THE BOJANOV–NAIDENOV EXTREMAL PROBLEM FOR THE k-TH DERIVATIVE OF ALGEBRAIC POLYNOMIALS: THE FINITE CUBE-VERTEX CHECK, SETTLED IN EVERY DEGREE $n \le 6$

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Abstract. Let π_n° be the real polynomials of degree at most n bounded by 1 on $[-1,1]$ and T_n the Chebyshev polynomial. Bojanov's problem, as recorded by Naidenov, asks whether $\int_{-1}^1 \varphi(|P^{(k)}|) < \int_{-1}^1 \varphi(|T_n^{(k)}|)$ for every strictly increasing convex φ, every $P \in \pi_n^\circ \setminus \{\pm T_n\}$ and every $2 \le k \le n-1$. Inside that range only $(n,k) = (4,2)$ was settled, by Zavalani (arXiv:2606.23020), who interpolates at the $n+1$ extremal points of T_n, passes by convexity to the cube of nodal values, and checks the 2^{n+1} vertices. We run that reduction in exact arithmetic over $\mathbb{Q}(\sqrt{D})$ at all ten pairs $3 \le n \le 6$, $2 \le k \le n-1$: at every vertex $y \in \{-1,1\}^{n+1}$ the integrated tails $\int(|P_y^{(k)}| - t)_+$ are dominated by those of $T_n^{(k)}$ at every level $t \ge 0$, the bound is attained at $\pm T_n$, and the extremal sup norm is $T_n^{(k)}(1)$ $(24, 80, 192, 200, 840, 1920, 420, 2688, 10368, 23040)$. Combined with the elementary convexity and tail-to-convex-order lemmas of the source, this gives the Bojanov–Naidenov inequality in its non-strict form — for every nondecreasing convex φ — at the nine previously open pairs, and under the weaker Duffin–Schaeffer hypothesis $|P(\cos(j\pi/n))| \le 1$, $j = 0, \dots, n$; we point out that the sup-norm content of every such statement is Duffin and Schaeffer's Theorem III of 1941, which the source proves at $(4,2)$, under the stronger hypothesis $\|P\|_{C[-1,1]} \le 1$, without citing it. A second finding concerns the method: the pointwise level-set domination through which the source proves its vertex bound is false at nine of the ten pairs, $(4,2)$ being the only one where it holds — certified at six pairs by explicit vertex witnesses whose k-th derivative stays away from zero on $[-1,1]$, and found by external computation at the other three. Each vertex is certified by a transport between two uniform partitions with degree-four Bernstein enclosures, whose only data per vertex is a pair of partition sizes; the certificates, the setup and the witnesses are verified in Lean 4 against *Mathlib*.

1. Introduction

1.1. The problem. Let π_n be the space of real algebraic polynomials of degree at most n, let $\pi_n^\circ = \{P \in \pi_n : \|P\|_{C[-1,1]} \le 1\}$, and let $T_n(x) = \cos(n \arccos x)$ be the Chebyshev polynomial of the first kind. Write Φ for the class of strictly increasing convex functions on $[0,\infty)$. In the open-problems section of the Sozopol 2010 proceedings, Naidenov [2] records the following question of Bojanov, verbatim:

Problem 1. Prove or disprove the inequality

$$\int_{-1}^{1} \varphi\big(|P^{(k)}(x)|\big)\,dx < \int_{-1}^{1} \varphi\big(|T_n^{(k)}(x)|\big)\,dx, \tag{1}$$

for every $\varphi \in \Phi$, $P \in \pi_n^\circ \setminus \{\pm T_n\}$ and $k \in \{2, \dots, n-1\}$.

Naidenov attributes the problem to Bojanov [3] and records what is known: for $k = n$, (1) is a consequence of a well-known property of T_n; for $k = 1$ it was proved by Bojanov [5] (the zbMATH review of that paper credits Kristiansen [6] with the result in a more general framework), the particular cases $\varphi(x) = \sqrt{1+x^2}$ and $\varphi(x) = x^p$, $p \ge 1$, having been obtained earlier in [4] and [3]; Problem 1 was solved by Bojanov and Rahman [8] on the subclass of polynomials having all their zeros in $[-1,1]$; and the case $k = 2$ was confirmed by Avvakumova [7] on an intermediate class between that subclass and π_n°. The papers of Kofanov [15, 16, 17] study a "Bojanov–Naidenov problem" for classes of differentiable functions on the real line, trigonometric polynomials and periodic splines, compared against the Euler spline; none of them concerns algebraic polynomials on $[-1,1]$.

Inside the range $2 \le k \le n-1$, one pair was settled before this paper: Zavalani [1] proved (1) for $(n,k) = (4,2)$, in the stronger integrated-tail form (Theorem 1 there). That paper names no further pair; it settles $(4,2)$ and says of its own method: "The argument is quite specific to degree four. … Thus the finite character of the proof is also its limitation: it gives a complete solution of this concrete intermediate case, but it is not a direct argument for higher degrees."

1.2. **The reduction.** Fix (n,k) and let $x_j = \cos(j\pi/n)$, $j = 0,\dots,n$, be the $n+1$ extremal points of T_n in $[-1,1]$, so that $T_n(x_j) = (-1)^j$. For $y \in \mathbb{R}^{n+1}$ let $P_y \in \pi_n$ be the interpolant, $P_y(x_j) = y_j$, and for $t \ge 0$ put

$$F_t(y) = \int_{-1}^{1} \big(|P_y^{(k)}(x)| - t\big)_+\,dx, \qquad G(t) = \int_{-1}^{1} \big(|T_n^{(k)}(x)| - t\big)_+\,dx. \tag{2}$$

The chain of [1] is: (1) if $P \in \pi_n^\circ$ then $y = (P(x_j))_j \in [-1,1]^{n+1}$ and $P = P_y$; (2) F_t is convex in y, so its maximum over the cube $[-1,1]^{n+1}$ is attained at a vertex $y \in \{-1,1\}^{n+1}$ (Lemma 3 of [1]); (3) the *finite check*: $F_t(y) \le G(t)$ for all 2^{n+1} vertices y and all $t \ge 0$; (4) a tail-to-convex-order lemma (Lemma 2 of [1]) turns the tail comparison into (1) for every nondecreasing convex φ, given the range bound $\|P^{(k)}\|_{C[-1,1]} \le \|T_n^{(k)}\|_{C[-1,1]}$ (Lemma 4 of [1]), and an L^1 strictness argument (Lemma 7 of [1]) gives the strict inequality for $P \ne \pm T_n$. Steps (1), (2) and (4) are elementary and uniform in (n,k). Step (3) is where the degree enters: at $(4,2)$ the source carries it out by hand, through the pointwise level-set domination

$$\Lambda_q(t) := \operatorname{meas}\{x \in [-1,1] : |q(x)| > t\} \;\le\; \Lambda_*(t) := \operatorname{meas}\{x \in [-1,1] : |T_n^{(k)}(x)| > t\}, \qquad t \ge 0, \tag{3}$$

for every vertex derivative $q = P_y^{(k)}$ (Lemma 6 of [1], equation (6.1) there), of which the tail comparison is the integrated consequence (equation (6.2) there, via $\int(|q| - t)_+ = \int_t^\infty \Lambda_q$).

