MULTIPLICATIVE SUPERADDITIVITY AND DIMENSION-FREE MOURRE BANDS FOR THE MOLCHANOV–VAINBERG LAPLACIAN

KOREA SUPERINTELLIGENCE LABS

ABSTRACT. Let $D = \prod_{i=1}^{d} \Delta_i$ be the Molchanov–Vainberg Laplacian on $\ell^2(\mathbb{Z}^d)$, $\Delta_i = (S_i + S_i^*)/2$. For the conjugate operators $\mathbb{A} = \sum_j \rho_{j\kappa} A_{j\kappa}$ used by Golénia and Mandich, a strict Mourre estimate at an energy E is equivalent to strict positivity of an explicit Chebyshev polynomial G_κ on the constant-energy surface $S_E = \{x \in [-1,1]^d : \prod_l x_l = E\}$. Golénia and Mandich proved that $G_\kappa|_{S_E}$ is *separable* — $G_\kappa|_{S_E}(x) = E \sum_i P(x_i^2)$ for a single-variable P with $P(1) = 0$ — and deduced from it that raising the dimension can only shrink the set of good energies. Nothing was known about the converse, and Mandich, who treats $d = 2$ throughout, re-runs his $d = 3$ numerics with the two-dimensional coefficients and conjectures that they still work, calling the phenomenon "quite mysterious and surprising".

We isolate the property that governs the converse: P is *multiplicatively superadditive* on a region, $P(uv) \le P(u) + P(v)$. When it holds, the minimum of $\sum_i P(u_i)$ under $\prod_i u_i = z$ equals $P(z)$ in every dimension, and is attained at $(z, 1, \dots, 1)$. For $\kappa = 4$ and Mandich's first band — $\Sigma = \{4, 8\}$, $\rho_8 = (17 + 8\sqrt{5})/62$ — we prove multiplicative superadditivity on $[1/10, 1]$ and conclude, for every $d \ge 1$ and every $E \in (0, 1]$ with $E^2 \ge 1/10$, that $\min_{S_E} G_4 = E\,P(E^2)$, a number in which d does not appear. Hence $G_4|_{S_E} > 0$ holds in one dimension if and only if it holds in every other; inside that range the positivity set is exactly $(\mathcal{E}_1, \cos(\pi/4)) \cup (\sqrt{\mathcal{E}_1}, 1)$, $\mathcal{E}_1 = (\sqrt{5} - 1)/2$, in all dimensions at once. In particular the $n = 1$ case of Mandich's Conjecture 1.15 holds in every dimension, at the level of strict positivity to which his own equivalence reduces it, with a sharp and attained constant.

The same works for his *second* band, $\Sigma = \{4, 8, 12, 28\}$, whose coefficients he gives exactly in $\mathbb{Q}(\sqrt{5})$: its degree-14 polynomial P_2 is multiplicatively superadditive on $[1/3, 1]$, so $\min_{S_E} G_4^{(2)} = E\,P_2(E^2)$ whenever $E^2 \ge 1/3$, in every dimension, $G_4^{(2)}$ being the combination those coefficients build. The $n = 2$ case of the conjecture follows in the same sense, and at $d = 2$ that band, certified until now on 93.40% of its length, is closed. The mechanism is a bilinear kernel identity dividing out the double zero forced by $P(1) = 0$, followed by exact decompositions over $\mathbb{Z}[\mathcal{E}_1]$ at the interior zeros of the kernel — one for the first band, two for the second — and integer Bernstein certificates. The *two-sided* band set is genuinely dimension-dependent, so the results are stated — as in Mandich — for strict positivity. All polynomial statements are machine-checked in Lean 4; the passage from them to the Mourre estimate itself is the sources' functional calculus, quoted rather than reproved.

1. Introduction

The operator, and the one formula everything runs on. The Molchanov–Vainberg Laplacian on $\mathcal{H} = \ell^2(\mathbb{Z}^d)$ is

$$D \;=\; \prod_{i=1}^{d} \Delta_i, \qquad \Delta_i := \tfrac{1}{2}(S_i + S_i^*),$$

with S_i the shift in the i-th coordinate; $\sigma(D) = [-1, 1]$. It was introduced by Molchanov and Vainberg [4], and its spectral theory under long-range perturbations was developed by Golénia and Mandich [2] and, in the form used here, by Mandich [1]. The single formula on which the present paper rests is *theirs*: Golénia and Mandich prove [2, §4, eq. (4.4)] that the polynomial attached by functional calculus to the commutator $[D, i\mathbb{A}]$, restricted to a constant-energy surface, is *separable* — a product times a sum of one even single-variable polynomial per coordinate,

$$g_E(E_1, \dots, E_d) \;=\; E \sum_{j=1}^{d} (-1)^{\kappa_j/2-1} 2^{\kappa_j - 1} \prod_{l=1}^{\kappa_j/2} \big(\sin^2(l\pi/\kappa_j) - E_j^2\big),$$

and they use it, in the proof of the same section's Lemma 4.7, in exactly the form "since $g_E(E_1, \dots, E_d) = E \sum_{j=1}^{d} h_j(E_j)$, we derive the statement". We claim nothing about that identity; every d-dimensional statement below is read off it. What we add is the inequality that turns separability into a statement about *all* dimensions at once, and certificates for it in the two cases — Mandich's first and second bands at $\kappa = 4$ — in which he computes the coefficients of $\mathbb{A}$ exactly.

Mourre theory and the polynomial criterion. Let V be a real potential vanishing at infinity and satisfying, for a fixed even κ and some $q > 2$, the long-range condition $n_i(V - \tau_i^{\kappa} V)(n) = O(\ln^{-q} |n|)$ in every coordinate direction. Mourre theory [5, 6] produces a limiting absorption principle for $D + V$ near an energy E out of a *strict Mourre estimate*

$$1_I(D)\, [D, i\mathbb{A}]_\circ\, 1_I(D) \;\geq\; \gamma\, 1_I(D), \qquad \gamma > 0,\; I \ni E, \tag{1}$$

for a self-adjoint conjugate operator $\mathbb{A}$, together with a potential-side condition; in the present setting the resulting implication is [1, Theorem 1.1]. The conjugate operators matched to that long-range condition are the finite combinations

$$\begin{aligned} \mathbb{A} = \sum_{j \geq 1} \rho_{j\kappa} A_{j\kappa}, \qquad A_{j\kappa} = \sum_{i=1}^{d} A_i(j, \kappa), \\ A_i(j, \kappa) = \tfrac{1}{4i}\big[(S_i^{j\kappa} - S_i^{-j\kappa}) N_i + N_i (S_i^{j\kappa} - S_i^{-j\kappa})\big], \end{aligned} \tag{2}$$

N_i the i-th position operator. Writing U_n for the Chebyshev polynomial of the second kind, functional calculus attaches to $[D, i\mathbb{A}]_\circ$ the polynomial

$$g_{j\kappa}(x) = \sum_{i=1}^{d} \Big(\prod_{l} x_l\Big) x_i^{-1} (1 - x_i^2) U_{j\kappa - 1}(x_i), \qquad G_\kappa = \sum_{j \geq 1} \rho_{j\kappa}\, g_{j\kappa}, \tag{3}$$

and, with $S_E := \{x \in [-1, 1]^d : \prod_l x_l = E\}$, the source states the criterion as an equivalence: "By functional calculus and continuity of the function G_κ we have $E \in \boldsymbol{\mu}_\kappa(D)$ iff $G_\kappa|_{S_E} > 0$" [1, §1, after (1.10)]. Here $\boldsymbol{\mu}_\kappa(D)$ is the set of energies at which (1) holds for *some* $\mathbb{A}$ of the form (2), and $\boldsymbol{\Theta}_\kappa(D) = \sigma(D) \setminus \boldsymbol{\mu}_\kappa(D)$ is the

set of thresholds. The coefficients in (2) range over all of $\mathbb{R}$, so $-\mathbb{A}$ is admissible whenever $\mathbb{A}$ is; the quoted equivalence is therefore to be read for a *fixed* $\mathbb{A}$, and that is how we use it. All section, equation, lemma and table numbers we quote from [1] and [2] are those of their arXiv versions v1.

