as a dataset [9]. Their Theorem 1.1 reads, verbatim: "Of knots that have crossing number at most 14, at least 24,504 (41%) have Möbius band fillings, while at least 22,412 (37%) have $\gamma_4(K) > 1$."

For the linking-form obstruction they quote, as their Theorem 2.4, Corollary 2.7 of Murakami–Yasuhara [12]: if K is slice or bounds a Möbius band in B^4, then the linking form λ on $H_1(D_K;\mathbb{Z})$ splits as an orthogonal direct sum $(G_1,\lambda_1)\oplus(G_2,\lambda_2)$ in which λ_1 is represented by the 1×1 matrix $[\pm 1/|G_1|]$ and λ_2 is metabolic. They then write: "The statement of the obstruction in Theorem 2.4 requires some work in order to practically implement it. In particular, we need to understand how to compute the linking form and how to test it for the existence of the required direct sum splitting. For algorithmic ease, we choose to restrict to the following special case" — their Corollary 2.5, which assumes that $\det(K)$ is square-free and concludes that λ is $[\pm 1/\det(K)]$ on some generator. Their Algorithm 1 accordingly returns the string `"det U not square-free"` and stops; its other early exit returns `"Need to find generator by hand"`. In the `Linking Form` column of their published dataset [9] the two exits are recorded as `det not square-free` and `find generator by hand`, occurring 14,216 and 103 times: 14,319 **knots on which the Murakami–Yasuhara obstruction was never evaluated**.

That is a large gap, and it is not where the mathematics stops. Square-freeness is a convenience of implementation. Gilmer and Livingston had already published, fifteen years earlier, the same conclusion whenever $H_1(D_K;\mathbb{Z})$ is cyclic of order a product of primes all with odd exponent [4, Corollary 3] — a strictly weaker hypothesis, since a square-free determinant already forces $H_1(D_K;\mathbb{Z})$ cyclic with every exponent 1 — and Fairchild [1] published the case $H_1(D_K;\mathbb{Z})\cong\mathbb{Z}/p^2q$ with p prime and q a product of primes with all exponents odd.

1.3. **Results.** Of the 14,319 skipped knots, 2,329 have at most 13 crossings — so that KnotInfo publishes a planar diagram code for them — and have cyclic $H_1(D_K;\mathbb{Z})$. We evaluate the Murakami–Yasuhara criterion on all of them. It obstructs 489; of those, 371 were already obstructed by the Arf criterion above, none carries a band in [9], and 118 had been left undetermined by the whole of Lee–Sabloff's pipeline: their Arf test was silent, their linking-form test did not run, their band search found nothing, and their second-pass ribbon check did not settle them.

Theorem 1.1 (Theorem 5.1 below)**.** *For each of the* 118 *knots listed in Table 1,* $\gamma_4(K)\geq 2$*. Combined with the upper bounds recorded in KnotInfo, this gives* $\gamma_4(K)=2$ *for* 91 *of them and* $\gamma_4(K)\in\{2,3\}$ *for the remaining* 27*.*

Every one of the 118 was an open entry of KnotInfo's column on the day the computation was run: 91 carried the range $[1,2]$ and 27 the range $[1,3]$. Five of the knots have 11 crossings, six have 12, and 107 have 13. Since the 118 are disjoint from the knots counted by Lee–Sabloff's Theorem 1.1 — that count is exactly the set of knots their Arf or linking-form tests obstructed — the survey total improves:

Corollary 1.2 (Corollary 5.2 below)**.** *Of the knots of crossing number at most* 14*, at least* 22,530 *have* $\gamma_4(K)>1$*.*

1.4. **What is new here, and what is not.** The obstruction argument used for each of the 118 knots is published, and we claim no part of it. Sorting the 118 by the arithmetic of their determinants:

determinants	published statement that applies	knots
135, 351, 375, 459 (all prime exponents odd)	Gilmer–Livingston [4, Cor. 3]	87
315, 495, 585 (of the form p^2q)	Fairchild [1, Lemma]	23
15, 105, 273, 357, 455, 465 (square-free)	Lee–Sabloff [8, Cor. 2.5]	8

For the last eight knots even Lee–Sabloff's own corollary applies: their Algorithm 1 skipped them not for arithmetic reasons but because none of the standard basis vectors of the Goeritz lattice happened to generate $H_1(D_K;\mathbb{Z})$, so the algorithm returned "Need to find generator by hand". The contribution of this paper is therefore of a specific and limited kind: these published criteria had never been run on these knots, and running them requires a Goeritz matrix for each diagram, a generator search wider than a sweep over basis vectors, and the correct enumeration of admissible splittings. We supply those three ingredients, machine-check each of the 118 verdicts, and report the values.

Two further items are recorded because they are needed, not because they are new. The first (Section 3) is the uniform decidable form of the Murakami–Yasuhara criterion for cyclic $H_1(D_K;\mathbb{Z})$: there exist n_1, m with $\Delta = n_1 m^2$, $\gcd(n_1, m) = 1$ and $at^2 \equiv \pm 1 \pmod{n_1}$ solvable, where λ is a/Δ on a generator. This is a routine unification of [4, Cor. 3] (the branch $m = 1$) and [1, Lemma] (the branch $\Delta = p^2 q$); we did not find the uniform statement in print, but we do not claim it as new mathematics, and both special cases carry their published proofs. The second is a demonstration that the square-free hypothesis in [8, Cor. 2.5] is load-bearing: applying that corollary's arithmetic verbatim to a non-square-free determinant "obstructs" 1,002 of the skipped knots, wrongly for 513 of them, and 322 of those 513 carry an explicit Möbius band in Lee–Sabloff's own dataset. The smallest witness is $11n_{39}$, which has determinant 25 and published $\gamma_4 = 1$.

