SUMS OF AT MOST FIFTY POSITIVE NINTH POWERS: THE EXCEPTIONAL VALUE 5 360 377 770

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Abstract. Vaughan and Wooley proved that every sufficiently large positive integer is a sum of at most 50 positive ninth powers, without an effective threshold. Benfield and Lippard, who computed the corresponding largest exception for the exponents $k = 5, 6, 7, 8$, left the ninth-power case open, remarking that the computation was out of reach for the machines available to them. We prove that 5 360 377 770 is not a sum of at most 50 positive ninth powers, that it *is* a sum of 51 of them, so that its minimal ninth-power representation length is exactly 51, and that it is the only exception in a window of 20 001 consecutive integers centred on it. Consequently, if every integer above 5 360 377 770 is a sum of at most 50 positive ninth powers — the effective form of the Vaughan–Wooley bound, which the literature does not supply — then 5 360 377 770 is exactly the largest integer that is not, answering the question of Benfield and Lippard conditionally. The refutation is not a failed search but a proved one: the depth-first search we use carries a reachability prune, and the statement that the prune discards no representation is itself part of the formal development, so that a search returning *false* refutes representability rather than merely failing to find it. The same machinery reproduces, as theorems, the four values already computed by Benfield and Lippard. Everything is machine-checked in Lean 4.

1. Introduction

For $k \geq 2$ let $G(k)$ denote the least s such that every sufficiently large positive integer is a sum of at most s positive k-th powers. The sharpest bounds available for small exponents are those of Vaughan and Wooley; the ninth-power bound

$$G(9) \leq 50$$

is Theorem 1.1 of *Further improvements in Waring's problem, IV: Higher powers* [2], whose table of upper bounds $H(k)$ for $9 \leq k \leq 20$ begins $H(9) = 50$, $H(10) = 59$. The bound is ineffective in the sense that matters here: it is a statement about *sufficiently large* integers, and no threshold beyond which it holds has ever been computed. (On p. 204 the same paper records $G(5) \leq 17$, $G(6) \leq 24$, $G(7) \leq 33$, $G(8) \leq 42$, $G(9) \leq 51$ as what one obtains by collecting the conclusions of the earlier papers [3, 4, 5] of the same series; the improvement of the ninth-power value from 51 to 50 is the content of its own Theorem 1.1. See Remark 1.1.)

Benfield and Lippard [1] revisit, for $5 \leq k \leq 9$, a computational programme of Zenkin [7] on the generalized Waring problem: for each exponent they determine which integers fail to be a sum of a prescribed number of positive k-th powers. The number they prescribe is the Vaughan–Wooley bound, and we write b_k for it:

$$b_5 = 17, \qquad b_6 = 24, \qquad b_7 = 33, \qquad b_8 = 42, \qquad b_9 = 50.$$

Because the Vaughan–Wooley bounds are ineffective, a statement of the form "every integer greater than M is a sum of at most b_k positive k-th powers" is, for them, a conjecture rather than a theorem. What is finite, and what they compute, is the largest integer that is *not* such a sum; write M_k for that value. For $k = 5, 6, 7, 8$ they obtain

$$M_5 = 87\,918, \qquad M_6 = 1\,414\,564, \qquad M_7 = 9\,930\,770, \qquad M_8 = 858\,367\,748,$$

appearing as the values in their Conjectures 5.2, 6.2, 7.2 and 8.2 respectively (all numbering here is that of v2, in which every theorem-like environment shares one counter numbered by section); we write $M_9 = 5\,360\,377\,770$ for the ninth-power value obtained below. For $k = 9$ they stop. Their §9 records $G(9) \leq 50$ and then asks, verbatim:

> **Question 9.2.** What is the largest positive integer that is *not* the sum of at most 50 positive ninth powers?

immediately preceded by "Unlike the previous cases, computation to establish the smallest such number is out of reach for the modest computers the authors are using", and followed by "The nature of the proof technique used in the later sections requires an exact value for the largest such number, but in the case of ninths, is still unknown." The question is still open in the current version of [1]: at the time of writing that paper stands at v2 of 31 March 2025, Crossref records no journal version of it, and OpenAlex records no work citing it.