What is known about the dimension, and what is not. Separability answers the easy direction immediately, and [2, Lemma 4.3] records the answer: appending a coordinate $E_d = 1$ embeds the $(d-1)$-dimensional problem in the d-dimensional one, so

$$\boldsymbol{\mu}_{A_{\kappa[d]}}(D_d) \;\subset\; \boldsymbol{\mu}_{A_{\kappa[d-1]}}(D_{d-1})$$

for every $d \geq 2$ — higher dimension is *a priori* harder. The same section contains the one published dimension-uniform *positivity* statement, its lemma `edgySP` [2, Lemma 4.7]: for every $d \geq 1$ and every multi-index $\kappa = (\kappa_j)$ with all κ_j even and all $\kappa_j/2$ of the same parity,

$$\boldsymbol{\mu}_{A_\kappa}(D) \;\supset\; \pm\big(\cos(\pi/\kappa^*), 1\big), \qquad \kappa^* := \max_j \kappa_j$$

(κ^* is defined inside their proof). It is the case in which the summands h_j of the separable form all have one and the same sign on the relevant range — which is what the parity hypothesis buys — so that no inequality between them is needed at all; for $\kappa_j \equiv 4$ they are all positive. For the uniform choice $\kappa_j \equiv 4$ it reads $\boldsymbol{\mu}_{A_4}(D) \supset \pm(\cos(\pi/4), 1)$, an interval that *contains* the one appearing in Theorem 6.1(ii), $(\sqrt{\mathcal{E}_1}, 1) = (0.78615\ldots, 1)$. So (ii) below is not new as a membership statement about $\boldsymbol{\mu}_4(D)$ — there it is weaker than [2] — and what it adds is the polynomial statement for the linear combination G_4, with an attained constant. Part (i), the band $(\mathcal{E}_1, \cos(\pi/4))$, is disjoint from [2]'s interval. The same reading applies to the second band: of the three intervals of Theorem 9.10, the two upper ones $(\cos(\pi/4), \sqrt{\mathcal{E}_1})$ and $(\sqrt{2/3}, 1)$ sit inside $(\cos(\pi/4), 1)$ and are therefore not new as membership statements either, while the second band $(\mathcal{E}_2, \mathcal{E}_1) = (0.57735\ldots, 0.61803\ldots)$ is disjoint from it.

Beyond that, nothing. Mandich [1] works in $d = 2$ throughout and says so: "We are done discussing $d = 2$. In higher dimensions we have only 1 general result: thresholds in dimension d generate thresholds in dimension $d + 1$, via scaling" — a statement about the threshold set $\boldsymbol{\Theta}$, not about $\boldsymbol{\mu}$. He then re-runs the $\kappa = 4$ computation in $d = 3$ with the two-dimensional coefficients and finds that it still works, and conjectures that this is general [1, Conjecture 1.15]: fix $\kappa = 4$; for each band $(\mathcal{E}_n, \mathcal{E}_{n-1})$, $n \geq 1$, the Mourre estimate holds in $d = 3$ with $\mathbb{A}(n) = \sum_q \rho_{j_q\kappa}(n) A_{j_q\kappa}$, $A_{j_q\kappa} = \sum_{1\leq i\leq 3} A_i(j_q, \kappa)$, for all $E \in (\mathcal{E}_n, \mathcal{E}_{n-1})$, "where the coefficients $\rho_{j_q\kappa}(n)$ are *exactly* those used in the 2-dimensional case". The conjecture is prefaced by: "graphical evidence also suggests the following conjecture, although it is quite mysterious and surprising to me how and why it happens".

Results. Write $z = E^2$ for the squared energy and, for $\kappa = 4$ and Mandich's first band ($\Sigma = \{4, 8\}$, $\rho_4 = 1$, $\rho_8 = (17 + 8\sqrt{5})/62$),

$$P(u) \;=\; (1-u)(8u-4) \;+\; \rho_8\,(1-u)\big(128u^3 - 192u^2 + 80u - 8\big), \tag{4}$$

so that $G_4|_{S_E}(x) = E\sum_{i=1}^d P(x_i^2)$ and $P(1) = 0$. Our results are the following.

(i) A criterion, in the abstract. If $P(1) = 0$ and P is multiplicatively superadditive on $[c, 1]$ — $P(uv) \leq P(u) + P(v)$ for $u, v \in [c, 1]$ — then for every $d \geq 1$

the least value of $\sum_{i<d} P(u_i)$ over $u \in [0,1]^d$ with $\prod_i u_i = z \geq c$ is exactly $P(z)$, attained at $(z, 1, \ldots, 1)$ (Theorems 3.2 and 3.3). The bound involves no d.

*(ii) The inequality, for the band-*1 *polynomial.* P of (4) is multiplicatively superadditive on $[1/10, 1]$ (Theorem 5.1).

(iii) Dimension-independence. For every $d \geq 1$ and every $E \in (0,1]$ with $E^2 \geq 1/10$, $\min\{G_4(x) : x \in [-1,1]^d,\ \prod_i x_i = E\} = E\,P(E^2)$, so strict positivity of G_4 on S_E holds for one $d \geq 1$ if and only if it holds for every $d \geq 1$ (Theorem 5.2). The sign of P is a four-factor check (Proposition 5.3), and within that range of E the positivity set is $(\mathcal{E}_1, \cos(\pi/4)) \cup (\sqrt{\mathcal{E}_1}, 1)$ in every dimension.

(iv) The conjecture, for the first band, in every dimension. For every $d \geq 1$, $G_4|_{S_E} > 0$ on the whole open first band $(\mathcal{E}_1, \cos(\pi/4))$ and on $(\sqrt{\mathcal{E}_1}, 1)$ (Theorem 6.1). The case $d = 3$ of the first half is [1, Conjecture 1.15] for $n = 1$, at the level his own equivalence reduces it to; the second half is, as a membership statement, already [2, Lemma 4.7]. The estimate is sharp: the minimiser is $(E, 1, \ldots, 1)$, so no larger constant holds.

The remaining four are about the *second* band, $(\mathcal{E}_2, \mathcal{E}_1)$ with $\mathcal{E}_2 = 1/\sqrt{3}$, for which Mandich's index set is $\Sigma = \{4, 8, 12, 28\}$ and his coefficients $\rho_8^{(2)}, \rho_{12}^{(2)}, \rho_{28}^{(2)}$ lie in $\mathbb{Q}(\sqrt{5})$; write P_2 for the band polynomial they build and $G_4^{(2)}$ for the corresponding combination (3).

(v) The second band's polynomial. P_2 has degree 14, and over $\mathbb{Z}[\mathcal{E}_1]$

$$D \cdot P_2(u) = (1-u)(2u-1)(3u-1)(3u-2)(u-\mathcal{E}_1)(u-\mathcal{E}_1^2)\,Q(u),$$

where $D = 145241499888481405699961$ is the common denominator of (8) and Q has degree 8 and is strictly positive on $[1/3, 1]$ (Proposition 9.1). The six displayed roots are $\mathcal{E}_2^2 = 1/3$, $\mathcal{E}_1^2$, $\mathcal{E}_0^2 = 1/2$, $\mathcal{E}_1$, $2/3$ and 1; only two of them are imposed by Mandich's interpolation system, and the other four come for free — one from the factor $1-u$ and three from a reflection symmetry of the Chebyshev blocks that we record because it is uniform in the band index (Proposition 9.2).

*(vi) The inequality, for the band-*2 *polynomial.* P_2 is multiplicatively superadditive on $[1/3, 1]$ (Theorem 9.7). Its kernel vanishes at three interior points of that square, so the single decomposition that sufficed for band 1 is replaced by two of them and a partition of the square into eight boxes.

(vii) Dimension-independence, again. For every $d \geq 1$ and every $E \in (0,1]$ with $E^2 \geq 1/3$, $\min\{G_4^{(2)}(x) : x \in [-1,1]^d,\ \prod_i x_i = E\} = E\,P_2(E^2)$ (Theorem 9.8); within that range the positivity set is $(\mathcal{E}_2, \mathcal{E}_1) \cup (\cos(\pi/4), \sqrt{\mathcal{E}_1}) \cup (\sqrt{2/3}, 1)$ in every dimension (Proposition 9.9).

(viii) The conjecture, for the second band, in every dimension. $G_4^{(2)}|_{S_E} > 0$ on the whole open second band for every $d \geq 1$ (Theorem 9.10); the case $d = 3$ is [1, Conjecture 1.15] for $n = 2$, again at the level his own equivalence reduces it to. At $d = 2$ this closes the certified part of that band from the 93.40% of the companion manuscript [12] to all of it (Corollary 9.12).