1.5. **Second half: the arithmetic behind six corrected table entries.** Section 6 is independent of the above and re-verifies published material. Lee–Sabloff's §4.3.2 corrects six entries of KnotInfo that came from Fairchild [1]: $11n_{92}$, $11n_{125}$, $11n_{131}$ and $11n_{161}$ do bound Möbius bands, and the bands drawn in [1] for $11n_{36}$ and $11n_{57}$ do not produce the knots they are claimed to produce. The correction has an arithmetic core — two Alexander polynomials differ; a signature is non-zero; a determinant is not a square; the Arf criterion fires on the figure-8 knot — and we recompute all of it from KnotInfo's Seifert matrices, with signatures certified by explicit congruences rather than produced by an unverified eigenvalue routine. The identification of the knot that Fairchild's band actually produces is Lee–Sabloff's and is cited, not reproved: their bands exist only as figures, so there is no band datum to transcribe, and certifying the output of a band move would require unknot recognition or an identification of a diagram with a table entry, neither of which is a finite invariant computation.

We note in passing, since it is not recorded elsewhere, that the six KnotInfo entries in question moved between the monthly mirror released on 1 August 2026 and a live reading on 22 August 2026, and that KnotInfo's description page for the column now cites [8].

1.6. **Organization.** Section 2 fixes notation. Section 3 states the criterion in the form we evaluate and compares it with [8, Cor. 2.5]. Section 4 describes the passage from a planar diagram code to a Goeritz matrix and the external checks it passed. Section 5 contains the main theorem, the table, a worked example and the controls. Section 6 is the re-verification of §4.3.2 of [8]. Section 7 says exactly what has been formally verified.

2. Preliminaries

2.1. **Goeritz matrices and the linking form.** Let D be a connected diagram of a knot K with n crossings. Its underlying 4-valent plane graph has $n+2$ faces, which admit a checkerboard colouring; colour them black and white. Each crossing joins two white faces and carries a sign $\eta = \pm 1$ according to which pair of opposite corners is white. Numbering the white faces $X_0, \ldots, X_{w-1}$, the *Goeritz matrix* of the coloured diagram is the symmetric integer matrix U of size $w - 1$ obtained from

$$g_{xy} = -\sum_{c \,\text{joining}\, X_x, X_y} \eta(c) \quad (x \neq y), \qquad g_{xx} = -\sum_{y \neq x} g_{xy},$$

by deleting the last row and column. Write D_K for the double cover of S^3 branched over K. By Gordon–Litherland [5], U is a presentation matrix for $H_1(D_K;\mathbb{Z})$, so $|\det U| = \det(K)$, and the linking form

$$\lambda\colon H_1(D_K;\mathbb{Z}) \times H_1(D_K;\mathbb{Z}) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

is represented by $-U^{-1}$ with respect to the generators corresponding to the white faces. Concretely, for integer vectors u, v,

$$\lambda([u],[v]) \;=\; -\,\frac{u^{\mathsf{T}}\,\mathrm{adj}(U)\,v}{\det U} \;\in\; \mathbb{Q}/\mathbb{Z}.$$

Both the colouring and the orientation convention in a diagram code are ambiguous up to a global sign; replacing U by $-U$ replaces λ by $-\lambda$ and changes neither $|\det U|$ nor anything we test.

We use one more elementary fact throughout. Let U be an $m \times m$ integer matrix and v an integer vector. By Cramer's rule the maximal minors of the $m \times (m+1)$ matrix $[\,U \mid v\,]$ are $\det U$ together with the entries of $\operatorname{adj}(U)v$, and the cokernel of $[\,U \mid v\,]$ is trivial exactly when the gcd of those minors is 1. Hence

$$\gcd\bigl(\det U,\ (\operatorname{adj}(U)v)_1, \ldots, (\operatorname{adj}(U)v)_m\bigr) = 1 \iff [v] \text{ generates } \operatorname{coker} U = H_1(D_K;\mathbb{Z}). \tag{1}$$

A single gcd therefore certifies both that $H_1(D_K;\mathbb{Z})$ is cyclic and that a given class generates it; no Smith normal form is needed, and nothing about how v was found is trusted.

2.2. **The Murakami–Yasuhara obstruction.** We restate the obstruction in the form [8] quotes it.

Theorem 2.1 (Murakami–Yasuhara [12, Cor. 2.7]; [8, Thm. 2.4])**.** *Let $K \subset S^3$ be a knot that is either slice or bounds a Möbius band in B^4. Then the linking form λ on $H_1(D_K;\mathbb{Z})$ splits as an orthogonal direct sum $(G_1, \lambda_1) \oplus (G_2, \lambda_2)$ in which* (i) *λ_1 is represented by the 1×1 matrix $[\pm 1/|G_1|]$, and* (ii) *λ_2 is metabolic.* (These are the conditions [8] numbers (1) and (2); we relabel them to keep them apart from the numbered displays below.)

Here a form (G_2, λ_2) is *metabolic* when it admits a subgroup H with $|H|^2 = |G_2|$ on which λ_2 vanishes identically. Murakami and Yasuhara state the result for knots with $\Gamma^*(K) \le 1$, where Γ^* is their minimum first Betti number over all surfaces in B^4 bounded by K; a properly embedded orientable surface with one boundary component has even first Betti number, so $\Gamma^*(K) \le 1$ is exactly "slice or bounds a Möbius band".

2.3. **Seifert-matrix invariants.** For Section 6 we also use the classical route through a Seifert matrix V of K: the Alexander polynomial is $\Delta_K(t) \doteq \det(V - tV^\mathsf{T})$ up to units $\pm t^k$, the determinant is $\det(K) = |\det(V + V^\mathsf{T})|$, and the signature is $\sigma(K) = \operatorname{sign}(V + V^\mathsf{T})$. We use Murasugi's theorem [11] that a slice knot has vanishing signature, the Fox–Milnor condition [2] in the form that a slice knot has square determinant, and Murasugi's mod-8 criterion [10] reading the Arf invariant off the determinant: $\operatorname{Arf}(K) = 0$ when $\det(K) \equiv \pm 1 \pmod 8$ and $\operatorname{Arf}(K) = 1$ when $\det(K) \equiv \pm 3 \pmod 8$. Signatures are never computed here by a numerical routine. For each symmetric matrix $S = V + V^\mathsf{T}$ we exhibit an integer matrix P with $\det P \neq 0$ and $P^\mathsf{T} S P = \operatorname{diag}(d)$; Sylvester's law of inertia then determines $\operatorname{sign}(S)$ from the signs of the entries of d, and the certificate (P, d) is verified by integer matrix multiplication alone.

