THE ZERO TRUNCATED PRODUCT ON P_K IS HYPERBOLIC EXACTLY WHEN THE MEAN IS NOT A GAUSS NODE: THE '?' CELLS OF THE ASSOCIATIVE-TRUNCATED-PRODUCT TABLE

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Abstract. Meghaichi and Xing (arXiv:2606.12632) discretise nonlinear conservation laws with uncertainty by replacing the multiplication on the space P_K of polynomials of degree at most K in the random variable by an *associative truncated product* (ATP), and they tabulate five such products against four properties; three cells of their table are printed '?'. We observe that every ATP on P_K is the quotient product $p * q = pq \bmod \omega$ for a unique monic ω of degree $K+1$, with $\omega = \phi_1\phi_K - \phi_1 * \phi_K$ up to a scalar, so that hyperbolicity, symmetry and positive definiteness become statements about the roots of one polynomial. For the *zero product*, $\omega = \pi_1\pi_K$, the two '?' cells are settled exactly: it is hyperbolic if and only if $\pi_K(\mu) \neq 0$, that is, iff the mean μ of the weight is not a node of its K-point Gauss rule — a repeated root of ω always produces a Jordan chain — and it is positive definite whenever μ and the Gauss nodes lie in Ξ, whether or not it is hyperbolic, because the eigenvalues of every multiplication operator are the values of the multiplier at the roots of ω with no diagonalisability assumption. Consequently $K = 1$ is never hyperbolic and $K = 2$ always ($\pi_2(\mu) = -\operatorname{Var} w$), for a symmetric weight hyperbolicity is equivalent to K even, and the two cells are independent: the Legendre weight at $K = 3$ is defective but positive definite, while the uniform weight on $[-2,-1] \cup [1,2]$ at $K = 2$ and an explicit four-point weight at $K = 3$ (an odd K that is hyperbolic) are hyperbolic but not positive definite. The third '?' cell follows from the same dictionary and a lemma of the source: the collocation product at $K+1$ distinct nodes is symmetric iff the nodes are the zeros of $\pi_{K+1} + t\,\pi_K$ for some real t. The two criteria, the cells and the controls are verified in Lean 4 against *Mathlib*, without compiled evaluation.

1. Introduction

Let w be a probability density on a set $\Xi \subset \mathbb{R}$, let $\{\phi_k\}_{k\geq 0}$ be the orthonormal polynomials of w with $\phi_0 \equiv 1$, and let P_K be the polynomials of degree at most K. An intrusive stochastic Galerkin (SG) discretisation of a nonlinear conservation law $u_t + f(u)_x = 0$ with a random parameter $\xi \sim w$ expands u in the ϕ_k and needs a *discrete multiplication* on P_K to evaluate f. The standard choice, the pseudospectral (Galerkin) product $\Pi_K(pq)$, is not associative, and this is tied to the well-known loss of hyperbolicity of the SG system: the blocks of the flux Jacobian do not commute. Meghaichi and Xing [1] propose instead to use an *associative* truncated product on the same space P_K and develop the SG method on that basis. In one stochastic dimension they show that an ATP is determined by the single element $\phi_1 * \phi_K$ (their Theorem 3.5), that a symmetric ATP is determined by one scalar (Lemma 3.7), and that if $\phi_1\phi_K - \phi_1 * \phi_K$ has $K+1$ distinct real roots then the product is uniformly hyperbolic, and is interpolation at those roots when they lie in Ξ (Theorem 3.9). Their Section 3.3 discusses four further products — zero, L^∞, collocation and their associative symmetric (AS) product — and their Table 3.1 records, for these and the pseudospectral product, whether each is associative, hyperbolic, symmetric and positive definite. Three cells are printed '?', with the legend

> "Here, the symbol '?' indicates that the property may or may not hold, depending on the distribution w and the degree of the polynomials K."

Two of the three are the Hyperbolic and the Positive-definite cells of the *zero product* $*_0$, the ATP defined by $\phi_1 * \phi_K = 0$; the third is the Symmetric cell of the collocation product. About the zero product the source says only

> "Moreover, by Lemma 3.8(d), the eigenvalues of M^1 are of the form $\phi_1(\zeta)$, where ζ is a root of $\phi_1\phi_K$. However, this does not imply that it is hyperbolic since $\phi_1\phi_K$ may have

> double roots, as is the case with Legendre polynomials for odd values of K, and it is possible that some of these eigenvalues are defective."

(Statement numbers throughout are those of the arXiv PDF of version 3, from which both quotations above were checked verbatim.)

This paper settles the two cells of the zero-product row exactly and derives the third cell from the same mechanism. The mechanism is a change of viewpoint that the source does not take. Write π_k for the monic orthogonal polynomial of degree k and μ for the mean of w, so that $\pi_1 = x - \mu$ and the roots of π_K are the nodes of the K-point Gauss rule of w.

- **The dictionary** (Proposition 3.1). Every ATP on P_K is $p * q = (pq) \bmod \omega$ for a unique monic ω of degree $K+1$, and $\omega = \phi_1\phi_K - \phi_1 * \phi_K$ up to a nonzero scalar. So $(P_K, *)$ is the algebra $\mathbb{R}[x]/(\omega)$, and the columns of the table are properties of the roots of ω: hyperbolic iff ω has $K+1$ distinct real roots, symmetric iff $\omega \perp P_{K-1}$, positive definite when ω splits over $\mathbb{R}$ with all roots in Ξ. The zero product is $\omega = \pi_1\pi_K$.
- **Cell 1, hyperbolicity** (Theorems 4.1, 4.2 and Corollary 4.3). The zero product on P_K is hyperbolic if and only if $\pi_K(\mu) \neq 0$, i.e. iff the mean of w is not a Gauss node of w. The new half is the negative one: a repeated root of ω does not merely defeat the source's sufficient condition, it always destroys hyperbolicity, because multiplication by $x - c$ then has the Jordan chain $\omega/(x-c)^2 \mapsto \omega/(x-c) \mapsto 0$. Hence $K = 1$ is never hyperbolic and $K = 2$ always is, for every weight (Theorems 4.4 and 4.5; the mechanism is $\pi_2(\mu) = -\operatorname{Var} w < 0$); for a symmetric weight the criterion reads "K even" (Remark 4.6), which contains the source's Legendre remark; but the parity rule is not a general law (Theorem 6.7).
- **Cell 2, positive definiteness** (Theorems 5.1 and 5.2). For every $p \in P_K$ the eigenvalues of the multiplication operator $q \mapsto p * q$ are values of p at roots of ω, *with no diagonalisability assumption.* So the zero product is positive definite as soon as the mean and the Gauss nodes lie in Ξ — automatic when Ξ is an interval containing the support — and it fails as soon as a root leaves Ξ. The source's Lemma 3.8(c) gives the positive conclusion under the hypothesis that M^1 is diagonalisable, so it cannot reach the cells where the '?' is interesting.
- **Independence of the two cells** (Section 6). The Legendre weight at $K = 3$ is not hyperbolic but positive definite on $[-1, 1]$ (Theorems 6.2 and 6.4); the uniform weight on $[-2, -1] \cup [1, 2]$ at $K = 2$ is hyperbolic but not positive definite (Theorem 6.5); and an explicit four-point weight at $K = 3$ is hyperbolic — an odd K — and not positive definite (Theorem 6.7).
- **Cell 3, symmetry of the collocation product** (Corollary 7.1). The collocation product at $K+1$ distinct nodes is symmetric iff the nodes are the zeros of $\pi_{K+1} + t\,\pi_K$ for some real t, the quasi-orthogonal (Gauss and Gauss–Radau-type) node families; every symmetric ATP is such a collocation product, and $t = 0$ is the source's AS product.