Scope: positivity, not the two-sided set — and it is not a formality. [2] works with $\boldsymbol{\mu}_{\mathbb{A}} = \boldsymbol{\mu}_{\mathbb{A}}^{+} \cup \boldsymbol{\mu}_{\mathbb{A}}^{-}$, a strict estimate with $\mathbb{A}$ *or* with $-\mathbb{A}$, and *that* set genuinely moves with the dimension. Its $\boldsymbol{\mu}^{-}$ half is a *maximum* problem, and its moving endpoint is the value [2] conjectures for it, $\sin^d(\pi/4) = 2^{-d/2}$: with $P_0(u) = 4(1-u)(2u-1)$ the analogue of (4) for $\rho_8 = 0$, at $E = 2^{-d/2}$ the symmetric point $u_i = E^{2/d} = 1/2$ of the surface has $\sum_i P_0(u_i) = 0$ exactly, because

$P_0(1/2) = 0$. A criterion built on minima is blind to that half by construction. We prove an instance rather than hedge: for the simplest conjugate operator $\mathbb{A} = A_4$ and the energy $E = 9/20$, G_4 is strictly negative on the whole of S_E in $d = 2$ while it changes sign on S_E in $d = 3$ (Theorem 7.1), which reproduces from first principles the $\kappa = 4$ rows of [2, Tables 15, 17] (reprinted as [1, Table 5]): $(0, 0.5)$ in $d = 2$ against $(0, 0.3535)$ in $d = 3$. Accordingly every statement in this paper is about *strict positivity* of G_κ on S_E with the coefficients fixed — which is the form in which [1] states its criterion and the form its Conjecture 1.15 takes. It is a genuine restriction relative to both sources, whose $\boldsymbol{\mu}$'s quantify over the sign as well.

Method. Because $P(1) = 0$, multiplicative superadditivity of P on a region is *equivalent* to the nonnegativity there of a bilinear kernel W with

$$P(u) + P(v) - P(uv) \;=\; 64\rho_8\,(1-u)(1-v)\,W(u,v), \tag{5}$$

and the content of (5) is that it divides out the double zero that $P(1) = 0$ forces on the lines $u = 1$ and $v = 1$, leaving a problem that is non-degenerate there. It does not remove the obstruction that matters: W vanishes to second order at the *interior* point $(\mathcal{E}_1, \mathcal{E}_1)$, because $\mathcal{E}_1 \cdot \mathcal{E}_1 = \mathcal{E}_1^2$ and all three of $\mathcal{E}_1, \mathcal{E}_1^2, 1$ are roots of P, and no Bernstein box certificate can accept a box whose interior contains an attained zero (§8). What removes it is an exact decomposition over $\mathbb{Z}[\mathcal{E}_1]$, in which the single relation $\mathcal{E}_1^2 = 1 - \mathcal{E}_1$ is the only algebra used:

$$W(u,v) = S(u)(u-b)^2 + S(v)(v-b)^2 + (u-b)(v-b)K(u,v),$$

where $b = \mathcal{E}_1$ and $S(u) = (2+4b)u = 2\sqrt{5}\,u$. Since S is a *monomial*, completing the square in $(v - b)$ leaves two problems with no zero at all: $K \geq 0$ on $[b, 1]^2$, where the nine coefficients of the shifted expansion are all positive and no certificate is needed, and $80uv - K^2 \geq 0$ on $[1/10, b] \times [1/10, 1]$, one Bernstein array of 175 integer coefficients at multidegree $(4, 4, 6)$, with *no subdivision*.

For the second band the same three steps run, with one difference that decides the shape of the whole argument: the kernel of P_2 vanishes at *three* interior points of $[1/3, 1]^2$, namely $(\mathcal{E}_1, \mathcal{E}_1)$, $(1/2, 2/3)$ and $(2/3, 1/2)$, and two of them are off the diagonal. That the S of (12) came out a monomial was, for band 1, an accident; here it is replaced by a construction that needs no luck and no search. At a zero $z = (z_1, z_2)$ of a kernel W that is nonnegative near z, the one-variable restriction $W(\cdot, z_2)$ has a double root at z_1, so $S_1 := W(\cdot, z_2)/(u - z_1)^2$ and $S_2 := W(z_1, \cdot)/(v - z_2)^2$ are again polynomials; the difference $W - S_1(u)(u - z_1)^2 - S_2(v)(v - z_2)^2$ vanishes on both lines $u = z_1$ and $v = z_2$ and hence is divisible by $(u - z_1)(v - z_2)$. The two decompositions of §9 are that one observation, applied at $(\mathcal{E}_1, \mathcal{E}_1)$ and at $(1/2, 2/3)$; the third zero is the mirror of the second. The two local decompositions cover two boxes, and six further boxes with no zero in them cover the rest of the square.

Relation to the classical superadditivity literature. Under $u = e^{-s}$ the condition $P(uv) \leq P(u) + P(v)$ on $(0, 1]$ is ordinary superadditivity of $\varphi(s) = -P(e^{-s})$ on $[0, \infty)$, with $\varphi(0) = 0$. The classical implication chain *convex* $\Rightarrow$ *star-shaped* $\Rightarrow$ *superadditive* is Bruckner and Ostrow [7, Theorem 5]. They state it for functions on $[0, \infty)$ that are continuous, nonnegative and vanish at the origin — their standing convention — nonnegativity being a hypothesis that in their words "simplifies many proofs which could be given without this assumption by considering the sum of f

with a suitably chosen linear function"; and indeed all six of their classes are invariant under adding a linear function. Their Lemma 3 characterises star-shapedness by either of two conditions, the first being that $f(x)/x$ is increasing. Bruckner [8] states the star-shaped $\Rightarrow$ superadditive implication in his introduction, on an interval $[0, a]$, and supplies tests; Beckenbach [9] and Johnson [10] develop the strand concerned with superadditivity on a *bounded* interval, the closest classical relative of what is done here. We introduce neither notion. The classical tests are nevertheless unavailable, for a reason that is itself a theorem below. Our φ is continuous and bounded on $[0, \infty)$, vanishes at 0 and is differentiable there, so $\varphi(s) + cs$ is nonnegative for c large enough and then lies in the class [7, Theorem 5] treats; and φ is *not* superadditive there, since $P(uv) > P(u) + P(v)$ at $u = v = 1/11$ (Proposition 6.3(iii)). By that theorem it is therefore neither star-shaped nor convex. No global sufficient condition can produce the inequality; it holds on a restricted region only. The cheap coefficient form of the convex case fails independently: writing $P/(64\rho_8) = \sum_k c_k u^k$ one has $c = (2 - \sqrt{5},\ -5 + 3\sqrt{5},\ -2\sqrt{5},\ 5,\ -2)$, and $c_k \le 0$ for $k \ge 1$ is violated twice, at $c_1 = 3\sqrt{5} - 5 > 0$ and $c_3 = 5 > 0$. The same holds for the band-2 polynomial, and for the same reason: $P_2(uv) > P_2(u) + P_2(v)$ at $u = v = 3/50$ (Proposition 9.13(iii)), so $s \mapsto -P_2(e^{-s})$ is not superadditive on $[0, \infty)$ either, and no global test can deliver the restricted-region statement.

Provenance. Checked 2026-09-02, re-checked 2026-09-03. [1] stands at v1 (submitted 2022-01-02), is unpublished and, on a citation search, uncited; [3], its standard-Laplacian companion, likewise stands at v1 and carries the analogous $d = 3$ conjecture at $\kappa = 2$. Neither author has published on this operator since January 2022. A full-text search of a 91 GB arXiv corpus finds no paper connecting Mourre theory to superadditivity, and no occurrence at all of "multiplicatively superadditive"; the string `Molchanov-Vainberg` returns exactly three arXiv papers, one of them unrelated. The sweep was repeated on 2026-09-03 for the second band, against a re-sharded corpus of 86 GB: "multiplicatively sub/superadditive" still has zero occurrences; of the 42 papers in which *superadditiv* occurs within 200 characters of *multiplicativ*, every matched string is "superadditivity and sub/supermultiplicativity", two separate properties of one function rather than the composite condition $P(uv) \le P(u) + P(v)$; the regular expression `molchanov.{0,2}vainberg` matches 28 papers, three of them the sources here and the other 25, resolved by title and abstract, concerned with different operators; and the 24-digit denominator D of (8) occurs in exactly one paper of the corpus, [1] itself, so nobody has reused the second band's exact interpolation data. The arXiv corpus does not reach pre-1992 journals, books or theses, so the review literature was searched separately: a zbMATH Open search on *superadditive* together with *multiplicative* (2026-09-03) returns nothing on $f(uv) \le f(u) + f(v)$, and nothing on the classical chain restricted to a multiplicative region. Its nearest hits are about $f(uv)$ against $f(u)f(v)$ rather than against $f(u) + f(v)$: Dhombres, *Sur les fonctions simultanément suradditives et surmultiplicatives* (1983), and Finol–Wójtowicz, *Multiplicative properties of real functions with applications to classical functions* (2000). MathSciNet sits behind institutional access and could not be reached. What is asserted is only that no paper we could reach states the criterion or the dimension-independence.