3. The obstruction at an arbitrary determinant

Throughout this section $G = H_1(D_K;\mathbb{Z})$ is cyclic of order $\Delta = \det(K)$, with $\lambda(z, z) = a/\Delta$ on a generator z; nondegeneracy of λ forces $\gcd(a, \Delta) = 1$.

Proposition 3.1. *Let K be a knot with $H_1(D_K;\mathbb{Z})$ cyclic of order Δ and linking form a/Δ on a generator. If K is slice or bounds a Möbius band in B^4, then there are positive integers n_1, m with*

$$\Delta = n_1 m^2, \qquad \gcd(n_1, m) = 1, \qquad at^2 \equiv \pm 1 \pmod{n_1} \text{ for some } t. \tag{2}$$

Write $S(\Delta)$ for the set of n_1 occurring in such a factorisation. Then:

- (a) *$\Delta \in S(\Delta)$, so the failure of condition* (2) *always implies the failure of the corresponding condition of* [8, Cor. 2.5]*;*
- (b) *if Δ is square-free then $S(\Delta) = \{\Delta\}$, so on square-free determinants condition* (2) is [8, Cor. 2.5].

Proof. A direct sum decomposition of a cyclic group has coprime summands, so a splitting as in Theorem 2.1 is $\Delta = n_1 n_2$ with $\gcd(n_1, n_2) = 1$, $|G_1| = n_1$, $|G_2| = n_2$. Condition (ii) requires a subgroup of G_2 of order m with $m^2 = n_2$, so $n_2 = m^2$ is a perfect square and $\gcd(n_1, m) = 1$. Conversely, in a cyclic group of order m^2 the subgroup of order m is $m \cdot G_2$, and for $h = my$ and $h' = my'$ one has $\lambda_2(h, h') = m^2 \lambda_2(y, y') = m^2 \cdot (b/m^2) = b \equiv 0$ in $\mathbb{Q}/\mathbb{Z}$; so condition (ii) holds automatically as soon as $|G_2|$ is a perfect square, and it carries no arithmetic content in the cyclic case. For condition (i), G_1 is the subgroup $m^2 \cdot G$ of order n_1, whose generators are the classes km^2z with $\gcd(k, n_1) = 1$, and

$$\lambda(km^2z,\, km^2z) = \frac{k^2m^4a}{\Delta} = \frac{(km)^2a}{n_1} \;\in\; \mathbb{Q}/\mathbb{Z}.$$

Since $\gcd(m, n_1) = 1$, the product $t = km$ runs over all units modulo n_1 as k does, so condition (i) says precisely that $at^2 \equiv \pm 1 \pmod{n_1}$ is solvable. This proves condition (2). Part (a) of the list is the case $m = 1$, which is always admissible. For part (b), if $m \geq 2$ then $m^2 \mid \Delta$ contradicts square-freeness. □

Remark 3.2. Proposition 3.1 is not a new theorem, and we do not present it as one. Its $m = 1$ branch is Corollary 3 of Gilmer–Livingston [4], proved there for cyclic $H_1(D_K;\mathbb{Z})$ under the weaker hypothesis that every prime exponent of Δ is odd, by the argument above specialised to $m = 1$ ("since each prime occurs with odd exponent, no primary summand of $H_1(M(K))$ is metabolic"). Its branch $\Delta = p^2q$ is the Lemma of Fairchild [1]: for $H_1(D_K;\mathbb{Z}) \cong \mathbb{Z}/p^2q$ with p prime and q a product of primes all with odd exponent, a Möbius band filling forces $\lambda(a, a) = \pm 1/p^2q$ or $\lambda(a, a) = \pm 1/q$ on a generator a — which is exactly $n_1 \in \{p^2q, q\}$, the two admissible orders here. The divisor structure $\det K = \ell m^2$ also appears in Jabuka–Kelly [6], on the Donaldson-diagonalization side and without the coprimality refinement, which is forced only in the cyclic case. We searched for the uniform statement and did not find it in print (see §7); the honest description is that it is a routine unification of two published statements, put in a shape a machine can decide. Note in particular that "all prime exponents odd" is strictly weaker than "square-free", so [8, Cor. 2.5] is a weaker test than one published fifteen years earlier.

The extra clauses of condition (2) are not optional, and the source's own data proves it.

Proposition 3.3. *The square-free hypothesis of* [8, Cor. 2.5] *cannot be dropped without replacing the test by condition* (2). *The knot* $11n_{39}$ *has* $\det = 25$ *and, on the generator produced by our computation, linking form* $7/25$*; the residue* $7t^2$ *is never* ± 1 *modulo* 25*, so the arithmetic clause of* [8, Cor. 2.5] *fires — yet* $25 = 1 \cdot 5^2$ *is an admissible splitting with* $n_1 = 1$*, so condition* (2) *does not obstruct, and indeed KnotInfo publishes* $\gamma_4(11n_{39}) = 1$ *and* [9] *carries an explicit Möbius band for it.*

Proof and census. The five clauses for $11n_{39}$ — that the reader of Section 4 returns the stated 6×6 Goeritz matrix from KnotInfo's diagram code, that its determinant has absolute value 25, that e_1 generates by the gcd criterion (1), that the linking form is 7/25, and that the clause of [8, Cor. 2.5] holds while condition (2) fails — are decided by exhaustive evaluation; the residue sweep is over $t = 0, \ldots, 24$ and the splitting enumeration returns $S(25) = \{25, 1\}$. Beyond this single witness we ran the naive test across the 14,319 knots Algorithm 1 skipped, restricted to the 2,329 with a KnotInfo diagram and cyclic H_1: the clause $n_1 = \Delta$ alone "obstructs" 1,002 of them; for 513 of those some smaller admissible n_1 admits a solution, so Theorem 2.1 does not obstruct; and for 322 of those 513 the search of [8] found an explicit Möbius band. This census is a computation over the published tables carried out outside the formal development; the $11n_{39}$ witness is machine-checked. □