What is new and what is not. We are careful about attribution, since the objects are elementary. The '$\Leftarrow$' directions of the dictionary are the source's own results: $K+1$ distinct real roots give hyperbolicity and, for roots in Ξ, interpolation (Theorem 3.9 of [1]), roots in Ξ give positive eigenvalues under diagonalisability (Lemma 3.8(c)), and a root of $\phi_1 * \phi_K - \phi_1\phi_K$ gives an eigenvalue of M^1 (Lemma 3.8(d)); and "a symmetric ATP is determined by one scalar" is its Lemma 3.7. That an associative truncated product is the same thing as a commuting family of structure matrices is not new either: the source states that equivalence itself, in its Definition 3.2, and the Gerster–Herty line reads the non-associativity of the Galerkin product off the non-commuting matrices [11, 10], crediting the test "all M^k commute" to [9]; the identity $\mathsf{P}(\mathsf{P}(\hat{p})\hat{q}) = \mathsf{P}(\hat{p})\mathsf{P}(\hat{q})$ is proved in [10] under the hypothesis that P has constant eigenvectors, i.e. exactly in the uniformly hyperbolic case. (What the source itself credits is [11, 12] for uniformly hyperbolic products, "described as the case where $\mathsf{P}(\hat{p})$ has constant eigenvectors", and [9] for the $\mathcal{A}$gPC basis; it does not cite [10].) Once the dictionary is stated, its consequences are standard commutative algebra: the multiplication-by-x operator on $\mathbb{R}[x]/(\omega)$ is the companion matrix of ω, whose minimal polynomial is ω, and the eigenvalue statement is the univariate case of the eigenvalue (Stickelberger) theorem [3, Ch. 2, §4, Thm. (4.5)]. What we claim is: the identification itself, which the source does not make (it asserts "every choice of $\phi_1 * \phi_K$ corresponds to a unique ATP” but proves only uniqueness); the *existence* half of that parametrisation; the converse directions — a repeated root destroys hyperbolicity, a root outside Ξ destroys positive definiteness — and the eigenvalue statement without diagonalisability, which are what the defective cells need; the exact answers for the zero product; and the examples. The third cell is a one-step corollary of the source's Lemma 3.7 and classical quadrature theory (Jacobi's theorem and Golub's Gauss–Radau construction, [5, Thms. 2.1 and 3.2] and [4]; the quasi-orthogonal polynomials of Riesz, Fejér and Shohat [7, 6]) and is presented as such. As of 2026-09-07 the three cells are still ‘?’ in the live version 3 of the source, we found no work citing it, and a full-text sweep of 890,835 arXiv e-prints found the phrase “associative truncated” in three papers (the source and two on star products and quantum groups) and “collocation product” in the source alone.

Verification. The two criteria, the identification of the zero product, the cells and the controls are formalised in Lean 4 against *Mathlib*; every theorem below whose statement is reproduced in Appendix A is machine-checked, and Section 9 records the modules, the declaration counts and the axioms. The dictionary (Proposition 3.1) and the third cell (Corollary 7.1) are proved on paper only; the formal results about the zero product do not depend on the dictionary but on the source's uniqueness theorem (Lemma 3.2).

2. Preliminaries

Throughout, w is a probability measure on $\Xi \subset \mathbb{R}$ with finite moments $m_k = \int_\Xi \xi^k\, w(\xi)\, d\xi$ and with at least $K+1$ points in its support, so that $\langle f, g\rangle_w = \int_\Xi f g\, w$ is an inner product on P_K, the real polynomials of degree at most K, and P_K is identified with the space $P_K(\Xi)$ of the source. The mean is $\mu = m_1$ and the variance $\operatorname{Var} w = m_2 - m_1^2 > 0$. We write ϕ_k for the orthonormal polynomials of w (positive leading coefficients, $\phi_0 \equiv 1$) and π_k for the monic ones; $\pi_1 = x - \mu$, and the K roots of π_K are real, simple, and lie in the interior of the convex hull of the support of w ([2, Thm. 3.3.1] when w has infinitely many points of increase; the same sign-change proof covers a w with finitely many, of which there are at least $K+1$ by the standing assumption); they are the nodes of the K-point Gauss rule of w, and we call them the *Gauss nodes*. The coefficient vector of $p = \sum_k \hat{p}_k \phi_k \in P_K$ is $\hat{p} \in \mathbb{R}^{K+1}$.

Definition 2.1 ([1], Definitions 3.1 and 3.2)**.** A map $* : P_K \times P_K \to P_K$ is a *truncated product* if it is bilinear, commutative, and *consistent*: $p * q = pq$ whenever $\deg(pq) \le K$. It is an *associative truncated product* (ATP) if moreover $(p*q)*r = p*(q*r)$. Its structure matrices are $\mathsf{M}^k_{i+1,j+1} = \langle \phi_i, \phi_k * \phi_j\rangle_w$, $0 \le i, j, k \le K$, and $\mathsf{P}(\hat{p}) = \sum_k \hat{p}_k \mathsf{M}^k$ is the matrix of the multiplication operator $\mathsf{P}(p) : q \mapsto p * q$ in the basis $\{\phi_k\}$. The product is *symmetric* if $\langle p, q*r\rangle_w = \langle p*q, r\rangle_w$ for all p, q, r (equivalently every M^k is symmetric); *hyperbolic* if $\mathsf{P}(\hat{p})$ is diagonalisable with real eigenvalues for every $\hat{p} \in \mathbb{R}^{K+1}$; and *positive definite* if every eigenvalue of $\mathsf{P}(\hat{p})$ is positive whenever $p > 0$ on Ξ.

Diagonalisability and eigenvalues are properties of the linear operator $\mathsf{P}(p)$, not of the basis, so we may and do work with the operator $q \mapsto p * q$ on P_K directly. Concretely, the formal development uses the following basis-free forms, which are the definitions of Appendix A: $*$ is hyperbolic when for every $p \in P_K$ there is a finite family of polynomials in P_K that spans P_K and consists of eigenvectors of $\mathsf{P}(p)$ with real eigenvalues; $\lambda \in \mathbb{R}$ is an eigenvalue of $\mathsf{P}(p)$ when $p * q = \lambda q$ for some $0 \ne q \in P_K$; and $*$ is positive definite on Ξ when every real eigenvalue of $\mathsf{P}(p)$ is positive for every $p \in P_K$ with $p > 0$ on Ξ. (When ω below splits over $\mathbb{R}$, every eigenvalue of $\mathsf{P}(p)$ is real by Theorem 5.1, so the restriction to real eigenvalues costs nothing in the positive-definite cells we decide.)

Definition 2.2. For a monic $\omega \in \mathbb{R}[x]$ of degree $K+1$, the *quotient product* on P_K is

$$p *_\omega q := (pq) \bmod \omega,$$

the remainder of pq on division by ω.

The quotient product is bilinear, commutative, associative, and consistent: if $\deg(pq) < \deg \omega$ then $(pq) \bmod \omega = pq$. (These are the lemmas `tmul_comm`, `tmul_assoc`, `tmul_eq_mul`, `tmul_add` and `tmul_smul` of the formal development.) So every monic ω of degree $K+1$ defines an ATP on P_K, namely the algebra $\mathbb{R}[x]/(\omega)$ with P_K as the space of canonical representatives. For a finite set $s \subset \mathbb{R}$ and $x \in s$ we write $\ell_x(X) = \prod_{y\in s,\, y\neq x}(X-y)/(x-y)$ for the Lagrange basis polynomial at x, and $\omega_s = \prod_{y\in s}(X-y)$ for the nodal polynomial of s.

3. The dictionary

Proposition 3.1 (the dictionary; proved on paper, not formalised). *Let $K \geq 1$ and let $*$ be an associative truncated product on P_K. Put*

$$\omega := x \cdot x^K - x * x^K,$$

a monic polynomial of degree $K+1$. Then $p*q = (pq) \bmod \omega$ for all $p, q \in P_K$, and ω is the unique monic polynomial of degree $K+1$ with this property. In terms of the source's datum, $\omega = c^{-1}(\phi_1\phi_K - \phi_1 * \phi_K)$ where $c > 0$ is the leading coefficient of $\phi_1\phi_K$; conversely, for every prescribed $\phi_1 * \phi_K \in P_K$ the quotient product modulo this ω is an ATP with that datum.