This paper answers the finite half of Question 9.2, exactly and unconditionally, and the whole of it conditionally on the effective form of $G(9) \leq 50$.

Results. Write $R_k(j)$ for the set of integers expressible as a sum of at most j positive k-th powers, and $E_k(j) = \{n \geq 1 : n \notin R_k(j)\}$ for the exceptional set; Question 9.2 asks for $\max E_9(50)$. We prove the following.

(1) $5\,360\,377\,770 \in E_9(50)$: it is not a sum of at most 50 positive ninth powers (Theorem 4.1).
(2) It is a sum of 51 positive ninth powers, so its minimal ninth-power representation length is exactly 51 and the threshold 50 in Question 9.2 is precisely where it fails (Theorem 4.2). An explicit representation is

$$5\,360\,377\,770 = 12^9 + 8^9 + 5 \cdot 6^9 + 8 \cdot 5^9 + 4^9 + 5 \cdot 3^9 + 11 \cdot 2^9 + 19 \cdot 1^9.$$

(3) $5\,360\,377\,770$ is the *only* element of $E_9(50)$ in the window $[5\,360\,367\,770,\ 5\,360\,387\,770]$ of 20 001 consecutive integers, 10^4 on either side (Theorem 5.1).
(4) Conditionally: if every integer greater than $5\,360\,377\,770$ lies in $R_9(50)$, then $5\,360\,377\,770 = \max E_9(50)$, which is the answer to Question 9.2 (Theorem 6.1).

The four values of Benfield and Lippard are reproduced by the same machinery, in both directions — not a sum of b_k positive k-th powers, and a sum of $b_k + 1$ of them (Theorem 7.1) — and this reproduction was carried out *before* the ninth-power value was computed, as a check on the pipeline. A uniform pattern emerges that [1] does not state: each of the five maxima requires exactly one power more than the bound, M_k being a sum of $b_k + 1$ but not of b_k positive k-th powers (Remark 7.3).

The proof of (1) is a finite search, and the point of the formal development is that it is a search whose *completeness* is proved rather than assumed. Section 3 introduces a depth-first search search(k, i, r, b) over representations of r by at most b positive k-th powers with bases in $\{1, \ldots, i\}$, carrying a reachability prune: a partial state with budget b and largest available base i is abandoned when $b \cdot i^k < r$. Without the prune the ninth-power instance is a walk over the $\binom{62}{12} = 2\,160\,153\,123\,141$ multiplicity vectors $(c_1, \ldots, c_{12})$ with $\sum c_i \leq 50$; with it, the instance closes in 435 471 nodes. Lemma 3.2 shows the prune is sound, Proposition 3.3 shows that nothing representable is ever pruned away, and Lemma 3.5 shows that no base above 12 can occur, since $13^9 = 10\,604\,499\,373 > 5\,360\,377\,770$. Together these turn "the search returned *false*" into a refutation.

What is not proved. We do not prove that $5\,360\,377\,770$ is the largest exception. That statement needs an *effective* $G(9) \leq 50$: an explicit n_0 together with a proof that every $n > n_0$ is a sum of at most 50 positive ninth powers. No such n_0 appears in the literature — the Vaughan–Wooley bound is proved by the circle method for sufficiently large n — and its absence is precisely why the four analogous statements in [1] are conjectures rather than theorems. Theorem 6.1 hoists exactly that missing ingredient into a hypothesis and proves the rest. The hypothesis is not vacuous and is not refutable by any computation we ran: its conclusion holds for the 10^4 integers immediately above the value (Theorem 5.1) and, outside the formal development, for every $n \leq 3 \cdot 10^{10}$ (Remark 6.2).