Proposition 3.4. *The enumeration* $S(\Delta)$ *is neither trivial nor vacuous:* $S(21) = \{21\}$*,* $S(135) = \{135\}$*,* $S(375) = \{375\}$*, while* $S(25) = \{25, 1\}$*,* $S(99) = \{99, 11\}$*,* $S(315) = \{315, 35\}$*,* $S(585) = \{585, 65\}$ *and* $S(36) = \{36, 9, 4, 1\}$*. Consequently a knot whose determinant is a perfect square is never obstructed by condition* (2)*, since then* $1 \in S(\Delta)$ *and the condition modulo* 1 *is vacuously satisfiable; this is verified for every numerator at* $\Delta = 9$*,* 25 *and* 225*. Finally, condition* (2) *is invariant under* $\lambda \mapsto -\lambda$*, checked for every numerator at* $\Delta = 135$ *and* $\Delta = 315$*; so neither mirroring the knot nor exchanging the two checkerboard colours can change a verdict.*

4. Goeritz matrices from planar diagram codes

To evaluate condition (2) on thousands of knots one needs a Goeritz matrix for each, and what the tables publish is a planar diagram code: for a knot with n crossings, a list of n quadruples of edge labels, one per crossing, in rotational order starting from the incoming under-strand. This is KnotInfo's `pd_notation` column, available for every knot of at most 13 crossings.

The passage from such a code to a Goeritz matrix is elementary and we describe it only to fix what was implemented. Flatten the code into $4n$ *darts*, the dart $4i+k$ carrying the label in slot k of crossing i; each label occurs exactly twice, which defines an involution α pairing darts along edges. Call $4i+k$ the *corner* of crossing i between slots k and $k+1$, and let ν send the corner $4i+k$ to the corner of the dart $\alpha(4i+((k+1) \bmod 4))$. Then ν is a permutation of the $4n$ corners whose orbits are the $n+2$ faces of the diagram, and a face is named by the least corner index in its orbit. The four corners of a crossing alternate in colour, so a checkerboard colouring is one bit b_i per crossing with $\mathrm{colour}(4i+k) \equiv b_i+k \pmod 2$; the constraint system this imposes along ν is solved by propagation from $b_0=0$, which terminates in n rounds because a knot diagram is connected. The white corners of crossing i are then $4i+b_i$ and $4i+b_i+2$, the crossing's sign is $\eta=+1$ if $b_i=0$ and -1 otherwise, and the matrix of Section 2 is assembled and truncated. Write $\mathcal{G}(P)$ for the resulting matrix.

Proposition 4.1. *For each of the* 118 *knots of Table 1 and for each of the control knots* 3_1*,* 4_1*,* 6_1*,* $11n_{36}$*,* $11n_{39}$*,* $11n_{57}$*, the matrix* $\mathcal{G}(P)$ *computed from KnotInfo's published diagram code* P *is the explicit symmetric integer matrix* U *recorded in the accompanying development, and* $|\det U|$ *equals the determinant KnotInfo publishes for that knot.*

Proof and external gates. Both clauses are decided by evaluation; determinants are computed by Bareiss fraction-free elimination, which needs n^3 exact integer divisions and, unlike cofactor expansion, is affordable on dense matrices inside a proof kernel. Proposition 4.1 is a statement about the knots we use, and by itself it does not certify that $\mathcal{G}$ computes a Goeritz matrix in general; that reading is the cited content of [5], exactly as the reading of a Seifert matrix is cited in Section 2. What licenses it in practice is a set of checks run over the whole table before any of the present statements were proved, each of them a computation outside the formal development:

- $|\det \mathcal{G}(P)|$ equals KnotInfo's published determinant for 12,965 **of the** 12,965 knots of at most 13 crossings, with no exceptions;
- on the 9,629 knots where Algorithm 1 of [8] actually ran, the pipeline reproduces its published `Linking Form` column — 4,062 `Obstructed`, 5,508 `No obstruction`, 59 `det = 1` — with zero disagreements;
- the gcd criterion (1) agrees with KnotInfo's independent `torsion_numbers` column about which knots have cyclic $H_1(D_K;\mathbb{Z})$ on all 12,716 knots matched by name; 758 knots of at most 13 crossings are non-cyclic by both;
- computing λ instead from KnotInfo's `seifert_matrix` column, as $(V+V^{\mathsf{T}})^{-1}$, gives the same determinant and the same verdict on all 11,958 cyclic knots;
- the reader reproduces the Goeritz matrix that [12] print for 4_1 in their Example 2.8, up to the global sign discussed in Section 2, and the Goeritz matrix that [8] print for the pretzel knot $P(3,3,-5)$ in their Example 2.6, with the same determinant 21 and the same linking-form numerator 2;
- a second, independently written implementation using the opposite colour anchor and a shifted corner successor produced different matrices — their negatives — and identical verdicts on all 12,716 knots.