1.3. **What is proved here.** We carry out step (3) at every pair (n,k) with $3 \le n \le 6$ and $2 \le k \le n-1$ — ten pairs, which we call *cells* — in exact arithmetic over the real quadratic field in which the nodes live ($\mathbb{Q}$ for $n=3$, $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{3})$ for $n = 4,5,6$), and certify the result in Lean 4.

- *The vertex check holds at all ten cells* (Proposition 5.1 for the source's own $(4,2)$, Theorem 5.2 for the other nine): for every sign vector $y \in \{-1,1\}^{n+1}$ and every $t \ge 0$, $F_t(y) \le G(t)$. The alternating vertex is $\pm T_n$ and gives equality, and $T_n^{(k)}(1)$ — classically $\|T_n^{(k)}\|_{C[-1,1]}$, and the constant of Theorem 2.3 — takes the ten values $24, 80, 192, 200, 840, 1920, 420, 2688, 10368, 23040$ (Proposition 4.1).
- *Consequently* (Corollary 5.3), with the cited steps (2) and (4), the non-strict form of (1) holds at all ten cells for every nondecreasing convex φ, and it holds under the weaker hypothesis $|P(x_j)| \le 1$, $j = 0,\dots,n$, rather than $\|P\|_{C[-1,1]} \le 1$. That is the hypothesis under which Duffin and Schaeffer [9] refined the Markov inequality in 1941; we did not find a convex-order statement of this kind in print (Section 8). The strict inequality of Problem 1 for $P \ne \pm T_n$ follows by the source's own argument from a strict L^1 comparison at every non-Chebyshev vertex; the formal development certifies that comparison at one vertex per cell, the one an exact external ranking finds largest, and we say so precisely (Remark 5.4).
- *The sup-norm content is old.* At the vertices, each cell theorem implies $\|P_y^{(k)}\|_{C[-1,1]} \le T_n^{(k)}(1)$. Under the nodal hypothesis that bound is Theorem III of Duffin and Schaeffer [9], which is what the source's Lemma 4 establishes at $(4,2)$; the source does not cite it. What is new in the cell theorems is the integrated-tail half, that is, the increasing convex order.
- *The level-set route does not extend* (Theorem 6.1): the pointwise domination (3) is false at nine of the ten cells — $(4,2)$, the source's own, is the only cell where it holds at every vertex — with explicit certified vertex witnesses at six cells, namely vertices whose k-th derivative is bounded away from zero on $[-1,1]$ while $T_n^{(k)}$ has $n-k$ zeros there; at the remaining three the failure is of a subtler kind and was found by an external computation only (Remark 6.2). This is a statement about the method, not a defect of the source's theorem: at $(4,2)$ the two orders coincide, from $n=5$ on only the integrated form survives, and the integrated form is what the cell theorems prove.

- *Negative controls* (Theorem 7.1): at every cell the right-hand side cannot be shrunk by any factor $\lambda < 1$, and one non-Chebyshev vertex per cell, the one the external ranking finds largest in L^1 norm, is strictly below $T_n^{(k)}$ at $t = 0$.
- *The certificate* (Theorem 3.3): each vertex is certified by a transport between two uniform partitions of $[-1, 1]$ with degree-four Bernstein enclosures, whose only data is a pair (N_A, N_B) of partition sizes; everything else is recomputed by the checker. The largest sizes needed are $N_A = 128$, $N_B = 684$, both at $(6, 2)$.

The finite checks, the setup and the witnesses are verified in Lean 4 against *Mathlib*; Section 9 and Appendix A say exactly what is formal and what is cited.

1.4. **What is not claimed.** (i) Nothing about the sup norm is new: see above and Section 8. (ii) The cited steps (2) and (4) are not formalized; Corollary 5.3 is a theorem of this paper only together with Lemmas 2.1, 2.2 and Theorem 2.3 as cited. (iii) The strict inequality for $P \neq \pm T_n$ rests, at each cell, on an external exact ranking of the vertices by L^1 norm (Remark 5.4). (iv) For $k = n-1$ — the cells $(3, 2)$, $(4, 3)$, $(5, 4)$, $(6, 5)$ — the k-th derivative is linear and the extremal problem is a convex maximization over the admissible pairs (a_n, a_{n-1}) of leading coefficients, whose boundary is Zolotarev's first problem; a closed-form treatment of general $k = n - 1$ by that route is a live possibility that our search did not find in print (Section 8.3). (v) Three of the nine failures of (3) — at $(5, 2)$, $(6, 2)$ and $(6, 3)$ — were found by external computation and are reported as such (Remark 6.2); six are certified. (vi) One paper that could bear on the novelty grading was not read: Avvakumova [7], whose class definition and restriction to $k = 2$ we take from the zbMATH author summary, the full text being unobtainable (Section 1.5). (vii) Nothing is claimed for $n \ge 7$, where the nodes leave the quadratic fields.

1.5. **Where this was looked for.** Searches were run on 2026-09-07. In a local text corpus of arXiv e-prints, "Bojanov" occurs in 144 shards and "Bojanov" together with "Naidenov" in 22, mapping to 14 distinct papers, of which exactly one, [1], is about this problem (the others concern the weighted Bojanov–Chebyshev problem, Landau–Kolmogorov inequalities, the largest critical value of $T_n^{(k)}$, and weighted Lagrange interpolation). The live arXiv API under six phrasings ("Bojanov-Naidenov", "Bojanov Naidenov problem", "Bojanov–Naidenov", Naidenov+Chebyshev, Bojanov+Chebyshev, "Bojanov problem") returned only [1] and unrelated work; OpenAlex records 0 citations of [1] and 16 citing works of [8], none settling a specific degree; the zbMATH Open API under "Bojanov Naidenov problem" returned 15 hits, of which the only one about algebraic polynomials on $[-1, 1]$ is [1], every Kofanov entry being on the real line, trigonometric or spline; five web phrasings found nothing further; the Sofia 2023 open-problems collection arXiv:2312.07754 does not contain the problem; Semantic Scholar returned HTTP 429 throughout and is recorded as no signal. The two classes that could have covered a cell do not: [8] needs all zeros in $[-1, 1]$, and the class $H_n = \{p \in \pi_n^\circ : p' \text{ has all its zeros in } [-1, 1] \text{ except at most one}\}$ of [7] is for $k = 2$ only and misses π_5° and π_6°: $p(x) = (x^5 + x^3)/2$ and $p(x) = (x^6 + x^4)/2$ have sup norm 1 on $[-1, 1]$ and derivatives with two non-real zeros. Both the class and the restriction to $k = 2$ come from the zbMATH author summary of [7] (Zbl 1088.26503), which describes "the set $\mathcal{H}_n$ of real algebraic polynomials $p(x)$ of degree n whose first derivative has all its zeros in $I := [-1, 1]$ except at most one and $|p(x)| \le 1$ on I" and states $\int_I F(|p''|) \le \int_I F(|T_n''|)$ on it for "a wide class of convex functions" F, with equality only at $p = \pm T_n$. The article itself is unobtainable: the journal is defunct, there is no DOI and no scan, and no citing work states its theorem. Two items in Naidenov's own reference list that might record the problem, Bojanov's open-problem chapter *Polynomial inequalities* (1994) and his survey *Markov-type inequalities for polynomials and splines* (2002), were not consulted. The negative is to be read as "not found".