Remark 1.1 (attribution of the bounds b_k). [1] attributes $G(9) \le 50$ to Vaughan and Wooley's 1995 paper [3]. Theorem 1.1 of that paper reads

$$G(5) \le 17, \quad G(6) \le 25, \quad G(7) \le 33, \quad G(8) \le 43, \quad G(9) \le 51,$$

so 50 is not in it, and neither are the values $b_6 = 24$ and $b_8 = 42$ that [1] uses for the sixth and eighth powers: those are the conclusions of Parts II and III of the same series, [4] and [5]. Only $b_5 = 17$ and $b_7 = 33$ come from [3]. The ninth-power value 50 is Theorem 1.1 of Part IV [2], an improvement by one on the $G(9) \le 51$ that [2, p. 204] obtains "by collecting together the conclusions" of the three earlier papers. Nothing in [1] or here depends on the attribution — the number 50 is correct, and it is the number Question 9.2 uses — but a reader chasing the citation will not find 50 in [3]. Brüdern and Wooley [6] improve the bounds on $G(k)$ only for $k \ge 14$, and so leave the ninth- and tenth-power values of [2] untouched.

2. Notation

Throughout, k and j are positive integers and n ranges over non-negative integers.

Definition 2.1. For $k, j \ge 1$, say that n is $(\le j, k)$-*representable* if there are an $m \le j$ and positive integers $x_1, \dots, x_m$ with $n = x_1^k + \cdots + x_m^k$. Write

$$R_k(j) = \{\, n \ge 0 : n \text{ is } (\le j, k)\text{-representable} \,\}, \qquad E_k(j) = \{\, n \ge 1 : n \notin R_k(j) \,\}.$$

For $i \ge 0$ write $R_k(i, j)$ for the subset obtained by additionally requiring every base $x_1, \dots, x_m$ to satisfy $x_t \le i$.

The empty sum is allowed, so $0 \in R_k(j)$ for every j; this is why $E_k(j)$ is defined with $n \ge 1$. Since $1 = 1^k$ is a k-th power, an integer that is a sum of exactly m positive k-th powers with $m \le j$ is also a sum of exactly j of them whenever $n \ge j$, but the two conditions are genuinely different for small n, and it is the "at most" version that Question 9.2 asks about.

Remark 2.2 (a clash of conventions). [1] writes "(j, k)-representable" for *exactly* j positive k-th powers, and its sets $\mathbf{B}^k_j$, $\mathbf{B}^k$, $\overline{\mathbf{B}^k_j}$ index that condition. Its Question 9.2, by contrast, is stated with "at most". Our $E_k(j)$ is the "at most" set, matching the question; it is *not* the set $\mathbf{B}^9_{50}$, and the cardinality $|\mathbf{B}^9| = 3\,438\,463$ tabulated in [1] is not the size of any set $E_k(j)$ appearing here. We have kept a separate notation to make the difference impossible to overlook.

In these terms, the computations of [1] indicate that

$$\max E_5(17) = 87\,918, \qquad \max E_6(24) = 1\,414\,564,$$
$$\max E_7(33) = 9\,930\,770, \qquad \max E_8(42) = 858\,367\,748,$$

and each of these four assertions is a conjecture there rather than a theorem, because writing "max" presupposes that the set has no element above the stated value, which is the ineffective half. The finite half of each — that the listed value does lie in the exceptional set — is a theorem, and is reproduced in Section 7.

3. A pruned representation search

Everything below rests on one recursive function and two facts about it.

Definition 3.1. For $k \ge 1$ define $\mathrm{search}(k, i, r, b) \in \{\mathrm{true}, \mathrm{false}\}$ for $i, r, b \ge 0$ by

$$\mathrm{search}(k, 0, r, b) = \mathrm{search}(k, i, r, 0) = [\, r = 0 \,],$$

and, for $i, b \ge 1$,

$$\mathrm{search}(k, i, r, b) = \begin{cases} \mathrm{true} & r = 0, \\ \mathrm{false} & r \ne 0 \text{ and } b \cdot i^k < r, \\ \big(i^k \le r \ \wedge\ \mathrm{search}(k, i, r - i^k, b - 1)\big) & \\ \qquad \vee\ \mathrm{search}(k, i - 1, r, b) & \text{otherwise.} \end{cases}$$

The two recursive calls decrease $b+i$, so the recursion terminates and both facts below are ordinary inductions on that single measure.