Proof. Consistency with $p = 1$ gives $1*q = q$, so $(P_K, *)$ is a commutative unital $\mathbb{R}$-algebra of dimension $K+1$. Consistency again gives $x^{*i} = x^i$ for $i \leq K$ (induction on i: $x * x^{i-1} = x^i$ as $\deg x^i \leq K$), so the algebra is generated by x; hence it is $\mathbb{R}[x]/I$ for an ideal I with $\dim \mathbb{R}[x]/I = K+1$, i.e. $I = (\omega')$ with ω' monic of degree $K+1$, and the isomorphism $\mathbb{R}[x]/(\omega') \to (P_K, *)$ is the identity on P_K because it fixes every x^i, $i \leq K$. Concretely, let $T : q \mapsto x * q$. Then $Tq = (xq) \bmod \omega$ for all $q \in P_K$: for $\deg q < K$ both sides equal xq by consistency, and for $q = x^K$ the right-hand side is $x^{K+1} - \omega = x * x^K$ by the definition of ω. Associativity gives $x^i * q = T^i q$, and bilinearity gives $p * q = p(T)q = (pq) \bmod \omega$; in particular $\omega' = \omega$. Uniqueness: two monic polynomials of degree $K+1$ with the same remainder map on P_K have the same value of $x^{K+1} \bmod \omega$, hence are equal. For the translation, $\phi_1 * \phi_K = (\phi_1\phi_K) \bmod \omega = \phi_1\phi_K - c\,\omega$ since $\deg(\phi_1\phi_K) = K+1$; and conversely, given any $r \in P_K$ as the value of $\phi_1 * \phi_K$, $\omega := c^{-1}(\phi_1\phi_K - r)$ is monic of degree $K+1$ and the quotient product modulo ω is an ATP (Definition 2.2) with $\phi_1 *_\omega \phi_K = \phi_1\phi_K - c\,\omega = r$. $\square$

Under the dictionary the four columns of the source's table are properties of ω alone: the source's products are $\omega = \pi_1\pi_K$ (zero product), $\omega = \prod_\ell (x - \xi_\ell)$ (collocation at the nodes ξ_ℓ), $\omega = \pi_{K+1}$ (AS product), and, for the L^∞ product on a compact Ξ, ω proportional to the equioscillating error of the best approximation of $\phi_1\phi_K$ from P_K; and

column	criterion on ω
associative	always
hyperbolic	ω has $K+1$ distinct real roots (Theorem 4.1)
symmetric	$\omega \perp P_{K-1}$ in L^2_w (Corollary 7.1)
positive definite	$\Leftarrow \omega$ splits over $\mathbb{R}$ with all roots in Ξ (Theorems 5.1, 5.2)

The remainder of the paper proves these criteria and applies them to the zero product. Since Proposition 3.1 is not formalised, we record separately the identification the formal results actually rest on.

Lemma 3.2 (identification of the zero product). *Let $\mu \in \mathbb{R}$ and let ϕ_K be monic. In the quotient product modulo $\omega = (x-\mu)\phi_K$ one has $(x-\mu) *_\omega \phi_K = 0$. Consequently, when μ is the mean of w and $\phi_K = \pi_K$, the quotient product modulo $\pi_1\pi_K$ is an ATP with $\phi_1 * \phi_K = 0$, and by Theorem 3.5 of* [1] *(an ATP on $P_K(\Xi)$ is uniquely determined by $\phi_1 * \phi_K$) it is* the *zero product $*_0$.*

Proof. $(x-\mu)\phi_K = \omega$ and $\omega \bmod \omega = 0$ (formal: `tmul_zero_product`). The rest is bilinearity (ϕ_1, ϕ_K are positive multiples of $x-\mu$, π_K) and the source's uniqueness theorem. $\square$

We therefore write $\omega_0 = \pi_1\pi_K = (x-\mu)\pi_K$ for the polynomial of the zero product on P_K; it is the nodal polynomial of the K Gauss nodes together with the mean.

4. The Hyperbolic cell

Theorem 4.1 (hyperbolicity of a quotient product). *Let ω be monic of degree $K+1$.*

(i) *If $(x-c)^2$ divides ω for some $c \in \mathbb{R}$, then $*_\omega$ is not hyperbolic.*

(ii) *If $\omega = \omega_s$ is the nodal polynomial of a set s of $K+1$ distinct real numbers, then $*_\omega$ is hyperbolic: for every $p \in P_K$ and every $x \in s$, $p *_\omega \ell_x = p(x)\,\ell_x$, and the Lagrange basis $\{\ell_x\}_{x\in s}$ diagonalises every multiplication operator simultaneously.*

Hence $*_\omega$ is hyperbolic if and only if ω has $K+1$ distinct real roots.

Proof. (ii) is Theorem 3.9 of [1] in operator form, and we include it for completeness (formal: `hyperbolic_nodal`, `tmul_basis`). Fix $x \in s$. Then

$$p\,\ell_x - p(x)\,\ell_x = (p - p(x))\,\ell_x \quad \text{is divisible by} \quad (X - x)\cdot \omega_{s\setminus\{x\}} = \omega_s,$$

so $(p\ell_x) \bmod \omega_s = p(x)\ell_x$ since $\deg \ell_x = K < \deg \omega_s$; and the ℓ_x span P_K because every $q \in P_K$ is its own Lagrange interpolant, $q = \sum_{x\in s} q(x)\ell_x$.

(i) is the new half (formal: `not_hyperbolic_of_sq_dvd`). Suppose $*_\omega$ were hyperbolic and apply the definition to $p = x$ (note $\deg\omega \geq 2$): there are $b_i \in P_K$ and $d_i \in \mathbb{R}$ with $x * b_i = d_i b_i$ and $\sum_i c_i b_i = 1$ for some scalars c_i. An eigenvector of multiplication by x is an eigenvector of multiplication by any $g \in \mathbb{R}[x]$ with eigenvalue $g(d_i)$ (formal: `tmul_eval_of_eigen`; by induction on the degree, using associativity in the form $x * (x^k * b) = x^{k+1} * b$). Let $g = \prod_{y\in S}(X - y)$ over the finite set $S = \{d_i\}$ of eigenvalues, a squarefree monic polynomial. Then $g * b_i = g(d_i)b_i = 0$ for every i, hence $g * 1 = \sum_i c_i\,(g * b_i) = 0$, i.e. $g \bmod \omega = 0$, i.e. $\omega \mid g$. But $(x-c)^2 \mid \omega \mid g$ contradicts the squarefreeness of g. (Equivalently: $v = \omega/(x-c)^2$ satisfies $(x-c) * v = \omega/(x-c) \neq 0$ and $(x-c) * ((x-c) * v) = \omega \bmod \omega = 0$, a Jordan chain of length two for multiplication by $x - c$.)