2. Preliminaries, and the four cited steps

2.1. **Notation.** Throughout, (n, k) is one of the ten cells, $x_j = \cos(j\pi/n)$ $(j = 0, \dots, n)$ are the extremal points of T_n in decreasing order, $\ell_j \in \pi_n$ is the Lagrange basis, $\ell_j(x_i) = \delta_{ij}$, and for a sign vector $s \in \{-1, 1\}^{n+1}$ we write $P_s = \sum_j s_j \ell_j$, so that $P_s(x_j) = s_j$. The alternating vector $s^* = ((-1)^j)_j$ gives $P_{s^*} = T_n$ and $P_{-s^*} = -T_n$; we call $\pm s^*$ the *Chebyshev vertices.* F_t and G are as in (2), and

$(u)_+ = \max(u,0)$. The nodes lie in $\mathbb{Q}(\sqrt{D})$ with $D = 2,5,3$ for $n = 4,5,6$ and in $\mathbb{Q}$ for $n = 3$ (Table 1); in the formal development every polynomial has coefficients in $\mathbb{Q}(\sqrt{D})$, represented as pairs of rationals, and the sign of an element $a + b\sqrt{D}$ is decided by comparing a^2 with Db^2.

2.2. **The cited steps.** The following four statements are taken from [1] and [9] and are *not* part of the formal development. Statement numbers of [1] are those of its compiled e-print (`tectonic` on the e-print source, numbering read from the `.aux` file), and agree with the arXiv PDF.

Lemma 2.1 (tail order to convex order; [1, Lemma 2])**.** *Let f, g be measurable on a finite measure space with values in $[0,M]$, and suppose $\int (f-t)_+\,d\nu \le \int (g-t)_+\,d\nu$ for $0 \le t \le M$. Then $\int \varphi(f)\,d\nu \le \int \varphi(g)\,d\nu$ for every nondecreasing convex $\varphi : [0,M] \to \mathbb{R}$. If in addition $\int f\,d\nu < \int g\,d\nu$, the last inequality is strict for every strictly increasing convex φ.*

Lemma 2.2 (convexity, and the vertex maximum; [1, Lemma 3])**.** *For each $t \ge 0$ the map $F_t : \mathbb{R}^{n+1} \to \mathbb{R}$ is convex; consequently $\max_{y\in[-1,1]^{n+1}} F_t(y)$ is attained at a vertex of the cube.*

The source states this for $n = 4$; the proof (interpolation is linear in y, $u \mapsto (|u| - t)_+$ is convex, integration preserves convexity, and a point of the cube is a convex combination of vertices) applies verbatim for every (n,k). The source also notes, without drawing the conclusion we draw in Corollary 5.3, that "the nodal vectors coming from polynomials in π_4° form only a subset of $[-1,1]^5$. We shall nevertheless work on the full cube."

Theorem 2.3 (Duffin–Schaeffer 1941, Theorem III)**.** *Let f be a polynomial of degree n satisfying $|f(\cos(\nu\pi/n))| \le 1$; $\nu = 0,1,2,\dots,n$. Then for $p = 1,2,3,\dots,n$,*

$$|f^{(p)}(x)| \le \frac{n^2(n^2-1)(n^2-2^2)\cdots(n^2-(p-1)^2)}{1\cdot 3\cdot 5\cdots(2p-1)},$$

and the equality occurs only if $f \equiv \gamma T_n$, $|\gamma| = 1$.

This is their Theorem III as printed on p. 525 of [9] (its display is their equation (19)), transcribed with $\le$ for the $\leqq$ of the original; the polynomial there is allowed complex coefficients, and the restriction $-1 \le x \le 1$, under which the displayed bound is proved, appears in the proof rather than in the printed statement. The right-hand side is $T_n^{(p)}(1) = \|T_n^{(p)}\|_{C[-1,1]}$; it reproduces the ten values of Proposition 4.1. Lemma 4 of [1] is the case $(n,p) = (4,2)$ under the stronger hypothesis $P \in \pi_4^\circ$; but its proof passes to the nodal cube and to Table 1 of that paper through the convexity of $y \mapsto \|P_y''\|$, so what it in fact establishes is the case $(4,2)$ of Theorem 2.3. The source does not cite [9], and the string "Duffin" does not occur in its text.

Lemma 2.4 (the L^1 strictness argument; [1, Lemma 7])**.** *Suppose $F_0(v) < F_0(s^*)$ for every vertex $v \in \{-1,1\}^{n+1} \setminus \{\pm s^*\}$. Then $F_0(y) < F_0(s^*)$ for every $y \in [-1,1]^{n+1} \setminus \{\pm s^*\}$.*

This is the argument of the proof of Lemma 7 in [1], stated for general n: write $y = \sum_v \lambda_v v$ as a convex combination of vertices; if some non-Chebyshev vertex has positive weight, convexity of F_0 gives the strict inequality; otherwise $y = \alpha s^*$ with $|\alpha| < 1$ and $F_0(y) = |\alpha| F_0(s^*)$. At $(4,2)$ the source obtains the hypothesis from (3) and its Table 1; at the other cells (3) is not available (Theorem 6.1), and the hypothesis has to be certified vertex by vertex (Remark 5.4).

3. The certificate

For a polynomial p of degree at most 4 with coefficients in $\mathbb{Q}(\sqrt{D})$ and a cell $[a, a+h]$ with $a, h \in \mathbb{Q}$, $h > 0$, write $p(a+hu) = \sum_{i=0}^{4} b_i \binom{4}{i} u^i (1-u)^{4-i}$ for the Bernstein coefficients $b_0,\dots,b_4 \in \mathbb{Q}(\sqrt{D})$; then $\min_i b_i \le p \le \max_i b_i$ on the cell.

Definition 3.1 (cell bounds)**.** Let $N \ge 1$ and let $I_i = [-1+2i/N,\ -1+2(i+1)/N]$, $i = 0,\dots,N-1$, be the uniform partition of $[-1,1]$, and let $b_0,\dots,b_4$ be the Bernstein coefficients of p on I_i. The *certified upper bound* of $|p|$ on I_i is

$$c_i(p,N) = \max\big(\max_j b_j,\ -\min_j b_j\big),$$

and the *certified lower bound* of $|p|$ on I_i is $C_i(p,N) = \min_j b_j$ if all $b_j \ge 0$, $C_i(p,N) = -\max_j b_j$ if all $b_j \le 0$, and $C_i(p,N) = 0$ otherwise. Thus $|p| \le c_i$ and $|p| \ge C_i \ge 0$ on I_i.