□

5. One hundred and eighteen knots

Theorem 5.1. *Let K be any of the 118 knots named in Table 1. Then K is neither slice nor bounds a Möbius band in B^4; that is,*

$$\gamma_4(K) \ \geq \ 2.$$

Consequently, for the 91 knots of the table whose entry in KnotInfo's `smooth_4d_crosscap_number` *column was the range $[1, 2]$, one has $\gamma_4(K) = 2$; and for the remaining 27, whose entry was $[1, 3]$, one has $\gamma_4(K) \in \{2, 3\}$.*

Proof, and what rests on exhaustive computation. Fix a knot K of the table, and let P be KnotInfo's diagram code for it, $U = \mathcal{G}(P)$ the Goeritz matrix of Section 4, Δ the determinant KnotInfo publishes, and v, a the integer vector and residue recorded for K in the accompanying development. Five decidable clauses are evaluated for K:

1. $\mathcal{G}(P) = U$;
2. $|\det U| = \Delta$;
3. $\gcd\big(\det U,\ (\mathrm{adj}(U)v)_1, \ldots, (\mathrm{adj}(U)v)_m\big) = 1$, the entries of $\mathrm{adj}(U)v$ being computed by Cramer's rule as the m determinants of the matrices obtained from U by replacing one column with v, so that the adjugate is never carried as data;
4. $-v^{\mathsf{T}}\,\mathrm{adj}(U)\,v \equiv a \pmod{\Delta}$;
5. for every $n_1 \in S(\Delta)$ and every t with $0 \leq t < n_1$, $at^2 \not\equiv \pm 1 \pmod{n_1}$, where $S(\Delta)$ is produced by testing every $m \leq \Delta$ for $m^2 \mid \Delta$ and $\gcd(\Delta/m^2, m) = 1$.

Clauses (1) and (2) are Proposition 4.1. By the gcd criterion (1), clause (3) says that $H_1(D_K; \mathbb{Z})$ is cyclic of order Δ with generator $[v]$; by Gordon–Litherland [5], clause (4) then says that $\lambda([v], [v]) = a/\Delta$. Clause (5) is the negation of condition (2), so by Proposition 3.1 and Theorem 2.1 the knot K is neither slice nor bounds a Möbius band, which is the assertion $\gamma_4(K) \geq 2$. The upper bounds 2 and 3 in the second sentence are KnotInfo's, cited and not proved here; its description page derives them from the unknotting number together with a search over crossing changes.

All five clauses are decided by exhaustive evaluation for each of the 118 knots, inside a proof kernel and without appeal to compiled evaluation. The searches involved are small and completely described by their bounds: the largest determinant occurring in the table is 585, so the longest residue sweep in clause (5) runs over 585 values of t, no knot in the table has more than two admissible splitting orders, the largest Goeritz matrix is 10×10, and the longest diagram code has 13 crossings, hence 52 darts. The 118 knots are grouped into eleven blocks of ten and one of eight, the block verdicts are counted, and the counts are summed formally; the whole evaluation takes about twelve CPU-minutes.

Three things are *not* machine-checked and are cited. First, Theorem 2.1 and the Gordon–Litherland description of $H_1(D_K; \mathbb{Z})$ and λ in terms of a Goeritz matrix. Second, that $\mathcal{G}$ computes a Goeritz matrix of the diagram a code describes, for which the evidence is the gate list in the proof of Proposition 4.1. Third, KnotInfo's data itself: the diagram codes, the determinants and the upper bounds. Note also that the *selection* of these 118 knots — that they are exactly the knots among the 2,329 candidates on which the criterion fires and whose γ_4 the pipeline of [8] had left undetermined — is the outcome of a search over the published tables carried out outside the formal development. What is certified is each of the 118 knots individually, not the completeness of the list. □

Corollary 5.2. *Of the knots of crossing number at most 14, at least 22,530 have $\gamma_4(K) > 1$.*

Proof. The count 22,412 of [8, Thm. 1.1] is, in the authors' dataset, exactly the number of knots obstructed by their Arf test or by their linking-form test. Every knot of Table 1 is obstructed by neither: their `Linking Form` entries in [9] read `det not square-free` for 110 of them and `find generator by hand` for the other 8, and their `Arf obstructed` entries are all `False`. So the two sets are disjoint and $22{,}412 + 118 = 22{,}530$. (That the recorded 22,412 is reproduced by summing the two obstruction columns of [9] is a check on the published dataset, run outside the formal development.) □

Example 5.3. Take $K = 13a_{1100}$. KnotInfo's diagram code has 13 crossings, and the reader returns

$$U = \begin{pmatrix} 3 & -1 & -2 & 0 & 0 & 0 \\ -1 & 5 & -1 & -1 & 0 & 0 \\ -2 & -1 & 6 & 0 & 0 & -1 \\ 0 & -1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 0 & -1 & 2 \end{pmatrix}, \qquad \det U = 315,$$

matching KnotInfo's determinant. With $v = e_1$ one computes $\operatorname{adj}(U)e_1 = (178, 61, 79, 48, 35, 57)$, whose gcd with 315 is 1; so $H_1(D_K;\mathbb{Z}) \cong \mathbb{Z}/315$ with generator $[e_1]$, and $\lambda([e_1],[e_1]) = -178/315 = 137/315$. Now $315 = 3^2 \cdot 5 \cdot 7$, so $S(315) = \{315, 35\}$: the choice $m = 1$ and the choice $m = 3$, the latter admissible because $\gcd(35, 3) = 1$. Neither $137t^2 \equiv \pm 1 \pmod{315}$ nor $137t^2 \equiv \pm 1 \pmod{35}$ has a solution — the residues of $137t^2$ modulo 35 are exactly $\{0, 2, 7, 8, 15, 18, 22, 23, 25, 28, 30, 32\}$ — so no splitting as in Theorem 2.1 exists and $\gamma_4(13a_{1100}) \geq 2$. KnotInfo publishes the range $[1, 2]$, so $\gamma_4(13a_{1100}) = 2$. Since 315 has an even prime exponent, this is one of the 23 knots for which the applicable published statement is Fairchild's Lemma, with $p = 3$ and $q = 35$.

Remark 5.4. The uniform criterion does have reach beyond both published lemmas, but on this table that reach produced nothing new, and we record the measurement rather than suppress it. Among the 2,329 candidates, 154 have determinant in $\{81, 225, 441, 405, 567\}$, where neither [4, Cor. 3] nor [1, Lemma] applies. The criterion obstructs 13 of them, all with determinant $405 = 3^4 \cdot 5$ or $567 = 3^4 \cdot 7$, whose admissible splitting orders are $\{405, 5\}$ and $\{567, 7\}$. All 13 were already obstructed by the Arf criterion, so none of them contributes a new value.

5.1. **Controls.** A computation of this shape can be wrong in two directions, and both are tested.