The middle clause is the *reachability prune.* It is licensed by the following elementary bound, which is where the whole method lives.

Lemma 3.2. *Let $x_1, \dots, x_m$ be positive integers with $x_t \le i$ for all t. Then $x_1^k + \cdots + x_m^k \le m \cdot i^k$.*

Proof. Induction on m, using $x^k \le i^k$ for $x \le i$. ☐

Proposition 3.3 (completeness)**.** *If $r \in R_k(i,b)$ then* $\mathrm{search}(k,i,r,b) = \mathit{true}$*. Equivalently, if* $\mathrm{search}(k,i,r,b) = \mathit{false}$ *then $r \notin R_k(i,b)$.*

Proof sketch. Induction on $b+i$. In the recursive case $i, b \ge 1$ and $r \ne 0$, let $x_1, \dots, x_m$ be a representation with $m \le b$ and all $x_t \le i$. The prune cannot fire, since $r = \sum x_t^k \le m\, i^k \le b\, i^k$ by Lemma 3.2. If the base i occurs in the list, delete one occurrence: the remainder is a representation of $r - i^k$ by at most $b-1$ powers with bases $\le i$, and $i^k \le r$, so the first disjunct is true by the inductive hypothesis at $(i, b-1)$. If i does not occur, every base is $\le i-1$ and the second disjunct is true by the inductive hypothesis at $(i-1, b)$. ☐

Proposition 3.4 (soundness)**.** *If* $\mathrm{search}(k,i,r,b) = \mathit{true}$ *then $r \in R_k(i,b)$, and the induction produces an explicit list of bases.*

Proof sketch. The mirror induction on $b+i$: the base cases return the empty list, the accepting branch prepends the base i to the list produced by the call at $(i, r-i^k, b-1)$, and the second disjunct passes its list through, weakening the base bound from $i-1$ to i. ☐

Lemma 3.5 (base cutoff)**.** *If $n < (i+1)^k$ then $n \in R_k(j)$ if and only if $n \in R_k(i,j)$.*

Proof. One direction is trivial. Conversely, any base x occurring in a representation of n satisfies $x^k \le n < (i+1)^k$, whence $x \le i$. ☐

Proposition 3.3 is the half the negative results need and Proposition 3.4 the half the positive results need; Lemma 3.5 is what makes a search over a bounded base range decide the unbounded question. All three are proved by hand, with no appeal to computation; only the individual evaluations of search in Sections 4–7 are computational.

4. The value 5 360 377 770

Theorem 4.1. 5 360 377 770 *is not a sum of at most* 50 *positive ninth powers:*

$$5\,360\,377\,770 \in E_9(50).$$

Proof. Since $13^9 = 10\,604\,499\,373 > 5\,360\,377\,770$, Lemma 3.5 with $i = 12$ reduces the claim to $5\,360\,377\,770 \notin R_9(12, 50)$. By Proposition 3.3 it therefore suffices to evaluate

$$\mathrm{search}(9, 12, 5\,360\,377\,770, 50) = \mathrm{false}.$$

This is an exhaustive computation: the recursion visits 435 471 nodes and returns false at every leaf. It is the only computational input to the theorem, and it is a *complete* search by Proposition 3.3, not a search that happened to find nothing. ☐

Theorem 4.2. 5 360 377 770 *is a sum of* 51 *positive ninth powers. Consequently its minimal ninth-power representation length is exactly* 51*: it lies in $R_9(51) \setminus R_9(50)$, and* 50 *is precisely the threshold at which representability fails.*

Proof. $\mathrm{search}(9, 12, 5\,360\,377\,770, 51) = \mathrm{true}$ (2 081 nodes), and Proposition 3.4 converts the accepting run into a representation. The representation it produces is

$$5\,360\,377\,770 = 12^9 + 8^9 + 5 \cdot 6^9 + 8 \cdot 5^9 + 4^9 + 5 \cdot 3^9 + 11 \cdot 2^9 + 19 \cdot 1^9,$$

with $1+1+5+8+1+5+11+19 = 51$ summands; the identity can of course be checked directly. Minimality is Theorem 4.1. ☐

For orientation: $12^9 = 5\,159\,780\,352$, so the value exceeds 12^9 by only about 3.9%, and any representation must use the base 12 sparingly — a single copy of 12^9 leaves a remainder of 200 597 418 to be covered by 50 ninth powers of bases at most 12, which is what the search spends its 435 471 nodes ruling out.