Definition 3.2 (block transport check)**.** Let q, q^* have degree at most 4 and coefficients in $\mathbb{Q}(\sqrt{D})$, and let $1 \le N_A \le N_B$. Sort the list $(c_i(q,N_A))_{i<N_A}$ and the list $(C_j(q^*,N_B))_{j<N_B}$ in decreasing order. Walk the first list: the current c *consumes* elements C off the front of what remains of the second list, each of which must satisfy $C \ge c$, until the consumed block satisfies $N_B\, c \le N_A \sum C$; the check *succeeds* if every c finds such a block, and *fails* at the first c that meets an element $C < c$ before its block is complete, or exhausts the second list. We write $\mathrm{check}(D,q,q^*,N_A,N_B)$ for the outcome; it is a decidable computation in $\mathbb{Q}(\sqrt{D})$ whose only inputs beyond q, q^* are the two integers.

Theorem 3.3 (soundness of the certificate)**.** *Let $D \ge 0$ be rational, let q, q^* be polynomials of degree at most 4 with coefficients in $\mathbb{Q}(\sqrt{D})$, and let $N_A, N_B \in \mathbb{N}$. If* $\mathrm{check}(D,q,q^*,N_A,N_B)$ *succeeds, then for every $t \ge 0$*

$$\int_{-1}^{1} \big(|q(x)| - t\big)_+ \, dx \;\le\; \int_{-1}^{1} \big(|q^*(x)| - t\big)_+ \, dx.$$

Moreover, for every p as above and every $N \ge 1$, $\frac{2}{N}\sum_{j<N} C_j(p,N) \le \int_{-1}^{1} |p| \le \frac{2}{N}\sum_{i<N} c_i(p,N)$.

Proof sketch. The left integral is at most $\sum_i \frac{2}{N_A}(c_i - t)_+$ and the right integral is at least $\sum_j \frac{2}{N_B}(C_j - t)_+$, by integrating the cell bounds over each cell; both sums are invariant under reordering. For one c and the block B it consumes, with $r = |B|$ and $S = \sum_{C\in B} C$, the two conditions $C \ge c$ for $C \in B$ and $N_B c \le N_A S$ give

$$\tfrac{2}{N_A}(c-t)_+ \le \sum_{C\in B} \tfrac{2}{N_B}(C-t)_+ \qquad (t \ge 0):$$

for $t \ge c$ the left side vanishes and the right side is nonnegative; for $0 \le t \le c$ both sides are affine in t (every $C \ge c > t$), and the two endpoint cases $t = 0$ and $t = c$ are exactly the two conditions. Summing over the blocks, which are disjoint, and dropping the unconsumed C's (nonnegative terms) gives the claim. The L^1 bounds are the case $t = 0$ of the two cell estimates, with the sums evaluated exactly in $\mathbb{Q}(\sqrt{D})$. □

Remark 3.4. A first design used a one-to-one sorted matching, $N_B\, c_{(i)} \le N_A\, C_{(i)}$ for every $i < N_A$, and failed at three vertices of $(5,2)$, three of $(6,2)$ and one of $(6,3)$, by 6% to 13% of the required slack; letting one A-cell consume a block of B-cells closed the gap everywhere and shrank the partitions (the largest N_A at $(6,2)$ fell from 384 to 128). Recorded because the one-to-one matching is the obvious first thing to try and is genuinely too weak.

4. The ten cells: setup

Sign vectors are encoded by integers: for $0 \le m < 2^{n+1}$, the vector $s(m)$ has $s(m)_j = +1$ if bit j of m is set and -1 otherwise, so that $P_{s(m)} = \sum_j \pm \ell_j$. The Chebyshev vertex s^* is $m^* = 5, 21, 21, 85$ for $n = 3,4,5,6$ and $-s^*$ is $2^{n+1} - 1 - m^*$.

Proposition 4.1 (the setup, certified at all ten cells)**.** *At each of the ten cells: (i) the explicit coefficient list used for T_n is Mathlib's Chebyshev polynomial of the first kind,* `Polynomial.Chebyshev.T` $\mathbb{R}$ *n, as a real function; (ii) the $n+1$ nodes satisfy $T_n(x_j) = (-1)^j$; (iii) the listed polynomials ℓ_j satisfy $\ell_j(x_i) = \delta_{ij}$ for $0 \le i,j \le n$; (iv) for every $0 \le m < 2^{n+1}$ and every $i \le n$, $P_{s(m)}(x_i) = s(m)_i$; (v) $T_n^{(k)}$ is the explicit polynomial of Table 1, and the k-th derivative of P_{s^*} is that polynomial (so the bound of Theorem 5.2 is attained).*

Proof sketch. (i) is proved by unfolding the recursion $T_{m+2} = 2XT_{m+1} - T_m$ of Mathlib $n-1$ times and comparing coefficients; it uses no evaluation of compiled code. (ii)–(v) are finite identities in $\mathbb{Q}(\sqrt{D})$ — $n+1$ evaluations of T_n, $(n+1)^2$ evaluations of the Lagrange basis, $(n+1)2^{n+1}$ evaluations of the vertex interpolants, and k formal derivatives — discharged by evaluation, with a proved bridge from the exact $\mathbb{Q}(\sqrt{D})$ value to the real number $a + b\sqrt{D}$. The files also prove that the nodes decrease, $x_{j+1} \le x_j$. □

(n,k)	D	vertices	orbits	$T_n^{(k)}(x)$	$T_n^{(k)}(1)$
$(3,2)$	2	16	6	$24x$	24
$(4,2)$	2	32	10	$96x^2-16$	80
$(4,3)$	2	32	9	$192x$	192
$(5,2)$	5	64	20	$320x^3-120x$	200
$(5,3)$	5	64	20	$960x^2-120$	840
$(5,4)$	5	64	19	$1920x$	1920
$(6,2)$	3	128	36	$960x^4-576x^2+36$	420
$(6,3)$	3	128	36	$3840x^3-1152x$	2688
$(6,4)$	3	128	35	$11520x^2-1152$	10368
$(6,5)$	3	128	28	$23040x$	23040

TABLE 1. The ten cells. D is the radicand of the field of the nodes (for $n=3$ the nodes $1,\frac{1}{2},-\frac{1}{2},-1$ are rational and $D=2$ is a dummy); "orbits" is the number of classes of vertex derivatives under $q\mapsto -q$ and $q(x)\mapsto q(-x)$, an external count; $T_n^{(k)}$ and $T_n^{(k)}(1)$ are certified (Proposition 4.1), and $T_n^{(k)}(1)$ agrees with the constant of Theorem 2.3 in every row.

The values $T_n^{(k)}(1)$ in Table 1 are classical: they are V. A. Markov's constants $n^2(n^2-1^2)\cdots(n^2-(k-1)^2)/(1\cdot 3\cdots(2k-1))$, and the certification, together with the link to Mathlib's Chebyshev polynomial, is what is ours.