Proposition 5.5. *The following hold, each by exhaustive evaluation from the published diagram codes.*

(1) (Too large.) *The trefoil* 3_1 *bounds a Möbius band, so the test must stay silent on it, and it does: the reader returns the* 1×1 *matrix* $[3]$, e_1 *generates* $\mathbb{Z}/3$, $\lambda = 2/3$, *and* $2 \cdot 1^2 \equiv -1 \pmod 3$ *is exactly the value a splitting is allowed to have. At scale, over the knots the criterion can be evaluated on: of the* 5,997 *of them for which the search of* [8] *found a band it obstructs* 0, *and of the* 109 *with a published value* $\gamma_4 = 1$ *it obstructs* 0.

(2) (Non-vacuity.) *The figure-8 knot* 4_1 *is obstructed by the linking form alone: the reader returns* $\left(\begin{smallmatrix} 3 & -1 \\ -1 & 2 \end{smallmatrix}\right)$, $\lambda = 3/5$ *on a generator, and* $3t^2$ *is never* ± 1 *modulo* 5. *This is independent of the Arf route* [8, Ex. 2.2] *takes to the same conclusion, and it reproduces Example 2.8 of* [12], *whose Goeritz matrix is the negative of ours and whose value* $\pm 2/5$ *is ours, since* $3 \equiv -2 \pmod 5$. *At scale: of the* 56 *knots with a published value* $\gamma_4 = 2$, *the criterion obstructs* 50.

(3) (Too small.) *A perfect-square determinant can never be obstructed, as in Proposition 3.4; this is checked end to end on the slice knot* 6_1, *whose diagram code yields determinant* 9 *and* $\lambda = 2/9$, *and where the test correctly stays silent.*

(4) (Certificate specificity.) *A vector that does not generate is rejected: for* $11n_{39}$ *the fifth standard basis vector has* $\gcd(\det U, \operatorname{adj}(U)e_5) = 5$ *and fails the gcd criterion* (1), *as does the zero vector for every knot of determinant* > 1.

(5) (The two knots of §6.) $11n_{36}$ *and* $11n_{57}$ *have square-free determinants* 67 *and* 7, *so on them the criterion coincides with* [8, Cor. 2.5]*; it stays silent on both, reproducing from the diagram codes the sentence of* [8] *that they "were not able to definitively obstruct either the 11n36 or the 11n57". Their* γ_4 *remains open.*

6. The arithmetic of six corrected table entries

This section re-verifies published material and claims nothing new. Its subject is §4.3.2 of [8], which corrects six entries of KnotInfo inherited from [1], and whose arithmetic core we recompute from KnotInfo's Seifert matrices with no published scalar used as an input.

Proposition 6.1. *From the Seifert matrices KnotInfo publishes,*

$$\Delta_{10_{129}}(t) = 2 - 6t + 9t^2 - 6t^3 + 2t^4, \quad \Delta_{11n_{82}}(t) = 1 - 3t + 4t^2 - 3t^3 + 4t^4 - 3t^5 + t^6,$$

both agreeing with the Alexander polynomial vector KnotInfo publishes, and in particular $\det(10_{129}) = 25 \neq 19 = \det(11n_{82})$.

Lee–Sabloff report that the band drawn in [1] for $11n_{36}$, which is claimed there to produce 10_{129}, in fact produces $11n_{82}$, and that the two knots have different Alexander polynomials. Proposition 6.1 sharpens the arithmetic half of that observation: already the determinants differ, so no polynomial comparison of degree 6 is needed. The identification of the band's output as $11n_{82}$ is theirs and is cited. Determinants and Alexander polynomials here are computed by Bareiss elimination over $\mathbb{Z}$ and over $\mathbb{Z}[t]$ respectively; as the two routines share no code path, the identity $|\Delta_K(-1)| = \det(K)$, which we verify for all twelve knots involved, is a genuine cross-check.

Proposition 6.2. $11n_{82}$ *is not slice, by two independent obstructions. There is an explicit integer matrix* P *with* $\det P \neq 0$ *and*

$$P^{\mathsf{T}}(V + V^{\mathsf{T}})P = \operatorname{diag}(2, -2, 4, -4, -4, 36, -18, -342),$$

so by Sylvester's law of inertia $\sigma(11n_{82}) = -2 \neq 0$*, matching KnotInfo, and Murasugi's theorem obstructs sliceness. Independently* $\det(11n_{82}) = 19$ *is not a perfect square, so the Fox–Milnor condition obstructs it as well;* [8] *cite only the signature.*

Proposition 6.3. *For the figure-8 knot,* $\sigma(4_1) = 0$ *by a certified congruence, so the signature obstruction is silent and a second invariant is genuinely needed;* $\det(4_1) = 5$ *is not a perfect square, so* 4_1 *is not slice; and* $\det(4_1) \equiv 5 \pmod 8$ *gives* $\operatorname{Arf}(4_1) = 1$*, whence* $\sigma + 4\operatorname{Arf} = 4 \equiv 4 \pmod 8$ *and* [8, Thm. 2.1] *applies:* $\gamma_4(4_1) \geq 2$*. With the Klein bottle filling noted in* [8] *this is* $\gamma_4(4_1) = 2$*, the value KnotInfo publishes.*

Propositions 6.1–6.3 are the whole arithmetic content of the retraction in §4.3.2 of [8]: the band on $11n_{36}$ does not produce 10_{129}; what it produces is not slice; and the knot the band on $11n_{57}$ produces, the figure-8, is not slice and bounds no Möbius band either.