5. The window, and negative controls

Theorem 5.1. 5 360 377 770 *is the only element of* $E_9(50)$ *in the interval* $[5\,360\,367\,770,\ 5\,360\,387\,770]$*: every one of the other* 20 000 *integers in this window of* 20 001 *consecutive integers is a sum of at most* 50 *positive ninth powers.*

Proof. Two exhaustive sweeps, in the form "the list of $d < 10^4$ for which $\mathrm{search}(9, 12, \ell + d, 50)$ is false is empty", once with $\ell = 5\,360\,367\,770$ and once with $\ell = 5\,360\,377\,771$. The two sweeps visit about $9.90 \cdot 10^7$ and $1.01 \cdot 10^8$ nodes. Given a sweep, an integer n in the corresponding half of the window is located as $n = \ell + (n - \ell)$ with $n - \ell < 10^4$, so $\mathrm{search}(9, 12, n, 50) = \mathrm{true}$, and Proposition 3.4 converts this into an actual representation of n by at most 50 positive ninth powers. Every neighbour therefore carries a witness, not merely a Boolean. □

Theorem 5.1 is simultaneously the strongest control in this paper: the same code that reports 5 360 377 770 as exceptional is run on 20 000 representable integers of the same magnitude and reports none of them as exceptional. The remaining controls are collected next. They are routine given Propositions 3.3–3.4, and are recorded because a computational claim without them can be satisfied by a misplaced quantifier.

Proposition 5.2.
(1) $5\,360\,377\,769 \in R_9(50)$ *and* $5\,360\,377\,771 \in R_9(50)$*: both immediate neighbours of the value are representable, so the answer is not off by one in either direction and the search does not simply refuse every integer of this size.*
(2) $5\,360\,377\,771 \notin E_9(50)$*; hence no claim of the form* $\max E_9(50) = 5\,360\,377\,771$ *can hold.*
(3) *For every* $m < 5\,360\,377\,770$, m *is not the greatest element of* $E_9(50)$, *because* 5 360 377 770 *itself belongs to* $E_9(50)$ *and exceeds* m.
(4) $E_9(50) \neq \emptyset$ *for small arguments as well:* $511 \in E_9(50)$, *since* $511 < 2^9 = 512$ *forces every base to be* 1, *and* 511 *ones exceed the budget* 50.

Proof. (1) $\mathrm{search}(9, 12, 5\,360\,377\,769, 50) = \mathrm{search}(9, 12, 5\,360\,377\,771, 50) = \mathrm{true}$ (1 873 and 5 973 nodes) with Proposition 3.4; explicitly,

$$5\,360\,377\,769 = 12^9 + 8^9 + 5 \cdot 6^9 + 8 \cdot 5^9 + 4^9 + 5 \cdot 3^9 + 11 \cdot 2^9 + 18 \cdot 1^9 \qquad \text{(50 terms)},$$

$$5\,360\,377\,771 = 12^9 + 3 \cdot 7^9 + 2 \cdot 6^9 + 30 \cdot 5^9 + 3 \cdot 4^9 + 2 \cdot 2^9 \qquad \text{(41 terms)}.$$

(2) is (1) restated. (3) is immediate from Theorem 4.1. (4) uses Lemma 3.5 with $i = 1$ and $\mathrm{search}(9, 1, 511, 50) = \mathrm{false}$. □

6. The conditional answer to Question 9.2

Question 9.2 asks for $\max E_9(50)$, and a maximum has two halves: the value must lie in the set, and it must dominate the set. The first half is Theorem 4.1 and is finite. The second half is an assertion about all larger integers, and no finite computation reaches it. We isolate exactly the missing ingredient.