5. The vertex check

Proposition 5.1 (the source's cell $(4,2)$; not new)**.** *For every $s\in\{-1,1\}^5$ and every $t\geq 0$, $\int_{-1}^1(|P_s''|-t)_+\leq\int_{-1}^1(|T_4''|-t)_+$.*

This is equation (6.2) of Lemma 6 in [1], the finite step of its Theorem 1; what is ours is the certificate. Before any new cell was run, the source's data were reproduced by two independent implementations outside the formal development (one in PARI/GP with exact polynomial arithmetic in $\mathbb{Q}[y]/(y^2-2)$, one in Python over $\mathbb{Q}(\sqrt{2})$ with exact rationals): both enumerate the 32 vertices, reduce modulo $q\mapsto -q$ and $q(x)\mapsto q(-x)$, and return the 10 classes of Table 2 of [1] with the orbit sizes $2,4,4,4,4,2,4,4,2,2$ printed there (in that table's row order) and the representatives 0, $12\sqrt{2}x+4$, $24x^2+12x-2$, $48x^2+12\sqrt{2}x-8$, $24x^2+12(\sqrt{2}-1)x-6$, $48x^2-12$, $72x^2+12x-14$, $24x^2+12(\sqrt{2}+1)x-6$, $48x^2-4$, $96x^2-16$, with the uniform norms 0, $4+12\sqrt{2}$, 34, $40+12\sqrt{2}$, $6+12\sqrt{2}$, 36, 70, $30+12\sqrt{2}$, 44, 80 of its Table 1. The same computation confirms Lemma 6 of [1] numerically: at $(4,2)$, (3) holds at every one of the 32 vertices on a 400-point grid of $t\in[0,80]$, with equality at every level only at the two Chebyshev vertices; at each of the other 28 nonzero vertices the two sides agree on the grid only at $t=0$ and at $t=80$.

Theorem 5.2 (the vertex check at the nine new cells)**.** *Let (n,k) be one of $(3,2)$, $(4,3)$, $(5,2)$, $(5,3)$, $(5,4)$, $(6,2)$, $(6,3)$, $(6,4)$, $(6,5)$. For every sign vector $s\in\{-1,1\}^{n+1}$ and every $t\geq 0$,*

$$\int_{-1}^1\big(|P_s^{(k)}(x)|-t\big)_+\,dx\ \leq\ \int_{-1}^1\big(|T_n^{(k)}(x)|-t\big)_+\,dx. \tag{4}$$

The nine cells are certified separately: $(3,2)$, $(4,3)$, $(5,2)$, $(5,3)$, $(5,4)$, $(6,2)$, $(6,3)$, $(6,4)$, $(6,5)$.

Proof sketch. Fix a cell. For each of the 2^{n+1} vertices m, the formal k-th derivative q_m of $P_{s(m)}$ is computed in $\mathbb{Q}(\sqrt{D})$ (its coefficients above degree 4 are checked to vanish, so q_m has degree at most 4), and q_m is compared with $q^*=T_n^{(k)}$. For the two Chebyshev vertices $q_m=\pm q^*$ is verified as an identity of coefficient lists and (4) is $|-u|=|u|$. For every other vertex a pair (N_A,N_B) is stored and $\mathrm{check}(D,q_m,q^*,N_A,N_B)$ is evaluated; it succeeds, and Theorem 3.3 gives (4). The pairs were found by an external search and are the only per-vertex data; the largest needed in each cell are

(n,k)	$(3,2)$	$(4,2)$	$(4,3)$	$(5,2)$	$(5,3)$	$(5,4)$	$(6,2)$	$(6,3)$	$(6,4)$	$(6,5)$
$\max N_A$	8	20	10	64	32	16	128	48	40	16
$\max N_B$	36	158	69	327	181	54	684	376	220	88

The identification of the real function $x \mapsto P^{(k)}_{s(m)}(x)$ with the evaluation of q_m (iterated derivative of a polynomial function equals the formal derivative) and of $T_n^{(k)}$ with q^* are proved once, in the shared module. The exhaustive part is thus: all 2^{n+1} vertices of each cell, $16+32+32+64+64+64+128+128+128+128 = 784$ vertices in all, each certified by one evaluation of the check. $\square$

Corollary 5.3 (Bojanov–Naidenov at the ten cells, under the nodal hypothesis)**.** *Let (n,k) be one of the ten cells and let $P \in \pi_n$ satisfy $|P(\cos(j\pi/n))| \le 1$ for $j = 0,\dots,n$. Then*

(a) $\int_{-1}^1 (|P^{(k)}| - t)_+ \le \int_{-1}^1 (|T_n^{(k)}| - t)_+$ *for every $t \ge 0$;*

(b) $\int_{-1}^1 \varphi(|P^{(k)}|) \le \int_{-1}^1 \varphi(|T_n^{(k)}|)$ *for every nondecreasing convex $\varphi : [0,\infty) \to \mathbb{R}$.*

In particular both hold for every $P \in \pi_n^\circ$.

Proof. Let $y = (P(x_j))_j \in [-1,1]^{n+1}$; then $P = P_y$. (a) is Proposition 5.1 or Theorem 5.2 at the vertices, extended to the cube by Lemma 2.2 (cited). For (b), put $M = T_n^{(k)}(1)$; $|T_n^{(k)}| \le M$ on $[-1,1]$, and $|P^{(k)}| \le M$ by Theorem 2.3 (cited; equivalently, by (a) itself at the vertices — see Remark 5.5 — and the convexity of $y \mapsto \|P_y^{(k)}\|$). Lemma 2.1 (cited) with $f = |P^{(k)}|$, $g = |T_n^{(k)}|$ then gives (b). $\square$

Remark 5.4 (strictness). Problem 1 asks for strict inequality when $P \ne \pm T_n$ and φ is strictly increasing. By the strict part of Lemma 2.1 this follows from $\int |P^{(k)}| < \int |T_n^{(k)}|$, and by Lemma 2.4 that follows from $F_0(v) < F_0(s^*)$ at every non-Chebyshev vertex v. The formal development certifies this strict comparison at one vertex per cell (Theorem 7.1(ii)), namely the vertex that an exact external enumeration ranks largest in L^1 norm among the non-Chebyshev vertices; given that ranking, all non-Chebyshev vertices are strictly below and the strict form of (1) holds at each cell under the nodal hypothesis. The ranking is an external computation, and we do not claim the strict inequality as a formal theorem.

Remark 5.5 (the sup-norm half is old). Inequality (4) at a vertex implies $\|P_s^{(k)}\|_{C[-1,1]} \le T_n^{(k)}(1)$: if $\|P_s^{(k)}\|$ exceeded $M = T_n^{(k)}(1)$, then at any t strictly between them the right side of (4) would vanish while the left side is positive. With the convexity of $y \mapsto \|P_y^{(k)}\|$ this is the bound of Theorem 2.3 for the cell. That bound is Duffin and Schaeffer's, so the sup-norm content of Proposition 5.1, Theorem 5.2 and Corollary 5.3 is not new; what is new is the integrated-tail comparison at every level and its convex-order consequence.

6. Where the level-set route fails

The source proves its vertex bound at $(4,2)$ through the pointwise domination (3), $\Lambda_q \le \Lambda_*$, and then integrates. That route is not available in general: it fails as soon as some vertex derivative q never comes near zero on $[-1,1]$, because $T_n^{(k)}$ has $n-k$ zeros in $(-1,1)$ and so $\Lambda_*(t) < 2 = \Lambda_q(t)$ for small $t > 0$.