Proposition 6.4. *Run end to end from Seifert matrices — signature from a Sylvester certificate, Arf invariant from the determinant modulo* 8*, no published scalar as input — the criterion* $\sigma + 4\operatorname{Arf} \equiv 4 \pmod 8$ *fires on exactly three of the fourteen knots* 3_1, 4_1, 6_1, 6_3, 7_5, 8_{20}, 10_{129}, $11n_{36}$, $11n_{57}$, $11n_{82}$, $11n_{92}$, $11n_{125}$, $11n_{131}$, $11n_{161}$*: namely* 4_1 *and* 6_3 *with* $\sigma = 0$, $\operatorname{Arf} = 1$*, and* 7_5 *with* $\sigma = -4$, $\operatorname{Arf} = 0$*. It is silent on the other eleven, among them* $11n_{36}$ *(*$\sigma = -2$, $\operatorname{Arf} = 1$*) and* $11n_{57}$ *(*$\sigma = -6$, $\operatorname{Arf} = 0$*), which is precisely the position* [8] *report. The criterion is moreover invariant under* $\sigma \mapsto -\sigma$*, so no sign convention in a Seifert matrix can move any of these verdicts.*

The Arf criterion as implemented here reproduces the `Arf obstructed` column of [9] with zero disagreements on all 12,965 knots of at most 13 crossings — the entire overlap, since KnotInfo does not carry the 46,972 knots of 14 crossings. That check is a computation over the published tables, outside the formal development, and it is what licenses reading the predicate as the hypothesis of [8, Thm. 2.1] rather than as a guess at it.

Proposition 6.5. *For the pretzel knot* $P(3,3,-5)$ *and the Goeritz matrix* $U = \left(\begin{smallmatrix} -6 & 3 \\ 3 & 2 \end{smallmatrix}\right)$ *printed in* [8, Ex. 2.6]*:* $\det U = -21$*, so* $\det(P(3,3,-5)) = 21$*;* 21 *is square-free, so* [8, Cor. 2.5] *applies;* e_1 *generates* $H_1(D_K;\mathbb{Z}) \cong \mathbb{Z}/21$*, because* $Ux = qe_1$ *has no integral solution for any proper divisor* $q \in \{1,3,7\}$*, which by Cramer is a divisibility test on the first column* $(2,-3)$ *of* $\operatorname{adj}(U)$*; and* $2n^2 \not\equiv \pm 1 \pmod{21}$ *for* $n = 1, \ldots, 20$*, the residues being* $\{2, 8, 9, 11, 14, 15, 18\}$*. Hence* $\gamma_4(P(3,3,-5)) \geq 2$.

Outside the formal development we also recomputed $-U^{-1}$ for that matrix, obtaining

$$-U^{-1} = \begin{pmatrix} 2/21 & -1/7 \\ -1/7 & -2/7 \end{pmatrix},$$

exactly the matrix [8, Ex. 2.6] prints.

Proposition 6.6. *Controls for the Seifert-matrix route.* (Silence where it is forced.) 10_{129}*,* 6_1 *and* 8_{20} *are slice, so both obstructions must fail on them, and they do: certified signature* 0*, and determinants* $25 = 5^2$*,* $9 = 3^2$*,* $9 = 3^2$*.* (Separation.) *The trefoil is not slice —* $\sigma = -2$ *and* $\det = 3$ *is not a square — yet* $\gamma_4(3_1) = 1$*, and the Arf criterion correctly stays silent, so "not slice" is strictly weaker than "bounds no Möbius band".* (Too large and too small.) $\det(11n_{82})$ *is neither* 18 *nor* 20*, and its certified inertia is neither* 0 *— the value Murasugi's theorem would force were it slice — nor* -4*.* (Certificate specificity.) *The transformation matrices of* $11n_{36}$ *and* $11n_{57}$ *do not diagonalize the form of* $11n_{82}$*; a diagonal perturbed in its last entry is rejected; and the zero matrix is rejected for singularity even though its congruence product is diagonal.* (Separation of the Alexander computation.) *It distinguishes* $11n_{36}$ *from* $11n_{57}$*,* $11n_{82}$ *from* $11n_{92}$*, and* $11n_{125}$ *from* $11n_{131}$*.* (Non-vacuity of the linking-form clauses.) *The obstruction of Proposition 6.5 fails with numerator* 1 *and with numerator* 5*; the generator test fails for the adjugate column* $(3, 3)$*; and the square-freeness guard bites at* 63 *— the determinant of* $11n_{125}$ *and* $11n_{161}$*, which is why* [9] *records "det not square-free" for both — and at* 25*.*

Remark 6.7. Nothing topological is formalized in this paper. Every machine-checked statement is a statement about explicit integer matrices, lists and residues; the passage from those to knots — a Seifert matrix computes Δ_K, det, σ; a Goeritz matrix presents $H_1(D_K; \mathbb{Z})$ and λ; the theorems of Murasugi, Fox–Milnor, Yasuhara, Gilmer–Livingston and Murakami–Yasuhara — is classical, cited, and used exactly as it is used in [8]. In particular no band move and no knot recognition occurs anywhere: the four corrections of [8] in the upward direction, for $11n_{92}$, $11n_{125}$, $11n_{131}$ and $11n_{161}$, are constructions and are outside the scope of any finite invariant computation.

7. Verification

Every finite computation asserted above has been checked in Lean 4 against *Mathlib*, in a development free of `sorry`, of added axioms, and of compiled evaluation: all decisions are made by the proof kernel, and `#print axioms` reports `propext` alone for the 118-knot statement, at most `propext`, `Classical.choice` and `Quot.sound` elsewhere. The verified content is, precisely: the five clauses of the proof of Theorem 5.1 for each of the 118 knots and for each control knot of Propositions 3.3, 3.4 and 5.5; the implications of Proposition 3.1 relating $S(\Delta)$, square-freeness and the test of [8, Cor. 2.5], which are four lemmas proved rather than decided; and, for Section 6, the Alexander polynomials, determinants, Arf invariants and Sylvester certificates of fourteen knots together with every claim of Propositions 6.1–6.6. The whole check costs about twelve CPU-minutes. What is *not* verified is stated as such in the text: the topological input, which is cited; the numbers KnotInfo and [9] publish, which are input data; and the table-wide searches — the 12,965-knot determinant gate, the reproduction of the `Arf obstructed` and `Linking Form` columns, the cyclicity comparison, the independent Seifert-matrix route, the unsoundness census of Proposition 3.3, the scale controls of Proposition 5.5, and the selection of the 118 — each of which was carried out over the published tables in a separate implementation, before the formal statements were written, and each of which is reported above with its counts.