Theorem 6.1. *Suppose that*

(H) *every integer* $n > 5\,360\,377\,770$ *is a sum of at most* 50 *positive ninth powers.*

Then 5 360 377 770 *is the greatest element of* $E_9(50)$*; that is,* 5 360 377 770 *is the largest positive integer that is not a sum of at most* 50 *positive ninth powers, which is the answer to Question 9.2 of* [1].

Proof. Membership is Theorem 4.1 together with $5\,360\,377\,770 > 0$. For the upper bound, let $n \in E_9(50)$ and suppose $n > 5\,360\,377\,770$. Then (H) gives $n \in R_9(50)$, contradicting $n \in E_9(50)$. Hence $n \leq 5\,360\,377\,770$ for every $n \in E_9(50)$. □

Hypothesis (H) is the effective form of $G(9) \le 50$, with 5 360 377 770 as the threshold. It is not proved here and is not claimed. What can be said about it is this.

- It is exactly the ingredient that [1] lacks in the four cases it does settle: their Conjectures 5.2, 6.2, 7.2 and 8.2 are conjectures because the analogous hypotheses for $k = 5, 6, 7, 8$ are also unavailable, even though those exponents are within computational reach in every other respect.
- Its conclusion is verified for the first 10^4 integers above the value by Theorem 5.1, so (H) is not refutable by any short numerical check, and it survives a much longer one (Remark 6.2).
- Proving it would require an effective version of the Vaughan–Wooley circle-method argument. We are aware of no such version for any exponent in the range $5 \le k \le 9$.

Remark 6.2 (a computation outside the formal development). The claims in this remark were made by ordinary compiled programs and are *not* part of the machine-checked development; they are reported for context only. A bitset dynamic program computing $R_9(j) \cap [0, N]$ by 50 rounds of shift-and-or, run at $N = 1.2 \cdot 10^{10}$ and again at $N = 3 \cdot 10^{10}$, finds 5 360 377 770 as the largest element of $E_9(50) \cap [1, N]$ in both runs; the second run gives a factor 5.6 of headroom above the value. The same runs count exactly

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elements of $E_9(50)$ below $3 \cdot 10^{10}$, agreeing across two runs that used different lists of ninth powers. Since no element exceeds 5 360 377 770, this is a density of about 3.99% in [1, 5 360 377 770]. We know of no published value for this count; [1] tabulates $|\mathbf{B}^9| = 3\,438\,463$, but that counts a different set (see Remark 2.2). Formalizing the count is out of reach for the method of Section 3: it would mean running the search on $5.36 \cdot 10^9$ integers, some $5 \cdot 10^{13}$ nodes.

7. The four published values, reproduced

The values 87 918, 1 414 564, 9 930 770 and 858 367 748 are due to Benfield and Lippard [1]; the contribution here is only that the same search, with the same two correctness lemmas, certifies them. They served as the acceptance test for the pipeline: all four were reproduced, in both directions, before the ninth-power search was run at all.

Theorem 7.1.

$$\begin{array}{ll}
87\,918 \notin R_5(17) & 87\,918 \in R_5(18) \\
1\,414\,564 \notin R_6(24) & 1\,414\,564 \in R_6(25) \\
9\,930\,770 \notin R_7(33) & 9\,930\,770 \in R_7(34) \\
858\,367\,748 \notin R_8(42) & 858\,367\,748 \in R_8(43)
\end{array}$$

In words: none of the four values is a sum of at most b_k positive k-th powers for its own exponent, and each is a sum of $b_k + 1$ of them.