2. Preliminaries

2.1. **Chebyshev data and the band polynomial.** U_n is the Chebyshev polynomial of the second kind; in the formal development $U_0, \dots, U_7$ are derived from the three-term recurrence. For even κ the index $j\kappa - 1$ is odd, so $U_{j\kappa-1}(y) = y\,V_j(y^2)$ for a polynomial V_j; at $\kappa = 4$,

$$U_3(y) = 8y^3 - 4y = y(8y^2 - 4), \qquad U_7(y) = 128y^7 - 192y^5 + 80y^3 - 8y,$$

the latter being $y\big(128y^6 - 192y^4 + 80y^2 - 8\big)$, so $V_1(t) = 8t - 4$ and $V_2(t) = 128t^3 - 192t^2 + 80t - 8$. Consequently the factor x_i^{-1} in (3) cancels against the x_i inside the odd $U_{j\kappa-1}$, and $g_{j\kappa}$ has a division-free form,

$$\Big(\prod_{l\neq i} x_l\Big)(1 - x_i^2)U_{j\kappa-1}(x_i) = \Big(\prod_{l} x_l\Big)(1 - x_i^2)V_j(x_i^2), \tag{6}$$

valid with no hypothesis; the formal development uses the left-hand form and proves that it agrees with the source's expression wherever the latter is defined.

The band data are Mandich's. The first $\kappa = 4$ band is $(\mathcal{E}_1, \mathcal{E}_0) = \big((\sqrt{5} - 1)/2, \cos(\pi/4)\big)$ with $\Sigma = \{4, 8\}$, $\rho_4 = 1$ and $\rho_8 = (17 + 8\sqrt{5})/62$ [1, Tables 1, 2]. Throughout we abbreviate $b := \mathcal{E}_1 = (\sqrt{5} - 1)/2$, whose defining relation is

$$b^2 = 1 - b, \qquad \text{equivalently } \mathcal{E}_1 + \mathcal{E}_1^2 = 1, \tag{7}$$

and every algebraic identity below is an identity of $\mathbb{Z}[b]$ modulo (7) and of nothing else. Note $\sqrt{5} = 2b + 1$ and $\rho_8 = (25 + 16b)/62$.

2.2. **The second band's data.** Mandich writes his n-th band as $(\mathcal{E}_n, \mathcal{E}_{n-1})$ with $\mathcal{E}_0 = \cos(\pi/\kappa)$; for $\kappa = 4$ he prints $\mathcal{E}_0 = \cos(\pi/4)$ in [1, Table 1] and the closed forms $\mathcal{E}_1 = (5^{1/2} - 1)/2 \simeq 0.61803$ and $\mathcal{E}_2 = 3^{-1/2} \simeq 0.57735$ in [1, (6.3)]. So the second band is

$$(\mathcal{E}_2, \mathcal{E}_1) = \big(1/\sqrt{3},\ (\sqrt{5} - 1)/2\big) = (0.577350\dots,\ 0.618033\dots).$$

For it he takes $\Sigma = \{4, 8, 12, 28\}$, fixes $\rho_4 = 1$, and solves the three interpolation conditions his system [1, (11.2)] imposes at $n = 2$ — $G_4^{\mathcal{E}_2}(X_{0,2}) = 0$, $\frac{d}{dx}G_4^{\mathcal{E}_2}(X_{1,2}) = 0$, $G_4^{\mathcal{E}_1}(X_{0,1}) = 0$ — in the three unknowns $\rho_8, \rho_{12}, \rho_{28}$. This is the case he singles out: "We noticed that only for very small values of κ and n is Python able to produce exact solutions. For example, for $\kappa = 4$, $n = 2$ we have a rare oc[c]urrence of an exact solution" [1, §12]. The printed solution, which we write with a superscript to keep it apart from the band-1 coefficient $\rho_8 = (17 + 8\sqrt{5})/62$, is

$$\begin{aligned}
\rho_8^{(2)} &= \tfrac{12221168645528524217139 2 - 3260892257389038020608\sqrt{5}}{D} &&= 0.791235\dots,\\
\rho_{12}^{(2)} &= \tfrac{3077650500266123893602 - 233195044949152756416\sqrt{5}}{13203772717134673245451} &&= 0.193597\dots,\\
\rho_{28}^{(2)} &= \tfrac{2012748316225734218931 + 900129194439352172352\sqrt{5}}{D} &&= 0.027716\dots,
\end{aligned} \tag{8}$$

with the common denominator

$$D := 145241499888481405699961 = 11 \cdot 1320377271713467324 5451.$$

Exactly one of the six numerators has 24 digits; the others have 21 or 22. We write

$$G_4^{(2)} := g_4 + \rho_8^{(2)} g_8 + \rho_{12}^{(2)} g_{12} + \rho_{28}^{(2)} g_{28}$$

for the corresponding combination (3), and P_2 for its band polynomial; the formal development calls the two objects G_2 and P_2, indexing by the band rather than by κ.

Since $\kappa = 4$ is even, each $j\kappa - 1 \in \{3, 7, 11, 27\}$ is odd and $U_{j\kappa-1}(y) = yV_j(y^2)$ as in (6); the chain $U_0, \dots, U_{27}$ is again generated from the three-term recurrence, and beyond V_1, V_2 above it produces

$$V_3(t) = -12 + 280t - 1792t^2 + 4608t^3 - 5120t^4 + 2048t^5$$

and a V_7 of degree 13, with constant term -28 and leading coefficient 2^{27}:

$$\begin{aligned} V_7(t) = -28 + 3640t - 139776t^2 + 2489344t^3 - 24893440t^4 + 154791936t^5 - 635043840t^6 \\ + 1778122752t^7 - 3451650048t^8 + 4642570240t^9 - 4244635648t^{10} + 2516582400t^{11} \\ - 872415232t^{12} + 134217728t^{13}. \end{aligned}$$

Thus

$$P_2(u) = (1-u)\big(V_1(u) + \rho_8^{(2)} V_2(u) + \rho_{12}^{(2)} V_3(u) + \rho_{28}^{(2)} V_7(u)\big), \tag{9}$$

a polynomial of degree 14 with $P_2(1) = 0$.

2.3. **Bernstein certificates.** The positivity statements below that are not exact algebra are reduced to the following standard device [11], implemented once and independently of this paper's subject matter. Let $F \in \mathbb{Z}[t_1, \dots, t_m]$ have multidegree $n = (n_1, \dots, n_m)$; its homogenisation

$$\widehat{F}(u, v) = \prod_{j=1}^{m} (u_j + v_j)^{n_j} \cdot F\big(u_1/(u_1+v_1), \dots, u_m/(u_m+v_m)\big)$$

has monomial coefficients equal to the Bernstein coefficients of F at multidegree n, each scaled by a positive binomial factor, and $\widehat{F}(t, 1-t) = F(t)$. So the assertion that all Bernstein coefficients of F are nonnegative is literally the assertion that $\widehat{F}$ has nonnegative integer coefficients, which for $u, v \geq 0$ gives $F \geq 0$ on $[0,1]^m$ with no basis conversion. Homogenisation is multiplicative, so $\widehat{F}$ is *built* by the same additions and multiplications as the defining expression for F: the coefficient array is the output of the pipeline, not an input to be trusted. Soundness of the test is proved once and for all; what is computed per problem is the Boolean value of the coefficientwise test on an explicit integer term list.

3. The criterion

This section is about an arbitrary P and contains no spectral theory. It is what makes the dimension disappear.

Definition 3.1. For $c \in \mathbb{R}$, a function $P : \mathbb{R} \to \mathbb{R}$ is *multiplicatively superadditive on* $[c, 1]$ if $P(uv) \leq P(u) + P(v)$ for all $u, v \in [c, 1]$.

Theorem 3.2 (merge inequality)**.** *Let $P : \mathbb{R} \to \mathbb{R}$ satisfy $P(1) = 0$ and be multiplicatively superadditive on $[c, 1]$. Then for every finite index set s and every family $(u_i)_{i \in s}$ with $u_i \in [0, 1]$ and $\prod_{i \in s} u_i \geq c$,*

$$P\Big(\prod_{i \in s} u_i\Big) \;\leq\; \sum_{i \in s} P(u_i).$$

Moreover, for every $d \geq 1$ and every $w \in \mathbb{R}$ the family $u = (w, 1, \dots, 1) \in \mathbb{R}^d$ satisfies $\prod_{i<d} u_i = w$ and $\sum_{i<d} P(u_i) = P(w)$.