A search for prior determinations of γ_4 for these knots found none. KnotInfo's own attribution page for the column, read live on 2026-08-28, credits Jabuka–Kelly with 8- and 9-crossing knots, Ghanbarian [3] with 10-crossing knots, Fairchild with non-alternating 11-crossing knots, and then records that "Daniel Lee and Joshua Sabloff [9] provided extensive new data. In particular, they extended the KnotInfo data to include 11-crossing alternating knots as well as 12-crossing and 13-crossing knots"; the survey sentence of [8] and the corresponding sentence of [13] agree, and the complete forward-citation lists of [6] (15 citing papers), of [3] (8) and of [1] (3), read from Semantic Scholar on 2026-08-28, contain no further crossing-number table. We also searched the arXiv full-text corpus and the listing interface, targeted the eleven named 11- and 12-crossing knots directly, and found nothing. Not searched, and therefore not a clean negative: MathSciNet and zbMATH review text, and journal back issues that never reached the arXiv.

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TABLE 1. The 118 knots of Theorem 5.1: KnotInfo name, determinant, and the value of γ_4 that follows from Theorem 5.1 together with KnotInfo's upper bound. All 118 were open entries of KnotInfo's `smooth_4d_crosscap_number` column when the computation was run.

knot	det	γ_4	knot	det	γ_4	knot	det	γ_4
$11a_{6}$	135	2	$13a_{2307}$	375	2	$13n_{439}$	135	2
$11a_{264}$	135	2	$13a_{2315}$	357	2	$13n_{448}$	135	2
$11a_{280}$	105	2	$13a_{2484}$	495	2	$13n_{650}$	135	2
$11a_{305}$	135	$\{2,3\}$	$13a_{2501}$	315	2	$13n_{831}$	135	$\{2,3\}$
$11a_{318}$	135	$\{2,3\}$	$13a_{2544}$	351	2	$13n_{874}$	135	$\{2,3\}$
$12a_{389}$	315	2	$13a_{2642}$	459	$\{2,3\}$	$13n_{949}$	135	$\{2,3\}$
$12a_{426}$	273	2	$13a_{2672}$	465	2	$13n_{984}$	135	2
$12a_{923}$	135	$\{2,3\}$	$13a_{2717}$	135	2	$13n_{990}$	135	2
$12a_{981}$	135	2	$13a_{2742}$	315	2	$13n_{1055}$	135	2
$12n_{799}$	135	2	$13a_{2947}$	135	2	$13n_{1079}$	135	$\{2,3\}$
$12n_{850}$	15	$\{2,3\}$	$13a_{3459}$	459	$\{2,3\}$	$13n_{1157}$	135	2
$13a_{132}$	315	$\{2,3\}$	$13a_{3530}$	315	$\{2,3\}$	$13n_{1193}$	135	2
$13a_{191}$	315	$\{2,3\}$	$13a_{3641}$	375	2	$13n_{1197}$	135	2
$13a_{316}$	351	2	$13a_{3642}$	459	$\{2,3\}$	$13n_{1518}$	135	2
$13a_{349}$	351	2	$13a_{3718}$	375	2	$13n_{1530}$	135	2
$13a_{361}$	351	2	$13a_{3794}$	375	2	$13n_{2326}$	135	2
$13a_{395}$	351	2	$13a_{3858}$	459	$\{2,3\}$	$13n_{2341}$	135	2
$13a_{445}$	375	2	$13a_{3865}$	315	2	$13n_{2454}$	105	2
$13a_{635}$	375	2	$13a_{3913}$	375	2	$13n_{2670}$	135	2
$13a_{654}$	351	2	$13a_{3917}$	135	2	$13n_{2764}$	135	2
$13a_{662}$	351	2	$13a_{3931}$	315	2	$13n_{2850}$	135	2
$13a_{738}$	375	2	$13a_{4017}$	375	2	$13n_{3020}$	135	$\{2,3\}$
$13a_{740}$	375	2	$13a_{4055}$	351	2	$13n_{3266}$	135	2
$13a_{762}$	495	2	$13a_{4112}$	315	2	$13n_{3458}$	135	2
$13a_{765}$	351	2	$13a_{4116}$	315	$\{2,3\}$	$13n_{3512}$	135	2
$13a_{832}$	315	2	$13a_{4176}$	351	2	$13n_{3530}$	135	2
$13a_{968}$	315	2	$13a_{4182}$	375	2	$13n_{3790}$	135	2
$13a_{971}$	315	2	$13a_{4190}$	315	$\{2,3\}$	$13n_{3891}$	135	$\{2,3\}$
$13a_{1100}$	315	2	$13a_{4201}$	135	$\{2,3\}$	$13n_{4171}$	135	2
$13a_{1196}$	459	$\{2,3\}$	$13a_{4210}$	375	2	$13n_{4199}$	135	2
$13a_{1266}$	375	2	$13a_{4250}$	375	2	$13n_{4369}$	135	2
$13a_{1353}$	455	$\{2,3\}$	$13a_{4309}$	351	$\{2,3\}$	$13n_{4396}$	135	2
$13a_{1395}$	375	2	$13a_{4359}$	495	2	$13n_{4507}$	135	2
$13a_{1468}$	135	$\{2,3\}$	$13a_{4695}$	585	2	$13n_{4748}$	105	2
$13a_{1682}$	375	2	$13a_{4706}$	315	2	$13n_{4764}$	135	2
$13a_{1897}$	351	2	$13a_{4745}$	315	2	$13n_{4815}$	135	2
$13a_{1899}$	375	2	$13a_{4800}$	315	$\{2,3\}$	$13n_{4971}$	135	$\{2,3\}$
$13a_{1971}$	351	2	$13n_{385}$	135	2	$13n_{5060}$	135	$\{2,3\}$
$13a_{2202}$	315	2	$13n_{404}$	135	2			
$13a_{2207}$	375	2	$13n_{416}$	135	2			