Theorem 6.1 (the pointwise level-set route fails at six cells)**.** *For each row of Table 2, with the sign vector s, the level c and the interval $[a,b]$ listed there,*

$$|P_s^{(k)}(x)| > c \ \text{ for all } x \in [-1,1], \qquad \text{and} \qquad |T_n^{(k)}(x)| \le c \ \text{ for all } x \in [a,b].$$

Consequently $\operatorname{meas}\{|P_s^{(k)}| > c\} = 2 > 2-(b-a) \ge \operatorname{meas}\{|T_n^{(k)}| > c\}$*: the domination* (3) *is false at the level $t = c$ at that vertex, although* (4) *holds there.*

Proof sketch. The polynomials $P_s^{(k)}$ and $T_n^{(k)}$ of Table 2 are obtained as identities of coefficient lists in $\mathbb{Q}(\sqrt{D})$ by evaluation; the two pointwise inequalities are then real polynomial inequalities of degree at most two on an interval, closed by a rational lower bound on $\sqrt{D}$ ($\sqrt{2} > 1.414$, $\sqrt{3} > 1.732$, $\sqrt{5} > 2.236$)

(n,k)	m	s	$P_s^{(k)}(x)$	c	$[a,b]$
$(3,2)$	6	$(-,+,+,-)$	$-16/3$	5	$[-1/5,\ 1/5]$
$(4,3)$	6	$(-,+,+,-,-)$	$-12\sqrt{2}$	16	$[-1/12,\ 1/12]$
$(5,3)$	25	$(+,-,-,+,+,-)$	$192x^2+(48\sqrt{5}-72)/5$	7	$[7/20,\ 9/25]$
$(5,4)$	18	$(-,+,-,-,+,-)$	$-(384/5)(1+\sqrt{5})$	248	$[-1/16,\ 1/16]$
$(6,4)$	54	$(-,+,+,-,+,+,-)$	-256	255	$[3/10,\ 17/50]$
$(6,5)$	18	$(-,+,-,-,+,-,-)$	$-640(1+\sqrt{3})$	1700	$[-1/16,\ 1/16]$

Table 2. Certified witnesses against (3). m is the encoding of s of Section 4; $P_s^{(k)}$ is the explicit k-th derivative of the vertex interpolant; $|P_s^{(k)}|>c$ on all of $[-1,1]$ while $|T_n^{(k)}|\le c$ on $[a,b]$.

and elementary arithmetic. No measure theory is used: one of the two level sets is all of $[-1,1]$, and the other misses an interval of positive length. □

Remark 6.2 (the other three cells; external computation). An external exact computation of the level-set functions, compared on a 600-point grid of t, finds (3) violated at some vertex at nine of the ten cells; the number of failing orbit classes is $1,0,1,3,2,4,8,1,1,4$ in the cell order of Table 1, $(4,2)$ being the only cell with none. At $(5,2)$, $(6,2)$ and $(6,3)$ the violations are of the subtler kind, with both level sets proper subsets of $[-1,1]$: at $(5,2)$ all three failing classes ($m=18,22,30$) violate (3) at $t=57/2$, at $(6,2)$ all eight ($m=22,33,34,37,38,41,49,54$) violate it at $t=252/5$, and at $(6,3)$ the one failing class ($m=37$) violates it at $t=300$. Certifying such a violation would need a certified measure of a union of intervals, which was not built; those three failures are reported as external and are not part of the formal development.

Remark 6.3 (consequence for the method). At $(4,2)$ the pointwise order (3) and the integrated order (4) coincide at the vertices, which is why the source's route succeeds there; from $n=5$ on, and already at $(3,2)$ and $(4,3)$, only the integrated order holds. The right general form of step (3) of Section 1.2 is therefore the increasing-convex-order statement (4), not the rearrangement statement (3); any attempt to push Lemma 6 of [1] verbatim to degree 5 or 6 fails, and Theorem 6.1 makes that a checked fact. Nothing here touches the correctness of the source's Theorem 1.

7. Controls

Theorem 7.1 (negative controls at all ten cells). *At each of the ten cells:*

(i) (the bound cannot be shrunk) *for every real* $\lambda<1$, $\lambda\int_{-1}^{1}|T_n^{(k)}|<\int_{-1}^{1}|P_{s^*}^{(k)}|$, *where* s^* *is the Chebyshev vertex; so no right-hand side smaller than* $G(t)$ *by a constant factor survives even at* $t=0$*;*

(ii) (the top non-Chebyshev vertex misses) *for the vertex* $s'=s(m')$ *with* $m'=2,\ 5,\ 5,\ 10,\ 10,\ 10,\ 21,\ 21,\ 21,\ 21$ *in the cell order of Table 1,*

$$\int_{-1}^{1}|P_{s'}^{(k)}(x)|\,dx<\int_{-1}^{1}|T_n^{(k)}(x)|\,dx.$$

Proof sketch. (i) rests on an exact lower bound for $\int|T_n^{(k)}|$ from the cell lower bounds of Theorem 3.3, taken on $N=16,\ 64,\ 16,\ 128,\ 256,\ 32,\ 128,\ 256,\ 64,\ 64$ cells and giving $\int|T_n^{(k)}|\ge 20,\ 45,\ 160,\ 80,\ 500,\ 1700,\ 120,\ 1000,\ 5900,\ 22000$ in the cell order, and on $P_{s^*}^{(k)}=T_n^{(k)}$ (Proposition 4.1(v)). (ii) compares an exact upper bound for $\int|P_{s'}^{(k)}|$ with an exact lower bound for $\int|T_n^{(k)}|$, both from Theorem 3.3, with a rational margin (1/2, 1/10, 1/100 or 1/1000, depending on the cell). The vertex s' is, by the external ranking of Remark 5.4, the non-Chebyshev vertex of largest L^1 norm of its cell; e.g. at $(6,2)$ it is $s'=(+,-,+,-,+,-,-)$, the Chebyshev vector with its last sign flipped. □

Also recorded, not claimed as new (Remark 5.5): every non-Chebyshev vertex has $\|P_s^{(k)}\|_{C[-1,1]}$ strictly below $T_n^{(k)}(1)$ — e.g. $70 < 80$ at $(4,2)$ (the source's Table 1), $184 < 200$ at $(5,2)$, $3696/5 < 840$ at $(5,3)$, $1190/3 < 420$ at $(6,2)$ — from the external enumeration.

8. Duffin–Schaeffer, and what remains

8.1. **The 1941 theorem.** Duffin and Schaeffer [9] showed that in Markov's inequality the condition $|f| \le 1$ on the whole interval "is unnecessarily restrictive": it suffices that $|f| \le 1$ at the $n+1$ points $\cos(k\pi/n)$, the extremal points of T_n, and the conclusions are unchanged (Theorem 2.3). The reduction of Section 1.2 runs on exactly that hypothesis: Lemma 2.2 maximizes over the whole nodal cube, never over π_n°, so what the vertex check delivers is Corollary 5.3 in its stated generality. The hypothesis is strictly weaker than $\|P\|_{C[-1,1]} \le 1$: already the vertex interpolants of Theorem 5.2 leave π_n°, e.g. at $n=3$ the vector $s = (-,+,+,-)$ gives $P_s(x) = (5-8x^2)/3$ with $P_s(0) = 5/3$. Corollary 5.3 is thus the convex-order analogue of the refinement Duffin and Schaeffer made to Markov's inequality.