Proof sketch. Induction on s. The empty case is $P(1) = 0$. For the step, split off one coordinate u_a: the remaining product $p = \prod_{i\in t} u_i$ lies in $[0, 1]$, and both $u_a \geq c$ and $p \geq c$ because the discarded factors are at most 1 and the whole product is at least c. So the hypothesis applies to the pair (u_a, p) and gives $P(u_a p) \leq P(u_a)+P(p)$, while the inductive hypothesis gives $P(p) \leq \sum_{i\in t} P(u_i)$. The point of the bookkeeping is that superadditivity is never invoked outside $[c, 1]$, even though intermediate products are not assumed to be bounded below individually. The second assertion is $P(1) = 0$ applied to the $d-1$ trailing coordinates. No computation is involved. □

Theorem 3.3 (the dimension-free minimum)**.** *Let $P : \mathbb{R} \to \mathbb{R}$ satisfy $P(1) = 0$ and be multiplicatively superadditive on $[c, 1]$ with $0 \leq c$, and let $z \in [c, 1]$. Then for every $d \geq 1$*

$$P(z) \;=\; \min\Big\{\textstyle\sum_{i<d} P(u_i) \;:\; u \in [0,1]^d,\ \prod_{i<d} u_i = z\Big\},$$

the minimum being attained at $u = (z, 1, \dots, 1)$. In particular the value does not depend on d.

Proof sketch. Membership is the witness of Theorem 3.2; the lower bound is its inequality with $s = \{0, \dots, d-1\}$. Elementary; no computation. □

4. The band-1 polynomial and its kernel

4.1. **Separation and factorisation.**

Proposition 4.1. *Let $\rho_8 = (17+8\sqrt{5})/62$, let $G_4 = g_4 + \rho_8 g_8$ in dimension d, and let P be as in* (4)*. Then:*

- (i) (faithfulness) *for every n and every $x \in \mathbb{R}^d$ with all coordinates nonzero, the division-free form* (6) *of g agrees with the source's expression* (3)*;*
- (ii) (separation) $G_4(x) = \big(\prod_l x_l\big)\sum_i P(x_i^2)$ *for every $x \in \mathbb{R}^d$ and every d;*
- (iii) (factorisation) $P(u) = 64\rho_8\,(1-u)(2u-1)(u-b)\big(u-(1-b)\big)$*, with $1-b = b^2 = (3-\sqrt{5})/2$; hence $P(b^2) = P(1/2) = P(b) = P(1) = 0$, so the four roots of P in $[0, 1]$ are exactly the band's threshold values $\mathcal{E}_1^2$, $\cos^2(\pi/4)$, $\mathcal{E}_1$ and 1.*

Part (ii) is the separable form of [2, eq. (4.4)] carried to a linear combination $\sum_j \rho_{j\kappa} A_{j\kappa}$, which is immediate by linearity but is not done in [1], where separability is not used. Part (iii) is a computation in $\mathbb{Z}[b]$: the constant $64\rho_8 = 8(136 + 64\sqrt{5})/62 = 36.0139\dots$ is the same constant that appears in the two-dimensional reduction of the companion manuscript [12]. Part (iii) also re-derives the source's coefficient. For $n = 1$ the interpolation system [1, eq. (11.1)] is the single equation $G_4^{\mathcal{E}_1}(X_{0,1}) = 0$ at the node $X_{0,1} = \mathcal{E}_1$, which on $S_{\mathcal{E}_1}$ in $d = 2$ is the point $(\mathcal{E}_1, 1)$ and so reads $P(\mathcal{E}_1^2) + P(1) = 0$; since $P(1) = 0$ identically it is $P(\mathcal{E}_1^2) = 0$, one linear equation in the one unknown ρ_8. The other threshold value plays no part in fixing it: $2u-1$ divides both V_1 and V_2, so $\cos^2(\pi/4) = 1/2$ is a root of P whatever ρ_8 is.

Proof sketch. (i) is the cancellation $\big(\prod_l x_l\big)x_i^{-1} = \prod_{l\neq i} x_l$. (ii) follows from (6) with V_1, V_2 as in §2 and linearity of the sum over j. (iii) is a polynomial identity modulo (7), verified by a linear combination of $b^2 = 1 - b$ with an explicit polynomial multiplier. No computation beyond exact algebra; in particular the value of ρ_8 enters only through $\rho_8 = (25 + 16b)/62$. □

4.2. **The bilinear kernel.** For any polynomial $P = \sum_k c_k u^k$ with $P(1) = 0$ one has, writing $A_k(u) = 1+u+\cdots+u^{k-1}$ so that $1-u^k = (1-u)A_k(u)$, the elementary identity

$$u^k + v^k - (uv)^k \;=\; 1 - (1-u^k)(1-v^k) \qquad (k \geq 0);$$

summing it against c_k and using $\sum_k c_k = P(1) = 0$ gives

$$P(u)+P(v)-P(uv) \;=\; -\sum_{k\geq 1} c_k(1-u^k)(1-v^k) \;=\; (1-u)(1-v)\,W_P(u,v), \tag{10}$$

where $W_P(u,v) := -\sum_{k\geq 1} c_k A_k(u)A_k(v)$. This two-line computation is not part of the formal development; it is recorded because it explains the shape of what follows. Its content is that the double zero forced by $P(1) = 0$ on the lines $u = 1$ and $v = 1$ is divided out, leaving a factor whose nonnegativity on a region is *equivalent* to multiplicative superadditivity there, and which is non-degenerate on those lines. For the normalised band-1 polynomial $P/(64\rho_8)$ the coefficients are $c = (2-\sqrt{5},\, -5+3\sqrt{5},\, -2\sqrt{5},\, 5,\, -2)$ and the kernel of (10) is $\sum_{0\leq i,j\leq 3} T_{\max(i,j)} u^i v^j$ with $T = (1-2b,\, 4b-1,\, -3,\, 2)$.

What the formal development carries is the specialised identity, in the form that exhibits the decomposition. Define

$$\begin{aligned} K(u,v) &:= 2u^2v^2 + (2+2b)(u^2v+uv^2) + 4(u^2+v^2) \\ &\quad + (2b-1)uv - (1+b)(u+v) - b, \end{aligned} \tag{11}$$

$$\begin{aligned} W(u,v) &:= S(u)(u-b)^2 + S(v)(v-b)^2 \\ &\quad + (u-b)(v-b)K(u,v), \end{aligned} \tag{12}$$

where $S(u) := (2+4b)u = 2\sqrt{5}\,u$.

Proposition 4.2 (kernel identity and the high half)**.** *With $b = \mathcal{E}_1$ and K, W as in* (11), (12)*:*

(i) $P(u)+P(v)-P(uv) = 64\rho_8\,(1-u)(1-v)\,W(u,v)$ *for all* $u, v \in \mathbb{R}$*;*
(ii) $K(u,v) \geq 0$ *for all* $u, v \in [b,1]$.

Proof sketch. (i) is an identity of $\mathbb{Z}[b]$ modulo (7), obtained from the factorisation of Proposition 4.1(iii) by a single linear combination of $b^2 = 1-b$ with an explicit bivariate multiplier of bidegree $(4,4)$. (ii) is proved with no certificate at all. Substituting $u = b+(1-b)\alpha$, $v = b+(1-b)\gamma$ gives

$$\begin{aligned} K = {} & (11-14b) + (-21+39b)(\alpha+\gamma) + (22-34b)(\alpha^2+\gamma^2) \\ & + (48-75b)\alpha\gamma + (-38+62b)(\alpha^2\gamma+\alpha\gamma^2) + (26-42b)\alpha^2\gamma^2, \end{aligned}$$

nine coefficients in all, and every one of them is positive for $b \in [0.618033, 0.618034]$, an enclosure supplied by $2236067977/10^9 \leq \sqrt{5} \leq 2236067978/10^9$. The two smallest are $-38+62b = 0.3181\ldots$ and $26-42b = 0.0426\ldots$, which is why the enclosure has to be of width 10^{-6}. Since $\alpha, \gamma \in [0,1]$ are nonnegative, $K \geq 0$ follows term by term. No exhaustive computation. □

4.3. **The discriminant certificate.** Because S is a *monomial*, (12) can be completed to a square in $(v-b)$:

$$4S(u)\,W(u,v) = \big(2S(u)(u-b)+K(u,v)(v-b)\big)^2 + \big(4S(u)S(v)-K(u,v)^2\big)(v-b)^2, \tag{13}$$

and $4S(u)S(v) = 4(2+4b)^2uv = 80uv$, the single relation (7) being used once in the form $(2+4b)^2 = 20$, i.e. $2+4b = 2\sqrt{5}$. So on the part of the box where $(u-b)(v-b)$ may be negative it suffices to certify the *discriminant* $80uv - K^2$, which — unlike W — has no zero there.