The 'iff': if ω has $K+1$ distinct real roots it is their nodal polynomial, and (ii) applies; otherwise ω has a repeated root or a non-real root. In the first case (i) applies. In the second, the minimal polynomial of the operator $T : q \mapsto x *_\omega q$ is ω (T is the companion matrix of ω: $g(T)1 = g \bmod \omega$), so T has a non-real eigenvalue and $\mathsf{P}(x)$ is not diagonalisable over $\mathbb{R}$. The second case is not formalised; it is not needed for the zero product, whose ω_0 splits over $\mathbb{R}$. □

Theorem 4.2 (the Hyperbolic cell of the zero product)**.** *Let $\mu \in \mathbb{R}$ and let ϕ_K be a monic polynomial, and let $*$ be the quotient product modulo $\omega = (x-\mu)\phi_K$.*

(i) *If $\phi_K(\mu) = 0$, then $*$ is not hyperbolic.*
(ii) *If $\phi_K = \omega_t$ is the nodal polynomial of a finite set $t \subset \mathbb{R}$ and $\mu \notin t$, then $*$ is hyperbolic.*

Proof. (i) $\phi_K(\mu) = 0$ gives $(x-\mu) \mid \phi_K$, so $(x-\mu)^2 \mid \omega$, and Theorem 4.1(i) applies (formal: `zero_not_hyperbolic`). (ii) $(x-\mu)\,\omega_t = \omega_{t\cup\{\mu\}}$ when $\mu \notin t$, and Theorem 4.1(ii) applies (formal: `zero_hyperbolic`). □

Corollary 4.3. *The zero product on P_K is hyperbolic if and only if $\pi_K(\mu) \neq 0$, that is, if and only if the mean of w is not a node of the K-point Gauss rule of w.*

Proof. π_K has K distinct real roots [2, Thm. 3.3.1], so $\pi_K = \omega_t$ with t its root set, and $\pi_K(\mu) \neq 0$ iff $\mu \notin t$. Apply Theorem 4.2 with $\phi_K = \pi_K$ and Lemma 3.2. (The classical real-rootedness of π_K is the only step outside the formal development.) □

So the source's "it is possible that some of these eigenvalues are defective" is not a possibility but a certainty: whenever $\phi_1\phi_K$ has a double root, the zero product is not hyperbolic. Two consequences hold for every weight.

Theorem 4.4 ($K = 1$)**.** *For every $\mu \in \mathbb{R}$, the quotient product modulo $(x-\mu)^2$ is not hyperbolic. Hence the zero product on P_1 is not hyperbolic, for every weight w.*

Proof. Here $\phi_1 = \phi_K$, so $\omega_0 = \pi_1^2$; Theorem 4.2(i) with $\phi_K = x - \mu$ (formal: `zero_K1_not_hyperbolic`). □

Theorem 4.5 ($K = 2$)**.** (i) *Let $m : \mathbb{N} \to \mathbb{R}$ with $m_0 = 1$ and let $x^2 + ax + b$ be orthogonal to 1 for the moment form $\langle f, g\rangle_m = \sum_{i,j\leq 2} f_i g_j\, m_{i+j}$ (i.e. $m_2 + am_1 + b = 0$). Then $(x^2+ax+b)(m_1) = m_1^2 - m_2$.*

(ii) *If $r < \mu < s$, the quotient product modulo $(x-\mu)(x-r)(x-s)$ is hyperbolic.*

Consequently the zero product on P_2 is hyperbolic for every weight w (with $\operatorname{Var} w > 0$).

Proof. (i) is a two-line identity (formal: `eval_phi_two_eq`); (ii) is Theorem 4.1(ii) for the three distinct points μ, r, s (formal: `zero_K2_hyperbolic`). For the consequence: $\pi_2 = x^2 + ax + b$ satisfies $\langle \pi_2, 1\rangle_w = m_2 + am_1 + b = 0$, so by (i) $\pi_2(\mu) = m_1^2 - m_2 = -\operatorname{Var} w < 0$; a monic quadratic negative at μ has two real roots $r < \mu < s$, and (ii) applies. Only the orthogonality relation $\langle \pi_2, 1\rangle_w = 0$ is used, so this is not a symmetry argument: for the Laguerre weight e^{-x} on $[0, \infty)$, $\pi_2 = x^2 - 4x + 2$, $\mu = 1$ and $\pi_2(1) = -1 = -\operatorname{Var} w$. □

Remark 4.6 (symmetric weights). If w is symmetric about its mean, then $\pi_K(\mu+y) = (-1)^K \pi_K(\mu-y)$, so $\pi_K(\mu) = 0$ for odd K, while for even K the root μ would leave $K-1$ further roots, distinct and paired by the symmetry, an odd number — impossible. By Corollary 4.3, *for a symmetric weight the zero product on P_K is hyperbolic if and only if K is even*: this covers the Legendre, both Chebyshev, the Gegenbauer and the Hermite weights and contains the source's remark on Legendre polynomials for odd K. The parity rule is not a general law: Theorem 6.7 exhibits a weight with $K = 3$ hyperbolic, and Remark 4.7 lists Laguerre and a Jacobi weight, hyperbolic at every odd $K \geq 3$ up to 12. This remark is outside the formal development.

Remark 4.7 (the exact table; a computation outside the formal development). Corollary 4.3 makes each cell an exact rational evaluation of $\pi_K(\mu)$ from the moments of w (monic orthogonal polynomials by the Stieltjes recurrence in exact rational arithmetic). We reproduced the following table from the founding record of this work; ✓ means hyperbolic, × means $\pi_K(\mu) = 0$ (not hyperbolic).

weight	μ	K=1	2	3	4	5	6	7	8	9	10	11	12
Legendre (uniform on $[-1, 1]$)	0	×	✓	×	✓	×	✓	×	✓	×	✓	×	✓
Chebyshev, first kind	0	×	✓	×	✓	×	✓	×	✓	×	✓	×	✓
Chebyshev, second kind	0	×	✓	×	✓	×	✓	×	✓	×	✓	×	✓
Hermite (standard Gaussian)	0	×	✓	×	✓	×	✓	×	✓	×	✓	×	✓
Laguerre (e^{-x} on $[0, \infty)$)	1	×	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
Jacobi $(1 - x)/2$ on $[-1, 1]$	$-1/3$	×	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
uniform on $[-2, -1] \cup [1, 2]$	0	×	✓	×	✓	×	✓	×	✓	×	✓	×	✓

For the record, $\pi_2(\mu)$ equals $-1/3, -1/2, -1/4, -1, -1, -2/9, -7/3$ in the seven rows, each equal to $-\operatorname{Var} w$ as Theorem 4.5 predicts, and the Laguerre cubic is $\pi_3 = x^3 - 9x^2 + 18x - 6$ with $\pi_3(1) = 4$. Of these cells only Legendre $K = 2, 3, 5$ and the two-gap weight at $K = 2$ are formalised (Section 6); certifying the hyperbolic Legendre cells $K \geq 4$ or the Laguerre cells $K \geq 3$ formally would need the Gauss nodes as real numbers or a real-rootedness certificate for π_K.

5. The Positive-definite cell

Theorem 5.1 (eigenvalues are values at the roots)**.** *Let $\omega = \prod_{x \in R}(X - x)$ for a finite multiset R of real numbers (repetitions allowed), and let $\Xi \subset \mathbb{R}$ contain every element of R. Then $*_\omega$ is positive definite on Ξ. The mechanism, one step inside the proof: if $p *_\omega q = \lambda q$ with $0 \neq q \in P_K$ and $\lambda \in \mathbb{R}$, then $\lambda = p(x)$ for some $x \in R$.*

Proof. (Formal: `posDef_of_roots_mem`; the eigenvalue step is `exists_root_of_dvd_mul` used inside its proof, and is not a separate Lean statement.) From $p * q = \lambda q$ and $\deg q < \deg \omega$ we get $\omega \mid (p - \lambda)q$ and $\omega \nmid q$. Induct on the multiset R, peeling one linear factor at a time: if $R = \{a\} \uplus R'$, then $(X - a) \mid (p - \lambda)q$, so either $(p - \lambda)(a) = 0$ — done with $x = a$ — or $q(a) = 0$, $q = (X - a)q'$, and cancelling $X - a$ gives $\omega' \mid (p - \lambda)q'$ with $\omega' \nmid q'$ for $\omega' = \prod_{x \in R'}(X - x)$; apply the induction hypothesis. If $p > 0$ on $\Xi \supseteq R$ then $\lambda = p(x) > 0$. Nothing is diagonalised, so the argument is indifferent to repeated roots; this is what distinguishes it from Lemma 3.8(c) of [1], whose hypothesis is that M^1 be diagonalisable. □