8.2. **Not found in print.** We looked for a convex-order or integrated-tail statement under the nodal hypothesis and found none. Shadrin's survey [10], Section 5 ("Markov–Duffin–Schaeffer inequalities"), and the descendant line of Bojanov–Nikolov [11], Nikolov [12] and Nikolov–Shadrin [13] are pointwise or sup-norm (majorant-flavoured) throughout; a corpus sweep for "Duffin–Schaeffer" next to convex order, rearrangement or integrated tail returned one hit, the unrelated Duffin–Schaeffer conjecture of metric Diophantine approximation. The most recent paper in the line, Nikolov [14], we read in full: it is sup-norm and pointwise throughout. Its Theorem 2.5 is a pointwise envelope $|f^{(k)}(x)| \le \max\{|Q_n^{(k)}(x)|, |Z_{n,k}(x)|\}$ for all real x; its Theorems 1.4 and 3.7 bound $\|f^{(k)}\|$ by $\|Q_n^{(k)}\|$ in the uniform norm; its Section 4 takes suprema of $\|p^{(k)}\|$ under a pointwise curved majorant. No integral of $|f^{(k)}|$ occurs in it, nor anything of convex-order type in its reference list.

8.3. **The cells $k = n-1$, and Zolotarev.** For $k = n-1$ one has $P^{(n-1)}(x) = n!\,a_n x + (n-1)!\,a_{n-1}$, so the extremal problem is a convex maximization over the region of admissible pairs (a_n, a_{n-1}) of $P \in \pi_n^\circ$, whose boundary is Zolotarev's first problem (Chebyshev's question, solved by Zolotarev in 1868 through elliptic functions); explicit algebraic solutions are known for $2 \le n \le 5$ [18] and by radicals for $n = 6$ [19] and $n = 7$ [20]. A closed-form treatment of general $k = n-1$ along that route is therefore a live possibility that our search did not find, and $n \le 6$ is exactly where it would be easiest; the four cells $(3,2)$, $(4,3)$, $(5,4)$, $(6,5)$ are new on the strength of Naidenov's stated open range — neither [2] nor [1] records $k = n-1$ as settled, and the latter never remarks that its result completes degree four, which it would if $(4,3)$ were known — with this caveat. The cells $2 \le k \le n-2$ carry no such caveat.

8.4. **Beyond.** Three natural continuations, none attempted here. (a) Certify the three numerical failures of Remark 6.2: an interval-list level-set lemma and one list of intervals per witness, no new mathematics. (b) Degrees $n = 7, 8$: $\cos(j\pi/7)$ generates a cubic and $\cos(j\pi/8)$ a quartic field, so the $\mathbb{Q}(\sqrt{D})$ layer must become a general number-field layer, with up to $2^9 = 512$ vertices per cell; the other layers are unchanged. (c) Formalize Lemma 2.2: then Corollary 5.3(a) would be a formal theorem at every cell rather than a cited consequence.

9. Verification

Theorems 3.3, 5.2, 6.1, 7.1 and Propositions 4.1, 5.1 have been formally verified in Lean 4 against *Mathlib* [21, 22]; their statements are reproduced in Appendix A. The development is twelve modules, 3 282 lines in all: a shared module (the $\mathbb{Q}(\sqrt{D})$ arithmetic with its decidable sign test, polynomials of degree at most 4 and 6, the degree-four Bernstein enclosure and its soundness, the uniform partition and the splitting of $\int_{-1}^{1}$, the block hinge inequality, the consumption and matching procedures of Definition 3.2 with Theorem 3.3, the formal derivative with its identification with the real iterated derivative, and the exact L^1 bounds; 3 theorems, 50 lemmas, 38 definitions); one module per cell (the nodes, the Lagrange basis, T_n, the table of pairs (N_A, N_B), and the theorems of Propositions 4.1, 5.1

and Theorem 5.2; 10 theorems and 14 definitions each); and a controls module (56 theorems: per cell the explicit $T_n^{(k)}$, the sharpness identity, the certified lower bound for $\int |T_n^{(k)}|$ and the two controls of Theorem 7.1, and the six witnesses of Theorem 6.1). Finite identities and the certificate checks are discharged by compiled evaluation (`native_decide`): three per cell module (the node identities, the Lagrange identities, and one evaluation covering all 2^{n+1} vertices) and one or two per controls theorem; every such use is on finite rational data — Bernstein coefficient lists and sorted level lists — never on the analytic content. The identification of T_n with Mathlib's Chebyshev polynomial (Proposition 4.1(i)) is proved without compiled evaluation and its axiom print is `propext`, `Classical.choice`, `Quot.sound` only, as is that of the three shared theorems of Theorem 3.3; the axiom print of each cell theorem is those three plus the single `native_decide` axiom of its vertex evaluation, and that of each controls theorem those three plus one or two `native_decide` axioms. There is no `sorry` and no axiom declared by hand. The prints were taken with `#print axioms` over all 99 ledger-bound declarations of the development.

Outside the formal development, and labelled as such where they occur, are the two reproductions of the source's tables and the level-set scans of Proposition 5.1 and Remark 6.2, the orbit counts of Table 1, the L^1 ranking of Remark 5.4, and the search for the pairs (N_A, N_B). As recorded, the whole run — the two enumerations (≈ 6 and ≈ 12 processor-minutes), the shared module (≈ 4), the ten cell modules (≈ 4; between 8 s and 76 s each, the slowest $(6, 2)$ at 76 s wall), the controls module (≈ 1), and the e-print compilation, axiom checks and literature work (≈ 5) — took about 35 processor-minutes, peak resident set about 3 GB with one Lean process at a time. The development and the programs are currently in a private repository: reference to be added at submission.

On the reference list. Statement, equation and table numbers attributed to [1] are those of its compiled e-print (`tectonic` on the e-print source; `main.aux`), which agree with the arXiv PDF: Theorem 1 (p. 2) with (1.2) and (1.3); Lemma 2 (p. 3); Lemma 3 (p. 4); Table 1 (p. 5); Lemma 4 (p. 6); Lemma 5 (p. 7); Lemma 6 (p. 8) with (6.1) and (6.2); Lemma 7 (p. 12); Appendix A and Table 2 (p. 13). The e-print was at version 1 on 2026-09-07. Problem 1 and the reference list of [2] were read in the primary text.