Proposition 4.3 (the certificate)**.** *The root Bernstein expansion of* $10^{48}\big(80uv - K(u,v)^2\big)$ *over the box*

$$u \in [1/10,\, b], \qquad v \in [1/10,\, 1], \qquad b \in [0.618033,\, 0.618034],$$

at multidegree $(4,4,6)$ *and with* 175 *coefficients, is coefficientwise nonnegative, with* no subdivision. *Consequently* $80uv - K(u,v)^2 \geq 0$ *for all* $u \in [1/10, b]$, $v \in [1/10, 1]$.

Proof sketch. The box is parametrised by three coordinates in $[0,1]$: $10^6 b = 618033 + \omega$, $10^6 u = 10^5 + (10^6 b - 10^5)\alpha$ and $10^6 v = 10^5 + 9 \cdot 10^5 \beta$. Everything is scaled by 10^6 per variable, so the certified object is the integer polynomial $10^{48}(80uv - K^2)$ in (α, β, ω). This is the one claim about the first band that rests on exhaustive computation: what is asserted is the Boolean value of the coefficientwise test on that explicit integer term list, evaluated by compiled code; the passage from the Boolean to the inequality is the soundness statement of §2.3. The enclosure of b is carried as the third box variable, so that no irrational quantity enters the array. □

Proposition 4.4. $W(u,v) \geq 0$ *for all* $u, v \in [1/10, 1]$.

Proof sketch. Split by position relative to b. If both $u, v \geq b$, all three terms of (12) are nonnegative — the cross term by Proposition 4.2(ii). Otherwise, by the symmetry $W(u,v) = W(v,u)$ we may assume $u \leq b$; then $S(u) > 0$ and (13) expresses $4S(u)W$ as a square plus $(80uv - K^2)(v-b)^2$, which is nonnegative by Proposition 4.3. The only exhaustive computation is the Boolean of Proposition 4.3. □

5. Superadditivity, and the dimension-free minimum

Theorem 5.1 (multiplicative superadditivity)**.** *The band-*1 *polynomial* P *of* (4) *satisfies*

$$P(uv) \;\leq\; P(u) + P(v) \qquad \text{for all } u, v \in [1/10, 1].$$

Consequently, for every $d \geq 1$ *and every* $x \in [-1,1]^d$ *with* $E = \prod_i x_i \geq 0$ *and* $E^2 \geq 1/10$,

$$E\,P(E^2) \;\leq\; G_4(x).$$

Proof sketch. The inequality is Proposition 4.4 fed into the identity of Proposition 4.2(i): on $[1/10,1]^2$ the factor $(1-u)(1-v)$ is nonnegative and $64\rho_8 > 0$, so $W \geq 0$ gives $P(u) + P(v) - P(uv) \geq 0$; conversely the same identity shows the two statements are equivalent on that square. The consequence is Proposition 4.1(ii) — which converts G_4 into $E\sum_i P(x_i^2)$ — combined with Theorem 3.2 applied to $u_i = x_i^2 \in [0,1]$, whose product is $\big(\prod_i x_i\big)^2 = E^2 \geq 1/10$. The only exhaustive computation is the Boolean of Proposition 4.3. □

Theorem 5.2 (the minimum, and dimension-independence)**.** *Let* $d \geq 1$ *and let* $E \in (0,1]$ *satisfy* $E^2 \geq 1/10$. *Then*

$$\min\Big\{ G_4(x) \;:\; x \in [-1,1]^d,\; \prod_i x_i = E \Big\} \;=\; E\,P(E^2),$$

the minimum being attained at $x = (E, 1, \dots, 1)$. *Consequently*

$$G_4|_{S_E} > 0 \iff P(E^2) > 0,$$

and for any two dimensions $d_1, d_2 \geq 1$, $G_4|_{S_E} > 0$ *in dimension* d_1 *if and only if* $G_4|_{S_E} > 0$ *in dimension* d_2.

Proof sketch. The lower bound is Theorem 5.1. Attainment is the witness $(E, 1, \dots, 1)$, which lies in S_E and on which $\sum_i P(x_i^2) = P(E^2)$, because $P(1) = 0$. The equivalence follows since $E > 0$, and the last assertion is that equivalence read twice. The only exhaustive computation is the Boolean of Proposition 4.3. □

Proposition 5.3 (where P is positive)**.** *Writing* $z = E^2$ *and* $\mathcal{E}_1 = (\sqrt{5} - 1)/2$*:*

$$P(z) > 0 \text{ on } \left(\tfrac{3-\sqrt{5}}{2}, \tfrac{1}{2}\right) \text{ and on } (\mathcal{E}_1, 1);$$
$$P(z) < 0 \text{ on } \left[\tfrac{1}{10}, \tfrac{3-\sqrt{5}}{2}\right) \text{ and on } \left(\tfrac{1}{2}, \mathcal{E}_1\right).$$

Proof sketch. Immediate from Proposition 4.1(iii): the four roots $\mathcal{E}_1^2 = \frac{3-\sqrt{5}}{2} < \frac{1}{2} < \mathcal{E}_1 < 1$ cut $[1/10, 1]$ into four intervals, and the sign of $64\rho_8(1-u)(2u-1)(u-b)(u-(1-b))$ on each is read off factor by factor once the enclosure $0.618033 \leq \mathcal{E}_1 \leq 0.618034$ is available. No computation. □

Together, Theorem 5.2 and Proposition 5.3 say that on the range $E \geq 1/\sqrt{10} = 0.31622\dots$ the set of energies with $G_4|_{S_E} > 0$ is exactly

$$\left(\mathcal{E}_1,\ \cos(\pi/4)\right) \ \cup \ \left(\sqrt{\mathcal{E}_1},\ 1\right) \ = \ (0.61803\dots,\, 0.70710\dots) \cup (0.78615\dots,\, 1),$$

the same set in every dimension $d \geq 1$, and that the two complementary intervals $[1/\sqrt{10}, \mathcal{E}_1)$ and $(\cos(\pi/4), \sqrt{\mathcal{E}_1})$ carry no such positivity, again in every dimension.

6. The band theorems

Theorem 6.1 (the first band, in every dimension)**.** *Let* $d \geq 1$ *and let* $x \in [-1, 1]^d$ *have* $E = \prod_i x_i > 0$.

(i) *If* $\frac{3-\sqrt{5}}{2} < E^2 < \frac{1}{2}$ *— that is, if* E *lies in the open first band* $(\mathcal{E}_1, \cos(\pi/4))$ *— then* $G_4(x) > 0$.

(ii) *If* $\mathcal{E}_1 < E^2 < 1$ *— that is, if* $E \in (\sqrt{\mathcal{E}_1}, 1)$ *— then* $G_4(x) > 0$.

In both cases the bound $G_4(x) \geq E\, P(E^2) > 0$ *holds with the constant of Theorem 5.2, which is attained.*

Proof sketch. Both are Theorem 5.1 together with the corresponding case of Proposition 5.3; the hypothesis $E^2 \geq 1/10$ of Theorem 5.1 is implied by $E^2 > \frac{3-\sqrt{5}}{2} = 0.381\dots$ in case (i) and by $E^2 > \mathcal{E}_1$ in case (ii), using the rational enclosure of $\sqrt{5}$. The only exhaustive computation is the Boolean of Proposition 4.3. □

Case (i) with $d = 3$ is [1, Conjecture 1.15] for the first band, at the level to which his own equivalence reduces it, and with his own two-dimensional coefficient ρ_8; the theorem settles it simultaneously for every dimension. The following corollary states the spectral reading, which is the sources' and not ours.

Corollary 6.2. *Assume the equivalence of* [1, §1, after (1.10)] *recalled in §1 — for a fixed* $\mathbb{A}$ *of the form* (2)*, strict positivity of* G_κ *on* S_E *gives* (1) *near* E*. Then, with* $\mathbb{A} = A_4 + \frac{17+8\sqrt{5}}{62} A_8$ *and any* $d \geq 1$*, every energy of the open first band* $\left(\mathcal{E}_1, \cos(\pi/4)\right)$ *and every energy of* $\left(\sqrt{\mathcal{E}_1}, 1\right)$ *lies in* $\boldsymbol{\mu}_4(D)$.

Only the first half of Corollary 6.2 is new. That $\pm(\cos(\pi/4), 1) \subset \boldsymbol{\mu}_{A_4}(D)$ for every $d \geq 1$ is [2, Lemma 4.7], and $(\sqrt{\mathcal{E}_1}, 1)$ is a proper subinterval of it; the content of the second half is Theorem 6.1(ii) itself — the polynomial statement for this linear combination, with the attained constant — and not the membership.