Theorem 5.2 (a root outside Ξ)**.** *Let $s \subset \mathbb{R}$ be a finite set, $x \in s$, and let $p \in P_{|s|-1}$ be positive on Ξ with $p(x) \leq 0$. Then $*_{\omega_s}$ is not positive definite on Ξ.*

Proof. $\ell_x \neq 0$ is an eigenvector of $\mathsf{P}(p)$ with eigenvalue $p(x) \leq 0$ (Theorem 4.1(ii); formal: `not_posDef_of_eval_nonpos`). □

Corollary 5.3 (the Positive-definite cell of the zero product)**.** *The zero product on P_K is positive definite on Ξ whenever the mean μ and the K Gauss nodes of w lie in Ξ; in particular whenever Ξ is an interval containing the support of w, hyperbolic or not. If $\pi_K(\mu) \neq 0$ and some root of $\omega_0 = (x-\mu)\pi_K$ is a point where a polynomial of degree $\leq K$, positive on Ξ, is nonpositive — for a closed Ξ and $K \geq 2$, any root outside Ξ, via $p = (x-\xi)^2 - \varepsilon$ — then it is not positive definite on Ξ.*

Proof. Theorem 5.1 with R the roots of ω_0 (all real: μ and the Gauss nodes), and Theorem 5.2 with s the root set of ω_0; the location of the Gauss nodes inside the convex hull of the support is classical [2, Thm. 3.3.1], and the mean lies there because it is an average. The converse in the repeated-root case is not formalised; it holds by the same eigenvector, $\omega_0/(x-\xi)$, which satisfies $p * (\omega_0/(x-\xi)) = p(\xi)\,\omega_0/(x-\xi)$ for every $p \in P_K$. □

So the second ‘?’ is ✓ for every weight supported on an interval Ξ, the defective cells included, and it is genuinely a ‘?’ only for disconnected or discrete Ξ. In particular the two ‘?’ columns of the zero-product row are logically independent, and Section 6 realises both directions of the independence.

6. The cells

6.1. **The Legendre weight.** Let w be the uniform weight on $[-1,1]$, with moments $m_k = 1/(k+1)$ for even k and $m_k = 0$ for odd k, mean 0, and monic Legendre polynomials

$$\pi_2 = x^2 - \tfrac{1}{3}, \qquad \pi_3 = x^3 - \tfrac{3}{5}x, \qquad \pi_5 = x^5 - \tfrac{10}{9}x^3 + \tfrac{5}{21}x.$$

These are textbook values [2]; what the formal development needs is the certificate that they are the orthogonal polynomials of *this* weight, through the moment form $\langle f,g\rangle_m = \sum_{i,j<N} f_i g_j\, m_{i+j}$ (which equals $\langle f,g\rangle_w$ for $\deg f, \deg g < N$).

Proposition 6.1 (Legendre certificates)**.** *With m the Legendre moments, $\pi_2 \perp \{1,x\}$ (with $N=3$), $\pi_3 \perp \{1,x,x^2\}$ ($N=4$) and $\pi_5 \perp \{1,x,\ldots,x^4\}$ ($N=6$) for the moment form.*

Proof. Finite rational identities (formal: `legPi2_orth`, `legPi3_orth`, `legPi5_orth`, discharged by `norm_num`). □

Theorem 6.2 (Legendre, $K=3$ and $K=5$: not hyperbolic)**.** *The quotient products modulo $x\,\pi_3$ and modulo $x\,\pi_5$ are not hyperbolic. Hence the zero product of the Legendre weight on P_3 and on P_5 is not hyperbolic.*

Proof. $\pi_3(0) = \pi_5(0) = 0$ and Theorem 4.2(i) (formal: `legendre_three_not_hyperbolic`, `legendre_five_not_hyperbolic`). This is the configuration the source names as the place where its criterion fails; the new content is the conclusion. □

Proposition 6.3 (Legendre, $K=2$: hyperbolic)**.** *The quotient product modulo $x\,(x+1/\sqrt{3})(x-1/\sqrt{3})$ is hyperbolic, and $(x+1/\sqrt{3})(x-1/\sqrt{3}) = \pi_2$. Hence the zero product of the Legendre weight on P_2 is hyperbolic, with nodes $0, \pm 1/\sqrt{3}$ — the two-point Gauss–Legendre nodes together with the mean.*

Proof. Theorem 4.5(ii) with $r = -1/\sqrt{3} < 0 < s = 1/\sqrt{3}$ (formal: `legendre_two_hyperbolic`, `legPi2_factor`). An instance of Theorem 4.5 with classical data. □

Theorem 6.4 (Legendre, $K=3$: defective but positive definite)**.** *The quotient product modulo $x\,\pi_3 = x^2(x^2 - 3/5)$ is positive definite on $\Xi = [-1,1]$. Together with Theorem 6.2: the zero product of the Legendre weight on P_3 is not hyperbolic and is positive definite on $[-1,1]$.*

Proof. $\omega_0 = \prod_{x\in R}(X-x)$ with $R = \{0, 0, \sqrt{3/5}, -\sqrt{3/5}\} \subset [-1,1]$; Theorem 5.1 (formal: `legendre_three_posDef`). □

6.2. **A disconnected support.** Let w be the uniform weight on $\Xi = [-2,-1] \cup [1,2]$, with moments $m_k = (2^{k+1}-1)/(k+1)$ for even k and 0 for odd k, mean 0, and $\pi_2 = x^2 - 7/3$.

Theorem 6.5 (two-gap weight, $K = 2$: hyperbolic but not positive definite)**.** *$\pi_2 = x^2 - 7/3$ is orthogonal to 1 and x for the moment form of w ($N = 3$), and the quotient product modulo $x\,\pi_2$ is not positive definite on $[-2,-1] \cup [1,2]$. Since $x\,\pi_2 = x(x^2 - 7/3)$ has the three distinct real roots $0, \pm\sqrt{7/3}$, the zero product of w on P_2 is hyperbolic (Theorem 4.5(ii)) and not positive definite.*

Proof. The certificate is a rational identity (formal: `twogapPi2_orth`). Hyperbolicity is Theorem 4.5(ii) with $r = -\sqrt{7/3} < 0 < s = \sqrt{7/3}$; the factorisation $x^2 - 7/3 = (x-r)(x-s)$ that feeds it is a paper step (its Legendre analogue `legPi2_factor` is formal), so the two Lean declarations for this weight are the orthogonality certificate and the failure of positive definiteness. The mean 0 is a root of ω_0 and lies in the gap; $p = x^2 - 1/2$ is positive on Ξ (where $x^2 \geq 1$) and $p(0) = -1/2$, so Theorem 5.2 applies with $s = \{0, \pm\sqrt{7/3}\}$ (formal: `twogap_two_not_posDef`). This is the exact sense in which the source's legend "depending on the distribution w" is true of the Positive-definite cell. □

6.3. **A four-point weight with an odd K hyperbolic.** To show that the parity rule of Remark 4.6 is not a law we need a weight, exact over $\mathbb{Q}$, with $\pi_K(\mu) \neq 0$ at an odd K. Laguerre at $K = 3$ has $\pi_3(1) = 4 \neq 0$, but its Gauss nodes are the roots of $x^3 - 9x^2 + 18x - 6$ (an irreducible cubic with three real roots, not expressible by real radicals), so a formal certificate of hyperbolicity would need a real-rootedness argument. Instead we build a weight with *rational* Gauss nodes by inverting the classical formula for an atomic weight: for atoms $y_0, \ldots, y_K$ with masses $m_i > 0$, π_K is proportional to $\sum_i \ell_i/(m_i\,\ell_i(y_i)^2)$ with $\ell_i = \prod_{j\neq i}(x - y_j)$, so prescribing a monic $\pi_K = g$ forces $m_j \propto 1/(g(y_j)\,\ell_j(y_j))$, which is positive for every node set interlacing the atoms. Atoms $\{0,1,3,7\}$ and nodes $\{1/2, 2, 4\}$ give:

Proposition 6.6 (the four-point weight)**.** *The weight with atoms $0,1,3,7$ and masses $195/1379$, $910/1379$, $273/1379$, $1/1379$ is a probability measure, has mean $248/197$, and $g = (x-1/2)(x-2)(x-4)$ is orthogonal to 1, x and x^2 for its inner product $\sum_i m_i\, f(y_i)g(y_i)$; so $\pi_3 = g$ and its three-point Gauss nodes are $1/2, 2, 4$.*

Proof. Three rational identities (formal: `dmass_prob`, `dmean`, `dPi3_orth`); the masses are visibly positive, and $\pi_3 = g$ then follows from the uniqueness of the monic orthogonal polynomial of degree 3, a paper step. The construction is standard interpolation algebra; the instance is ours. □

Theorem 6.7 (an odd K that is hyperbolic)**.** *For the four-point weight, the quotient product modulo $\omega_0 = (x - 248/197)\,\pi_3$ is hyperbolic and is not positive definite on $\Xi = \{0,1,3,7\}$. Hence the zero product of this weight on P_3 is hyperbolic although $K = 3$ is odd, and it is not positive definite.*

Proof. ω_0 has the four distinct rational roots $248/197, 1/2, 2, 4$, so Theorem 4.2(ii) applies (formal: `discrete_three_hyperbolic`). All four roots miss the support: $p = (x-2)^2 - 1/2$ takes the values $7/2, 1/2, 1/2, 49/2$ at $0,1,3,7$ and $-1/2$ at the node 2, so Theorem 5.2 applies (formal: `discrete_three_not_posDef`). With Theorem 6.5 this is a second instance of "hyperbolic but not positive definite"; with Theorem 6.2 it shows that the answer to the first '?' is a property of the pair (w, K) and not of K alone, as the source's legend asserts without proof. □

6.4. **Controls.**

Proposition 6.8 (controls and non-vacuity)**.**

(i) *For every finite $s \subset \mathbb{R}$, $x \in s$ and $p \in \mathbb{R}[x]$, $p(x)$ is an eigenvalue of $\mathsf{P}(p)$ for $*_{\omega_s}$ (so the eigenvalue predicate of Theorem 5.1 is not vacuous, and every root of a squarefree ω is realised as an eigenvalue, which is Lemma 3.8(d) of* [1] *for that case).*

(ii) *"The zero product is hyperbolic for every μ and every monic ϕ_K" is false (Legendre, $K = 3$), and "it is never hyperbolic" is false (Legendre, $K = 2$).*

(iii) *"Hyperbolic iff K even" is false: the four-point weight at $K = 3$ is hyperbolic while Legendre at $K = 3$ is not.*

(iv) *Real roots alone do not suffice: $x\,\pi_3 = \prod_{x\in R}(X - x)$ with $R = \{0, 0, \sqrt{3/5}, -\sqrt{3/5}\}$ splits over $\mathbb{R}$, yet the product is not hyperbolic — the double root, not a complex root, is what breaks it.*

Proof. (i) is Theorem 4.1(ii) at one node (formal: `eigenvalue_realised`); (ii)–(iv) combine the theorems above (formal: `control_not_always_hyperbolic`, `control_not_never_hyperbolic`, `control_parity_rule_fails`, `control_real_roots_not_enough`). □

7. The third ‘?’ cell: symmetry of the collocation product

The symmetric column of the table is decided by the same dictionary, together with the source's Lemma 3.7 and classical quadrature theory. Nothing in this section is formalised, and nothing in it is claimed as new beyond the sentence joining the two ingredients.

Corollary 7.1 (symmetric ATPs; proved on paper only)**.** *Let $*$ be an ATP on P_K with polynomial ω (Proposition 3.1).*

(i) *$*$ is symmetric if and only if $\omega \perp P_{K-1}$ in L^2_w, if and only if $\omega = \pi_{K+1} + t\,\pi_K$ for some $t \in \mathbb{R}$.*
(ii) *The collocation product at $K+1$ distinct nodes $\{\xi_\ell\}$ is symmetric if and only if $\{\xi_\ell\}$ is the zero set of $\pi_{K+1} + t\,\pi_K$ for some $t \in \mathbb{R}$; and every symmetric ATP on P_K is such a collocation product. The case $t = 0$ is the AS product of* [1]*; the two values of t for which an endpoint of a bounded support is a node give the Gauss–Radau rules.*

Proof. (i) $\mathsf{M}^1_{i+1,j+1} = \langle \phi_i, \phi_1 * \phi_j \rangle_w$. For $j \le K-1$ consistency makes the entry $\langle \phi_i, \phi_1\phi_j \rangle_w$, which is symmetric in (i,j) for $i,j \le K-1$; so M^1 is symmetric iff $\langle \phi_1 * \phi_K, \phi_i \rangle_w = \langle \phi_1\phi_i, \phi_K \rangle_w = \langle \phi_1\phi_K, \phi_i \rangle_w$ for $i \le K-1$, i.e. iff $\phi_1\phi_K - \phi_1 * \phi_K \perp P_{K-1}$, i.e. iff $\omega \perp P_{K-1}$. By Lemma 3.6 of [1] a symmetric M^1 makes every M^k symmetric, so symmetry of $*$ is equivalent to symmetry of M^1. Finally, expanding a monic ω of degree $K+1$ as $\pi_{K+1} + \sum_{j \le K} c_j \pi_j$, orthogonality to P_{K-1} kills c_j for $j \le K-1$. (ii) For the collocation product, $\omega = \prod_\ell (x - \xi_\ell)$, since the interpolation error of $\phi_1\phi_K$ at $K+1$ nodes is its leading coefficient times the nodal polynomial; apply (i). Conversely, $\pi_{K+1} + t\,\pi_K$ is the case $k = 2$ of the quasi-orthogonal polynomials of Riesz, Fejér and Shohat ([6] calls the case k “quasi-orthogonal of order $k-1$”, so this is order one there), and for every real t its $K+1$ zeros are real and simple, at most one of them outside the convex hull $[a,b]$ of the support. This is case (β) on p. 467 of Shohat [7], verbatim “all zeros are real and simple, with one, at most, $\le a$ or $\ge b$” (stated there for $A_1 \ne 0$; $t = 0$ is the Gauss case, [2, Thm. 3.3.1]). It also follows from [2, Thms. 3.3.1 and 3.3.2]: writing $\mu_1 < \cdots < \mu_K$ for the zeros of π_K, the separation theorem makes π_{K+1}, and hence $\pi_{K+1} + t\pi_K$, alternate in sign at the μ_i, so the latter has a zero in each (μ_i, μ_{i+1}) and, by its signs at $\pm\infty$, one in $(-\infty, \mu_1)$ and one in (μ_K, ∞) — $K+1$ zeros in $K+1$ disjoint intervals, hence all of them, all simple; and for $t \ge 0$ both π_{K+1} and π_K are positive on $[b, \infty)$, so only the zero in $(-\infty, \mu_1)$ can leave $[a,b]$, while for $t \le 0$ the mirror sign count on $(-\infty, a]$ leaves only the zero in (μ_K, ∞). So by Theorem 3.9 of [1] or Theorem 4.1(ii) every symmetric ATP is the collocation product at those zeros. For the last sentence, $(\pi_{K+1} + t\,\pi_K)(a) = 0$ exactly when $t = -\pi_{K+1}(a)/\pi_K(a)$, so by (i) and Jacobi's theorem [5, Thm. 2.1] the node polynomial of the $(K+1)$-point Gauss–Radau rule with the fixed node a is $\pi_{K+1} - \frac{\pi_{K+1}(a)}{\pi_K(a)}\pi_K$, which is of the stated form; Golub's eigenvalue construction of that rule is [4], printed as [5, Thm. 3.2]. □

Remark 7.2. By Theorems 5.1 and 5.2, the symmetric ATP with parameter t is positive definite on a closed bounded interval $\Xi = [a,b]$ containing the support iff all $K+1$ zeros of $\pi_{K+1} + t\,\pi_K$ lie in $[a,b]$. By the sign count in the proof above, for $t \ge 0$ no zero lies in $[b, \infty)$ and only the smallest can leave $[a,b]$, and it does not iff $(-1)^{K+1}(\pi_{K+1} + t\,\pi_K)(a) \ge 0$, i.e. iff $t \le -\pi_{K+1}(a)/\pi_K(a)$, a positive number; for $t \le 0$ the mirror argument gives $t \ge -\pi_{K+1}(b)/\pi_K(b)$, a negative one. So this happens exactly for t in the closed interval $\left[-\pi_{K+1}(b)/\pi_K(b),\ -\pi_{K+1}(a)/\pi_K(a)\right]$, which contains 0 and whose endpoints are the two Gauss–Radau rules.