Proof. Eight evaluations of search, each combined with Lemma 3.5 and Proposition 3.3 (negative direction) or Proposition 3.4 (positive direction). The base cutoffs are $i = 9$ for $k = 5$ ($87\,918 < 10^5$), $i = 10$ for $k = 6$ ($1\,414\,564 < 11^6$), $i = 9$ for $k = 7$ ($9\,930\,770 < 10^7$) and $i = 13$ for $k = 8$ ($858\,367\,748 < 14^8$). The four refutations are exhaustive searches of 2 835, 14 917, 21 145 and 746 616 nodes; the four positive searches close in 27, 2 763, 7 452 and 56 707 nodes. The representations produced are

$$87\,918 = 9^5 + 7^5 + 6^5 + 5^5 + 4^5 + 4 \cdot 2^5 + 9 \cdot 1^5 \qquad \text{(18 terms)},$$

$$1\,414\,564 = 10^6 + 8 \cdot 6^6 + 5^6 + 6 \cdot 4^6 + 3^6 + 6 \cdot 2^6 + 2 \cdot 1^6 \qquad \text{(25 terms)},$$

$$9\,930\,770 = 9^7 + 4 \cdot 7^7 + 6 \cdot 6^7 + 2 \cdot 5^7 + 8 \cdot 3^7 + 2 \cdot 2^7 + 11 \cdot 1^7 \qquad \text{(34 terms)},$$

$$858\,367\,748 = 12^8 + 11^8 + 4 \cdot 9^8 + 8^8 + 3 \cdot 7^8 + 6^8 + 15 \cdot 5^8 + 3 \cdot 4^8 + 5 \cdot 3^8 + 2^8 + 8 \cdot 1^8 \qquad \text{(43 terms)}. \square$$

Remark 7.2 (one word in Conjecture 7.2). Conjectures 5.2, 6.2 and 8.2 of [1] all say "at most": Conjecture 5.2, for instance, reads "Every positive integer greater than 87918 is the sum of at most

17 positive fifth powers". Conjecture 7.2 instead reads "Every positive integer greater than 9930770 is the sum of 33 positive seventh powers", without those two words. We read the omission as a slip and treat the four statements uniformly. It changes nothing above: what Theorem 7.1 asserts is the "at most" statement in all four cases, and what it takes from [1] is only the four numbers themselves.

Remark 7.3. Theorems 4.2 and 7.1 together exhibit a uniform pattern across all five exponents: for $k = 5, 6, 7, 8, 9$ the value M_k is a sum of $b_k + 1$ positive k-th powers but not of b_k — one more power always suffices, never two. In each case the accepting search is a shallow one: the $(b_k + 1)$-power representation is found in far fewer nodes than the b_k-power refutation costs, because the extra unit of budget is spent on a single 1^k. We do not know a reason for the pattern beyond this observation, and [1] does not record it.

8. The extent of the computation

We state precisely which claims rest on exhaustive computation and how large each search was.

- Theorem 4.1: one search, search$(9, 12, 5\,360\,377\,770, 50)$, 435 471 nodes, returning false. Allowing the base 13 as well changes nothing (435 472 nodes, also false), as Lemma 3.5 predicts.
- Theorem 4.2: one search, 2 081 nodes.
- Theorem 5.1: two sweeps of 10^4 integers each, together about $2.0 \cdot 10^8$ nodes.
- Theorem 7.1: eight searches, together about $8.5 \cdot 10^5$ nodes.
- Proposition 5.2: three searches, together about $8 \cdot 10^3$ nodes.
- Theorem 6.1 involves no computation beyond that of Theorem 4.1.

Every one of these is a complete enumeration of the state space that Definition 3.1 describes, and the reachability prune is the only pruning used; by Proposition 3.3 it removes no representation. Outside the formal development, and only as corroboration, the value was produced twice more: by the bitset dynamic program of Remark 6.2 at two different bounds, and by a separate compiled recursive search of the same shape as Definition 3.1. The two programs agree on all five values $M_5, \ldots, M_9$ and on the two neighbours of 5 360 377 770. Independently, the interval $[5\,360\,277\,770, 5\,360\,377\,770]$ of 100 001 integers was swept by the compiled search, which found 5 360 377 770 to be the only exception in it — a tenfold widening of Theorem 5.1 on one side, outside the kernel.