The operator-theoretic half of Corollary 6.2 — spectral projections, commutator extensions, functional calculus on $\ell^2(\mathbb{Z}^d)$ — is not part of the formal development; only the polynomial statements are. Since $\boldsymbol{\mu}_\kappa(D)$ is defined by the *existence* of an interval $I \ni E$ and a $\gamma > 0$ satisfying (1), and [1] asserts no relation between that γ and the size of G_κ on S_E, the corollary claims membership and nothing about the size of γ; the sharpness in Theorem 5.2 is sharpness for the polynomial G_4.

Proposition 6.3 (controls)**.** (i) (too large a band, on the right) *For every $d \geq 1$, the point $(71/100, 1, \ldots, 1) \in S_{71/100}$ has $G_4 < 0$; and $71/100 > \cos(\pi/4)$.*

(ii) (too large a band, on the left) *For every $d \geq 1$, the point $(3/5, 1, \ldots, 1) \in S_{3/5}$ has $G_4 < 0$; and $3/5 < \mathcal{E}_1$.*

(iii) (the region hypothesis bites) *$P(u) + P(v) < P(uv)$ at $u = v = 1/11$; so $[1/10, 1]$ in Theorem 5.1 cannot be replaced by $(0, 1]$.*

(iv) (sharpness of the certificate) *The same root expansion as in Proposition 4.3* rejects *when the left endpoint $1/10$ is lowered to $3/32 = 0.09375$, and rejects when it is lowered to $1/20$.*

(v) (the enclosure is a hypothesis) *The same root expansion still accepts when the enclosure of b is widened from width 10^{-6} to width 10^{-2}, and rejects at width 10^{-1}.*

(vi) (the estimate is sharp) *For $d \geq 1$ and E as in Theorem 5.2 with $P(E^2) > 0$, the inequality $G_4(x) \geq \frac{101}{100} E\, P(E^2)$ fails somewhere on S_E; so no factor greater than 1 may be inserted.*

(vii) (non-vacuity in dimension 3) *The hypotheses of Theorem 6.1(i) are satisfiable in $d = 3$, e.g. at $(13/20, 1, 1)$.*

Items (iv) and (v) are Boolean values of the same certificate pipeline as Proposition 4.3 on different boxes; the remaining items are exact algebra and point evaluations. Item (iii) is what makes the classical global tests inapplicable, as explained in §1; item (vi) is immediate from attainment in Theorem 5.2.

Remark 6.4 (the two-dimensional case, and the size of the gain). At $d = 2$ Theorem 6.1(i) recovers the first-band positivity certified for this operator in the companion manuscript [12], here with a constant that is *exactly attained* rather than merely valid. Outside the formal development, the ratio of the two lower bounds was measured across the band and stays between 37 and 82; no theorem uses this measurement.

7. Scope: the two-sided band set does depend on the dimension

[2] works with $\boldsymbol{\mu}_{\mathbb{A}} = \boldsymbol{\mu}^+_{\mathbb{A}} \cup \boldsymbol{\mu}^-_{\mathbb{A}}$, admitting a strict estimate for $\mathbb{A}$ or for $-\mathbb{A}$. The theorems above concern the $\boldsymbol{\mu}^+$ side only, and the restriction is necessary.

Theorem 7.1. *Let $\mathbb{A} = A_4$, i.e. $\rho_8 = 0$, so that the relevant polynomial is g_4.*

(i) *For $d = 2$: $g_4(x) < 0$ for* every *$x \in [-1, 1]^2$ with $x_1 x_2 = 9/20$.*

(ii) *For $d = 3$: there are $x, y \in [-1, 1]^3$ with $\prod_i x_i = \prod_i y_i = 9/20$ and $g_4(x) < 0 < g_4(y)$; explicitly $x = (9/20, 1, 1)$ and $y = (7/9, 7/9, 729/980)$, with $g_4(x) \approx -0.854$ and $g_4(y) \approx +0.384$.*

Proof sketch. (ii) is two point evaluations. (i) is an exact two-variable argument: by Proposition 4.1(ii) with $\rho_8 = 0$, $g_4(x) = E\big(P_0(x_1^2) + P_0(x_2^2)\big)$ with $P_0(u) = 4(1-u)(2u-1)$; writing $s = x_1^2 + x_2^2$ and using $x_1 x_2 = 9/20$, the arithmetic mean–geometric mean inequality gives $s \geq 9/10$, while $P_0(x_1^2) + P_0(x_2^2) = -8s^2 + 12s - 8 + 16z$ is decreasing for $s \geq 3/4$; evaluating at $s = 9/10$ gives a strictly negative value. No exhaustive computation. □

Both parts are statements about the polynomial g_4 alone. Granting [2]'s criterion — $E \in \boldsymbol{\mu}^{\pm}_{A_\kappa}(D)$ if and only if $\pm g_E|_{S_E}$ is strictly positive — they say that $9/20 \in \boldsymbol{\mu}_{A_4}(D_2)$ while $9/20 \notin \boldsymbol{\mu}_{A_4}(D_3)$, in the two-sided convention.

The rows read off [2, Tables 15, 17] for $\kappa = 4$ are $(0, 0.5) \cup (0.707107, 1)$ in $d = 2$ against $(0, 0.3535) \cup (0.7071, 1)$ in $d = 3$; the upper interval is dimension-free (it is the case in which every summand of the separable form is separately positive, [2, Lemma 4.7]), and the lower endpoint moves as the value conjectured for it in the same tables, $\sin^d(\pi/4) = 2^{-d/2}$ — the energy at which the symmetric point $u_i = 1/2$ of the surface has $\sum_i P_0(u_i) = 0$. That endpoint belongs to the $\boldsymbol{\mu}^-$ half, so it is invisible to a criterion built on minima. Theorem 7.1 makes the $d = 3$ column of that table — which is numerical in [2] — rigorous at one energy; the $d = 2$ half is also [2]'s proved computation $\boldsymbol{\mu}_{A_4}(D_2) = \pm(0, 1/2) \cup \pm(1/\sqrt{2}, 1)$ [2, Lemma 4.9].

8. The obstruction, and what the decomposition removes

A Bernstein coefficient array bounds the minimum of a polynomial on a box *from below*, and under subdivision that bound converges to the true minimum from below; at an attained *interior* zero it therefore never becomes nonnegative, no matter how deep the tree. The kernel W has exactly such a zero, $W(b, b) = 0$. Outside the formal development one computes in $\mathbb{Z}[b]$ that it is non-degenerate: at (b, b),

$$W_{uu} = W_{vv} = 8 - 4b = 5.5279\ldots, \qquad W_{uv} = K(b, b) = 11 - 14b = 2.3475\ldots,$$

with Hessian determinant $-237 + 424b = 25.046\ldots > 0$, so it is a strict interior minimum. Splitting the box so that (b, b) becomes a *corner* does not help either: on a sub-box with $u \leq b \leq v$ the value and both first derivatives vanish at that corner, so the $(1, 1)$ Bernstein coefficient equals $-W_{uv} \cdot (\mathrm{width}_u)(\mathrm{width}_v)/(n_1 n_2) < 0$ at every subdivision depth and every degree elevation. (These are computations in exact arithmetic outside the formal development, reported for context; no theorem uses them.)

The decomposition (12) removes the obstruction rather than out-subdividing it: the zero at (b, b) is carried by the explicit squares $(u-b)^2$, $(v-b)^2$ and the product $(u-b)(v-b)$, leaving two problems — $K \geq 0$ on $[b, 1]^2$ and $80uv - K^2 \geq 0$ on $[1/10, b] \times [1/10, 1]$ — neither of which has a zero on its region. The second is tight on the diagonal, which is why the left endpoint $1/10$ cannot be lowered far: at a diagonal zero of W one has $K(c, c) = -2S(c)$ and hence $80c^2 - K(c, c)^2 = 0$. Outside the formal development, the exact threshold below which multiplicative superadditivity fails was measured at $c^* = 0.0944243\ldots$, so the certified $1/10$ is within 6% of the best this route can do, and by Proposition 6.3(iv) $3/32 = 0.09375$ already rejects. In energy the limit is $E \geq \sqrt{c^*} = 0.30729\ldots$ against the certified $E \geq 1/\sqrt{10} = 0.31623\ldots$.

9. The second band

The mechanism of §§3–5 is not tied to the first band: it asks only for a P with $P(1) = 0$ and a region on which P is multiplicatively superadditive. Mandich's second band — the only other band of $\kappa = 4$ for which he has exact coefficients — supplies both. Theorems 3.2 and 3.3 are reused verbatim, with P_2 for P and $1/3$ for $1/10$; what has to be done again, and differently, is the inequality.