8. Remarks and open questions

Remark 8.1 (a citation in the source). In its Section 3.1 the source writes “This lack of associativity of the pseudospectral product is already known in the literature” and cites three works, the oldest of which is Debusschere, Najm, Pébay, Knio, Ghanem and Le Maître [8]. That paper was read in full for the present work (the authors' own copy: same title, authors, abstract and length as the published article), and it contains no statement about associativity: the string *associativ* does not occur in it (all five occurrences of *associat* are *associated*), and neither *commutative* nor *distributive* occurs. What it does contain, and what the source correctly cites it for in the opening of its Section 3, is the construction of discrete division, inverses, square roots and other operations from a discrete multiplication on a polynomial-chaos basis (its Section 2.2). We therefore do not cite it for non-associativity.

Remark 8.2 (an even K that is defective). Along the one-parameter family of five-atom weights on $\{0,1,2,3,4\}$ with masses proportional to $(1,1,1,1,t)$, exact rational evaluation gives $\pi_4(\mu) > 0$ at $t = 77/20$ and $\pi_4(\mu) < 0$ at $t = 39/10$ (we reproduced both signs), so by continuity some real t^* in between has $\pi_4(\mu) = 0$: even-K defective cells exist. No rational instance was found in the searches made for this work (five- and six-atom integer configurations with small masses, and one- and two-parameter rational sweeps, each stopped at fifteen minutes); the clean route is Gram–Schmidt over $\mathbb{Q}(t)$ followed by a rational-root test on the numerator of $\pi_4(\mu)(t)$. This remark is a computation outside the formal development.

Remark 8.3 (what remains). (a) The Chebyshev rows of Remark 4.7 for all K at once: $\omega_0 = x\,T_K$ up to a scalar, and $T_K(0) = \cos(K\pi/2)$, so the parity rule for that row is a two-line argument from the known real roots of T_K and U_K; it is not formalised here. (b) The hyperbolic Legendre cells $K \geq 4$ and Laguerre cells $K \geq 3$ are exact in Remark 4.7 but their formal certificates need the Gauss nodes as real numbers, i.e. a sign-alternation lemma (a monic f of degree n with $n+1$ rational points of alternating sign has n distinct real roots). (c) In several stochastic dimensions the source extends ATPs to Q_K by tensorisation; the dictionary should become $\mathbb{R}[x_1,\ldots,x_d]/I$ for a zero-dimensional ideal I, with hyperbolicity equivalent to I radical with all its points real. Nothing of (c) is done here.

9. Verification

Lemma 3.2, Theorems 4.1, 4.2, 4.4, 4.5(i)(ii), 5.1, 5.2, 6.2, 6.4, 6.5, 6.7 and Propositions 6.1, 6.3, 6.6, 6.8 rest on declarations formally verified in Lean 4 against *Mathlib* [13, 14] (Lean 4.32.0, Mathlib at its `v4.32.0` tag), whose statements are reproduced verbatim in Appendix A. Where a printed statement says more than its Lean counterpart, the proof names the extra step: the non-real-root case of Theorem 4.1, the eigenvalue mechanism displayed in Theorem 5.1, the quadratic-root step in Theorem 4.5, the square-root factorisation in Theorem 6.5, the identification $\pi_3 = g$ in Proposition 6.6, and the classical real-rootedness in Corollaries 4.3 and 5.3. The development is two modules, 729 lines in all. The first (336 lines; 4 definitions, 16 lemmas, 4 theorems) defines the quotient product $p *_\omega q = (pq) \bmod \omega$ on $\mathbb{R}[X]$ through Mathlib's division by a monic polynomial, proves its commutativity, associativity, bilinearity and consistency, defines hyperbolicity, eigenvalues and positive definiteness in the basis-free forms of Section 2, and proves the two criteria (Theorems 4.1 and 5.1, 5.2). The second (393 lines; 12 definitions, 8 lemmas, 26 theorems) specialises to $\omega = (X-\mu)\phi_K$, proves Theorems 4.2, 4.4, 4.5, defines the moment form and the atomic form of Section 6, and proves the certificates, the cells and the controls. Every proof is checked by the Lean kernel; there is no compiled evaluation (`native_decide`), no `sorry`, and no axiom declared by hand; the finite rational identities of Propositions 6.1 and 6.6 and of Theorem 6.5 are discharged by `norm_num`. The axiom print (`#print axioms`) of each of the 30 theorems of Appendix A is `propext`, `Classical.choice`, `Quot.sound`. Outside the formal development, and labelled as such where they occur, are Proposition 3.1, Corollary 7.1 with the remark following it, the classical facts about the zeros of π_K used in Corollaries 4.3 and 5.3, and the exact-rational computations of Remark 4.7 and of the five-atom family. As recorded, the exact-rational computations took about 4 processor-minutes and the Lean checks about 35 (five checks of the two modules at roughly 3 and 4 minutes each), with a peak resident set of about 3 GB per Lean process; two further exhaustive searches for a rational weight realising the remark on even K were stopped at their 900-second limits and are not counted in those figures. The development is currently in a private repository: repository reference to be added at submission.

Appendix A. The formal statements

The Lean statements corresponding to the results above, verbatim (statements only; proofs, docstrings and attributes omitted). `ℝ[X]` is Mathlib's polynomial ring, `%ₘ` its remainder on division by a monic polynomial, `Lagrange.nodal s id` the nodal polynomial ω_s of a finite set `s` of reals, `Lagrange.basis s id x` the Lagrange basis polynomial ℓ_x, `C c` the constant polynomial, and `p.degree` the degree in $\mathbb{N} \cup \{-\infty\}$; "$\deg p < \deg \omega$" is the formal rendering of $p \in P_K$ for $\deg \omega = K + 1$. `ipm m N f g` is the moment form $\sum_{i,j<N} f_i g_j\, m_{i+j}$ and `ipd y m f g` the atomic form $\sum_i m_i f(y_i) g(y_i)$.