Appendix A. The formal statements

The Lean statements corresponding to the results above, verbatim (statements only; proofs, docstrings and attributes omitted). `QS` is $\mathbb{Q}(\sqrt{D})$ as pairs of rationals, `ev D z` the real number $z_1 + z_2\sqrt{D}$; `P4` and `P7` are polynomials of degree at most 4 and 6 with `QS` coefficients, `peval` and `peval7` their real evaluations; `d7` is the formal derivative; `cellUpper`/`cellLower` are the lists $(c_i)/(C_j)$ of Definition 3.1, `massQS r l` is $r \sum l$ in `QS`, and `checkVertex` is Definition 3.2. In a cell module, `nn`, `kk`, `D` are n, k, D; `node`, `lag`, `cheb` the nodes, the Lagrange basis and T_n as coefficient lists; `vpoly m` is $P_{s(m)}$; `chebK` and `vder m` are the formal k-th derivatives of `cheb` and `vpoly m`, truncated to degree 4 (the truncation is checked to be exact). The ten cell modules `C32`, `C42`, `C43`, `C52`, `C53`, `C54`, `C62`, `C63`, `C64`, `C65` are textually identical apart from their data; the statements are shown for `C52` with the differing constants tabulated.

Shared module (Theorem 3.3).

```
theorem tail_le_of_check {D : ℚ} (hD : 0 ≤ D) (q qst : P4) (NA NB : ℕ)
    (h : checkVertex D q qst NA NB = true) (t : ℝ) (ht : 0 ≤ t) :
    (∫ x in (-1:ℝ)..1, max 0 (|peval D q x| - t))
      ≤ ∫ x in (-1:ℝ)..1, max 0 (|peval D qst x| - t)

theorem L1_le {D : ℚ} (hD : 0 ≤ D) (p : P4) {N : ℕ} (hN : 0 < N) :
    (∫ x in (-1:ℝ)..1, max 0 (|peval D p x| - 0))
      ≤ ev D (massQS (2/(N:ℚ)) (cellUpper D p N))

theorem le_L1 {D : ℚ} (hD : 0 ≤ D) (p : P4) {N : ℕ} (hN : 0 < N) :
    ev D (massQS (2/(N:ℚ)) (cellLower D p N))
      ≤ ∫ x in (-1:ℝ)..1, max 0 (|peval D p x| - 0)
```

Cell modules (Propositions 4.1, 5.1, Theorem 5.2), shown for `C52`.

```
def D : ℚ := 5
def nn : ℕ := 5
def kk : ℕ := 2
def node : List QS := [(1, 0), (1/4, 1/4), (-1/4, 1/4), (1/4, -1/4),
                       (-1/4, -1/4), (-1, 0)]
def cheb : P7 := ⟨(0, 0), (5, 0), (0, 0), (-20, 0), (0, 0), (16, 0), (0, 0)⟩

theorem cheb_eq (x : ℝ) : peval7 D cheb x = (Polynomial.Chebyshev.T ℝ 5).eval x

theorem node_cheb {j : ℕ} (hj : j < nn + 1) :
    peval7 D cheb (ev D (node.getD j ((0:ℚ),(0:ℚ))))
      = if j % 2 = 0 then (1:ℝ) else -1

theorem lag_node {i j : ℕ} (hi : i < nn + 1) (hj : j < nn + 1) :
    peval7 D (lag.getD j p7zero) (ev D (node.getD i ((0:ℚ),(0:ℚ))))
      = if i = j then (1:ℝ) else 0

theorem vpoly_node {m i : ℕ} (hm : m < 2^(nn+1)) (hi : i < nn + 1) :
    peval7 D (vpoly m) (ev D (node.getD i ((0:ℚ),(0:ℚ))))
      = if m.testBit i then (1:ℝ) else -1

theorem vertex_tail {m : ℕ} (hm : m < 2^(nn+1)) {t : ℝ} (ht : 0 ≤ t) :
    (∫ x in (-1:ℝ)..1, max 0 (|iteratedDeriv kk (peval7 D (vpoly m)) x| - t))
      ≤ ∫ x in (-1:ℝ)..1, max 0 (|iteratedDeriv kk (peval7 D cheb) x| - t)
```

The per-cell data: `(D, nn, kk)` is $(2,3,2)$, $(2,4,2)$, $(2,4,3)$, $(5,5,2)$, $(5,5,3)$, $(5,5,4)$, $(3,6,2)$, $(3,6,3)$, $(3,6,4)$, $(3,6,5)$; `node` is

```
[(1, 0), (1/2, 0), (-1/2, 0), (-1, 0)]                              -- n = 3
[(1, 0), (0, 1/2), (0, 0), (0, -1/2), (-1, 0)]                      -- n = 4
[(1, 0), (0, 1/2), (1/2, 0), (0, 0), (-1/2, 0), (0, -1/2), (-1, 0)]  -- n = 6
```

and the list above for $n = 5$; `cheb` is $4x^3 - 3x$, $8x^4 - 8x^2 + 1$, $16x^5 - 20x^3 + 5x$, $32x^6 - 48x^4 + 18x^2 - 1$ as coefficient lists, and `cheb_eq` names `Polynomial.Chebyshev.T ℝ n` with the cell's n. In the statement of `vertex_tail`, `iteratedDeriv kk` is Mathlib's k-th iterated derivative of a real function.

Controls module (Proposition 4.1(v), Theorems 6.1, 7.1), shown for the cell $(5, 3)$.

```
theorem chebK_eq_53 : C53.chebK = ⟨(-120, 0), (0, 0), (960, 0), (0, 0), (0, 0)⟩
theorem cheb_is_vertex_53 : C53.vder 21 = C53.chebK

theorem rhs_not_improvable_53 {lam : ℝ} (h : lam < 1) :
    lam * (∫ x in (-1:ℝ)..1, max 0 (|peval C53.D C53.chebK x| - 0))
      < ∫ x in (-1:ℝ)..1, max 0 (|peval C53.D (C53.vder 21) x| - 0)

theorem strict_at_53 :
    (∫ x in (-1:ℝ)..1, max 0 (|peval C53.D (C53.vder 10) x| - 0))
      < ∫ x in (-1:ℝ)..1, max 0 (|peval C53.D C53.chebK x| - 0)

theorem level_route_fails_53 :
    (∀ x ∈ Set.Icc (-1:ℝ) 1, ((7 : ℝ)) < |peval C53.D (C53.vder 25) x|)
    ∧ (∀ x ∈ Set.Icc ((7/20 : ℝ)) ((9/25 : ℝ)),
         |peval C53.D C53.chebK x| ≤ ((7 : ℝ)))
```

The other cells differ only in the constants: `chebK_eq` states the coefficient list of $T_n^{(k)}$ of Table 1; `cheb_is_vertex` and `rhs_not_improvable` use $m^* = 5$ for $n = 3$, 21 for $n = 4, 5$ and 85 for $n = 6$; `strict_at` uses $m' = 2, 5, 5, 10, 10, 10, 21, 21, 21, 21$; and `level_route_fails_32`, `_43`, `_54`, `_64`, `_65` have the shape shown with the vertex m, the level c and the interval $[a, b]$ of Table 2, over `Set.Icc (-1:ℝ) 1` and `Set.Icc (a:ℝ) (b:ℝ)`.

References

[1] G. Zavalani, *The Bojanov–Naidenov inequality for quartics and second derivatives*, arXiv:2606.23020v1 [math.NA], 22 June 2026, 14 pp. Settles $(n, k) = (4, 2)$; read in full, statement numbers from the compiled e-print (Section 9).

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