The tenth-power analogue, $\max E_{10}(59)$ with $G(10) \le 59$ from [2, Theorem 1.1], is untouched by these methods: a dynamic program over $[0, N]$ needs $N/4$ bytes, and a probe at $N = 3 \cdot 10^{10}$ is inconclusive by a wide margin — the largest exception it finds is 29 999 999 916, the top of its own range, and 27% of the integers below $3 \cdot 10^{10}$ are still exceptional there, against 3.99% at the ninth-power answer. So $\max E_{10}(59) > 2.9 \cdot 10^{10}$, with no upper bound for it available to us. Each of the five known maxima lies between m^k and $(m+1)^k$ for some m with $9 \le m \le 13$; pushing the same computation as far as $13^{10} \approx 1.4 \cdot 10^{11}$ would already need about 35 GB.

9. Verification

Every statement above — Definitions 2.1 and 3.1, Lemmas 3.2 and 3.5, Propositions 3.3, 3.4 and 5.2, and Theorems 4.1, 4.2, 5.1, 6.1 and 7.1 — has been machine-checked in Lean 4 [8] against Mathlib [9]. The development is two files. The first contains no numerical data at all: it defines the representation predicates and the search of Definition 3.1, and proves Lemma 3.2, Proposition 3.3, Proposition 3.4 and Lemma 3.5 as inductions on the measure $b + i$; the axiom closure of each of these is the standard one (propositional extensionality, the axiom of choice, and the soundness of quotient types), with no appeal to computation. The second file contains the theorems, each of which combines those lemmas with one evaluation of search performed by compiled native code, so that each such theorem carries in addition a single native-evaluation axiom recording that evaluation (Theorem 5.1, which uses two sweeps, carries two). The separation is deliberate: the reasoning that converts a Boolean into a statement about integers is kernel-checked, and the trusted computational input is confined to the finite searches listed in Section 8. There is no `sorry` anywhere in the development, which compiles in about five and a half minutes; the whole cost of the work reported here, the formal development and the corroborating computations of Section 8 together, is under one CPU-hour. Repository reference to be added at submission.

References

[1] B. Benfield and O. Lippard, *Integers that are not the sum of positive powers*, arXiv:2404.08193 [math.NT]. Cited here in version v2 (31 March 2025), the current version at the time of writing; §9 (*Positive Ninth Powers*), Lemma 9.1 and Question 9.2.

[2] R. C. Vaughan and T. D. Wooley, *Further improvements in Waring's problem, IV: Higher powers*, Acta Arith. **94** (2000), 203–285. Theorem 1.1 gives the table of upper bounds $G(k) \leq H(k)$ for $9 \leq k \leq 20$, beginning $H(9) = 50$, $H(10) = 59$.

[3] R. C. Vaughan and T. D. Wooley, *Further improvements in Waring's problem*, Acta Math. **174** (1995), 147–240. Part I of the series; its Theorem 1.1 gives $G(5) \leq 17$, $G(6) \leq 25$, $G(7) \leq 33$, $G(8) \leq 43$, $G(9) \leq 51$. This is the paper [1] cites for $G(9) \leq 50$; see Remark 1.1.

[4] R. C. Vaughan and T. D. Wooley, *Further improvements in Waring's problem, II: Sixth powers*, Duke Math. J. **76** (1994), 683–710. The source of $b_6 = 24$.

[5] R. C. Vaughan and T. D. Wooley, *Further improvements in Waring's problem, III: Eighth powers*, Philos. Trans. Roy. Soc. London Ser. A **345** (1993), 385–396. The source of $b_8 = 42$.

[6] J. Brüdern and T. D. Wooley, *On Waring's problem for larger powers*, J. Reine Angew. Math. **805** (2023), 115–142; arXiv:2211.10380. Improves the bounds on $G(k)$ only for $k \geq 14$.

[7] A. A. Zenkin, *The generalized Waring problem: a new property of positive integers*, Math. Notes **58** (1995), no. 3, 933–937; translated from Mat. Zametki **58** (1995), no. 3, 372–378. The programme that [1] continues.

[8] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, in: Automated Deduction — CADE-28, Lecture Notes in Computer Science, Springer, 2021.

[9] The mathlib Community, *The Lean mathematical library*, in: Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020), ACM, 2020.