Definitions.

```
def tmul (ω p q : ℝ[X]) : ℝ[X] := (p * q) %ₘ ω

def Hyperbolic (ω : ℝ[X]) : Prop :=
  ∀ p : ℝ[X], p.degree < ω.degree →
    ∃ (ι : Type) (t : Finset ι) (b : ι → ℝ[X]) (d : ι → ℝ),
      (∀ i ∈ t, (b i).degree < ω.degree) ∧
      (∀ i ∈ t, tmul ω p (b i) = d i • b i) ∧
      (∀ q : ℝ[X], q.degree < ω.degree → ∃ c : ι → ℝ, q = ∑ i ∈ t, c i • b i)

def IsEigenvalue (ω p : ℝ[X]) (μ : ℝ) : Prop :=
  ∃ q : ℝ[X], q ≠ 0 ∧ q.degree < ω.degree ∧ tmul ω p q = μ • q

def PosDef (ω : ℝ[X]) (Ξ : Set ℝ) : Prop :=
  ∀ p : ℝ[X], p.degree < ω.degree → (∀ x ∈ Ξ, 0 < p.eval x) →
    ∀ μ : ℝ, IsEigenvalue ω p μ → 0 < μ

def zeroOmega (μ : ℝ) (φK : ℝ[X]) : ℝ[X] := (X - C μ) * φK

def ipm (m : ℕ → ℝ) (N : ℕ) (f g : ℝ[X]) : ℝ :=
  ∑ i ∈ Finset.range N, ∑ j ∈ Finset.range N, f.coeff i * g.coeff j * m (i + j)

def ipd {n : ℕ} (y m : Fin n → ℝ) (f g : ℝ[X]) : ℝ :=
  ∑ i, m i * (f.eval (y i) * g.eval (y i))

def legMom (k : ℕ) : ℝ := if k % 2 = 0 then 1 / (k + 1) else 0
def legPi2 : ℝ[X] := X ^ 2 - C (1 / 3)
def legPi3 : ℝ[X] := X ^ 3 - C (3 / 5) * X
def legPi5 : ℝ[X] := X ^ 5 - C (10 / 9) * X ^ 3 + C (5 / 21) * X

def twogapMom (k : ℕ) : ℝ := if k % 2 = 0 then (2 ^ (k + 1) - 1) / (k + 1) else 0
def twogapPi2 : ℝ[X] := X ^ 2 - C (7 / 3)

def datoms : Fin 4 → ℝ := ![0, 1, 3, 7]
def dmass : Fin 4 → ℝ := ![195/1379, 910/1379, 273/1379, 1/1379]
def dPi3 : ℝ[X] := (X - C (1/2)) * ((X - C 2) * (X - C 4))
```

The criteria (Theorems 4.1, 5.1, 5.2).

```
theorem hyperbolic_nodal (s : Finset ℝ) : Hyperbolic (Lagrange.nodal s id)

theorem not_hyperbolic_of_sq_dvd (hω : ω.Monic) {c : ℝ} (hdvd : (X - C c) ^ 2 ∣ ω) :
    ¬ Hyperbolic ω

theorem posDef_of_roots_mem {Ξ : Set ℝ} {rts : Multiset ℝ}
    (hprod : ω = (rts.map (fun x : ℝ => X - C x)).prod) (hmem : ∀ x ∈ rts, x ∈ Ξ) :
    PosDef ω Ξ

theorem not_posDef_of_eval_nonpos {Ξ : Set ℝ} {s : Finset ℝ} {x : ℝ} (hx : x ∈ s)
```

```
    {p : ℝ[X]} (hp : p.degree < (Lagrange.nodal s id).degree) (hpos : ∀ y ∈ Ξ, 0 < p.eval y)
    (hneg : p.eval x ≤ 0) : ¬ PosDef (Lagrange.nodal s id) Ξ
```

The zero product (Lemma 3.2, Theorems 4.2, 4.4, 4.5).

```
theorem tmul_zero_product (μ : ℝ) {φK : ℝ[X]} (h : φK.Monic) :
    tmul (zeroOmega μ φK) (X - C μ) φK = 0

theorem zero_not_hyperbolic (μ : ℝ) {φK : ℝ[X]} (hm : φK.Monic) (h : φK.eval μ = 0) :
    ¬ Hyperbolic (zeroOmega μ φK)

theorem zero_hyperbolic (μ : ℝ) {t : Finset ℝ} (hμ : μ ∉ t) :
    Hyperbolic (zeroOmega μ (Lagrange.nodal t id))

theorem zero_K1_not_hyperbolic (μ : ℝ) : ¬ Hyperbolic (zeroOmega μ (X - C μ))

theorem zero_K2_hyperbolic {μ r s : ℝ} (hr : r < μ) (hs : μ < s) :
    Hyperbolic (zeroOmega μ ((X - C r) * (X - C s)))

theorem eval_phi_two_eq (m : ℕ → ℝ) (a b : ℝ) (h0 : m 0 = 1)
    (horth : ipm m 3 (X ^ 2 + C a * X + C b) 1 = 0) :
    (X ^ 2 + C a * X + C b).eval (m 1) = (m 1) ^ 2 - m 2
```

The Legendre cells (Proposition 6.1, Theorem 6.2, Proposition 6.3, Theorem 6.4).

```
theorem legPi2_orth : ipm legMom 3 legPi2 1 = 0 ∧ ipm legMom 3 legPi2 X = 0

theorem legPi3_orth : ipm legMom 4 legPi3 1 = 0 ∧ ipm legMom 4 legPi3 X = 0 ∧
    ipm legMom 4 legPi3 (X ^ 2) = 0

theorem legPi5_orth : ipm legMom 6 legPi5 1 = 0 ∧ ipm legMom 6 legPi5 X = 0 ∧
    ipm legMom 6 legPi5 (X ^ 2) = 0 ∧ ipm legMom 6 legPi5 (X ^ 3) = 0 ∧
    ipm legMom 6 legPi5 (X ^ 4) = 0

theorem legendre_three_not_hyperbolic : ¬ Hyperbolic (zeroOmega 0 legPi3)

theorem legendre_five_not_hyperbolic : ¬ Hyperbolic (zeroOmega 0 legPi5)

theorem legendre_two_hyperbolic :
    Hyperbolic (zeroOmega 0 ((X - C (-Real.sqrt (1/3))) * (X - C (Real.sqrt (1/3)))))

theorem legPi2_factor :
    (X - C (-Real.sqrt (1/3))) * (X - C (Real.sqrt (1/3))) = legPi2

theorem legendre_three_posDef : PosDef (zeroOmega 0 legPi3) (Set.Icc (-1 : ℝ) 1)
```

The two-gap and four-point weights (Theorem 6.5, Proposition 6.6, Theorem 6.7).

```
theorem twogapPi2_orth :
    ipm twogapMom 3 twogapPi2 1 = 0 ∧ ipm twogapMom 3 twogapPi2 X = 0

theorem twogap_two_not_posDef :
    ¬ PosDef (zeroOmega 0 twogapPi2) (Set.Icc (-2 : ℝ) (-1) ∪ Set.Icc 1 2)

theorem dmass_prob : ∑ i, dmass i = 1
```

```
theorem dmean : ipd datoms dmass X 1 = 248/197

theorem dPi3_orth : ipd datoms dmass dPi3 1 = 0 ∧ ipd datoms dmass dPi3 X = 0 ∧
    ipd datoms dmass dPi3 (X ^ 2) = 0

theorem discrete_three_hyperbolic : Hyperbolic (zeroOmega (248/197) dPi3)

theorem discrete_three_not_posDef :
    ¬ PosDef (zeroOmega (248/197) dPi3) ({0, 1, 3, 7} : Set ℝ)
```

Controls (Proposition 6.8).

```
theorem eigenvalue_realised (s : Finset ℝ) {x : ℝ} (hx : x ∈ s) (p : ℝ[X]) :
    IsEigenvalue (Lagrange.nodal s id) p (p.eval x)

theorem control_not_always_hyperbolic :
    ¬ (∀ (μ : ℝ) (φK : ℝ[X]), φK.Monic → Hyperbolic (zeroOmega μ φK))

theorem control_not_never_hyperbolic :
    ¬ (∀ (μ : ℝ) (φK : ℝ[X]), ¬ Hyperbolic (zeroOmega μ φK))

theorem control_parity_rule_fails :
    Hyperbolic (zeroOmega (248/197) dPi3) ∧ ¬ Hyperbolic (zeroOmega 0 legPi3)

theorem control_real_roots_not_enough :
    zeroOmega 0 legPi3
        = ((↑[(0 : ℝ), 0, Real.sqrt (3/5), -Real.sqrt (3/5)] : Multiset ℝ).map
            (fun x : ℝ => X - C x)).prod ∧
      ¬ Hyperbolic (zeroOmega 0 legPi3)
```

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