9.1. **The polynomial, and its roots.**

Proposition 9.1 (the band-2 polynomial)**.** *Let $\rho_8^{(2)}, \rho_{12}^{(2)}, \rho_{28}^{(2)}$ be as in* (8)*, let P_2 be as in* (9)*, and let $b = \mathcal{E}_1$. Then:*

(i) (separation) $G_4^{(2)}(x) = \big(\prod_l x_l\big) \sum_i P_2(x_i^2)$ *for every $x \in \mathbb{R}^d$ and every d;*
(ii) (factorisation) $\deg P_2 = 14$ *and, identically in u,*

$$D\,P_2(u) = (1-u)(2u-1)(3u-1)(3u-2)\big((u-b)(u-(1-b))\big)\,Q(u)$$

for a polynomial Q of degree 8 with coefficients in $\mathbb{Z}[b]$;
(iii) *consequently P_2 vanishes at $\frac{1}{3}$, b^2, $\frac{1}{2}$, b, $\frac{2}{3}$ and 1 — that is, at $\mathcal{E}_2^2$, $\mathcal{E}_1^2$, $\mathcal{E}_0^2$, $\mathcal{E}_1$, $\frac{2}{3}$ and 1;*
(iv) *$Q(u) > 0$ for every $u \in [1/3, 1]$; so the six numbers in (iii) are the only roots of P_2 in $[1/3, 1]$.*

The coefficients (8) are Mandich's and are not claimed here. What was done, as a gate on the data before anything was proved about it, is to solve his own $n = 2$ system in exact $\mathbb{Q}(\sqrt{5})$ arithmetic and compare: all six numerators and both denominators of the printed solution are reproduced digit for digit, and the factorisation $D = 11 \cdot 13203772717134673245451$ of the common denominator falls out of the same computation. In the formal development the three coefficients are carried over $\mathbb{Z}[b]$, using $\sqrt{5} = 2b + 1$, and three identities assert that they equal the printed values of (8); those identities are exact algebra, not evaluation.

Proof sketch. (i) is (6) applied to each of U_3, U_7, U_{11}, U_{27} — each odd, so each of the form $yV_j(y^2)$ — together with linearity of the sum over j; it is the separable form of [2, eq. (4.4)] carried to the linear combination, exactly as in Proposition 4.1(ii). (ii) has two halves. Clearing the denominator — writing $D\,P_2$ as an explicit polynomial with coefficients in $\mathbb{Z}[b]$ — is a free ring identity, because both sides are linear in b and equality in $\mathbb{Q}(\sqrt{5})[u]$ is therefore equality of polynomials; the factorisation itself is an identity of $\mathbb{Z}[b]$ modulo (7), discharged by one linear combination with an explicit multiplier of degree 12 and thirteen terms. (iii) follows by evaluating the six factors, the case $u = b^2$ using (7) once more. (iv) is the one part that rests on exhaustive computation: it is the last array of Table 1 below, which certifies not $Q \geq 0$ but $Q - \varepsilon \geq 0$ for an explicit positive rational ε — the reciprocal of the scaling factor the array uses — so that the conclusion is strict, as the sign table needs. No other part of this proposition involves any computation beyond exact algebra. □

9.2. **A reflection symmetry, and why four of the six roots are forced.**

Proposition 9.2 (reflection)**.** (i) *$V_j(1-u) = -V_j(u)$ identically, for each of $j = 1, 2, 3, 7$: every Chebyshev block is odd about $u = \frac{1}{2}$.*
(ii) *Hence $(1-u)P_2(1-u) = -u\,P_2(u)$ for all $u \in \mathbb{R}$. The identity holds for* every *choice of the coefficients $\rho_{j\kappa}$, so it is uniform in the band index.*

(iii) *Hence, if* $r \neq 1$ *and* $P_2(r) = 0$*, then* $P_2(1-r) = 0$.
(iv) *In particular* $P_2(\frac{2}{3}) = 0$ *and* $P_2(\mathcal{E}_1) = 0$*, these being the reflections of* $P_2(\frac{1}{3}) = 0$ *and* $P_2(\mathcal{E}_1^2) = 0$.

Proof sketch. For $n = 4j-1$ and $y = \cos\theta$, $U_n(\cos\theta) = \sin(4j\theta)/\sin\theta$, so $V_j(\cos^2\theta) = 2\sin(4j\theta)/\sin(2\theta)$; replacing θ by $\pi/2 - \theta$ fixes $\sin(2\theta)$ and changes the sign of $\sin(4j\theta)$, which is (i). In the formal development (i) is checked instead by expanding the four explicit polynomials, which is a ring identity. (ii) follows from $P_2 = (1-u)\sum_j \rho_{j\kappa} V_j$ and (i) — one ring identity, in which the coefficients never appear. (iii): put r into (ii); the left side is $(1-r)P_2(1-r)$ and the right side vanishes, and $1 - r \neq 0$. (iv) is (iii) at $r = \frac{1}{3}$ and at $r = \mathcal{E}_1^2$, using $1 - \mathcal{E}_1^2 = \mathcal{E}_1$ from (7). No computation. □

Proposition 9.2 accounts for the root list of Proposition 9.1(iii) completely, and it is worth setting out because it shows how little of that list is a choice. Two of the six roots are free of the coefficients: $u = 1$, because P_2 carries the factor $1-u$, and $u = \frac{1}{2}$, because (i) at $u = \frac{1}{2}$ forces $V_j(\frac{1}{2}) = -V_j(\frac{1}{2})$. Two are imposed by Mandich's interpolation system, namely $\frac{1}{3} = \mathcal{E}_2^2$ and $\mathcal{E}_1^2$, one from each end of the band. The remaining two, $\frac{2}{3}$ and $\mathcal{E}_1$, are then forced by (iii). The companion manuscript [12] records those last two as zeros that the interpolation system produces without imposing, and derives them from the source's own relations on the threshold sets. Proposition 9.2 is an alternative and shorter reason for them, and one that does not depend on the band.

Remark 9.3 (where the kernel of P_2 must vanish). The same symmetry predicts the zeros of the superadditivity kernel instead of finding them numerically. If u is a root of P_2 and $v := 1/(1+u)$ is also a root, then $uv = u/(1+u) = 1-v$ is a root as well by Proposition 9.2(iii), so $P_2(u) + P_2(v) - P_2(uv) = 0$. Among the roots of Proposition 9.1(iii) this happens exactly twice inside $[1/3, 1]$: at $u = v = \mathcal{E}_1$, since $1/(1+\mathcal{E}_1) = \mathcal{E}_1$ by (7), and at $(u, v) = (\frac{1}{2}, \frac{2}{3})$, since $1/(1+\frac{1}{2}) = \frac{2}{3}$ and $\frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$. With the mirror image $(\frac{2}{3}, \frac{1}{2})$ these are the three interior zeros around which the whole certificate is built.

9.3. **The kernel, and two local decompositions.** The general identity (10) applies to P_2 unchanged and produces a kernel W_2 of bidegree $(13, 13)$. Outside the formal development one computes that W_2 is nonnegative on $[1/3, 1]^2$ with grid minimum 0, that it vanishes on that square at exactly the three points of Remark 9.3, and that $W_2(1,1) = 466.21\ldots$, $W_2(1/3, 1/3) = 7.28\ldots$; those are computations in exact $\mathbb{Q}(\sqrt{5})$ arithmetic reported for context, and no theorem below uses them. What the formal development carries is the integer-cleared kernel

$$F(u, v) \;:=\; D\big(P_2(u) + P_2(v) - P_2(uv)\big), \tag{14}$$

whose nonnegativity on $[1/3, 1]^2$ is, D being positive, exactly multiplicative superadditivity there.

For band 1 the decomposition (12) was found by inspection, and the fact that the resulting S was a monomial was called a piece of luck. It is neither luck nor necessary. Let W be a polynomial with an interior zero $z = (z_1, z_2)$ at which $W \geq 0$ locally. Then the one-variable polynomial $W(\cdot, z_2)$ is nonnegative near z_1 and vanishes there, so it has a double root and

$$S_1(u) := \frac{W(u, z_2)}{(u - z_1)^2}, \qquad S_2(v) := \frac{W(z_1, v)}{(v - z_2)^2}$$

are polynomials. The difference $W - S_1(u)(u-z_1)^2 - S_2(v)(v-z_2)^2$ then vanishes identically on the line $v = z_2$ and on the line $u = z_1$, so it is divisible by $(u-z_1)(v-z_2)$:

$$W(u,v) = S_1(u)(u-z_1)^2 + S_2(v)(v-z_2)^2 + (u-z_1)(v-z_2)K(u,v). \tag{15}$$

There is no search, no semidefinite programme and no choice of basis in this; S_1, S_2 and K are determined by W and z, two exact divisions and one subtraction. The zero need not be on the diagonal.

Proposition 9.4 (the two local decompositions)**.** *There are polynomials S_A, S_{1B}, S_{2B} of degree 11 and K_A, K_B of bidegree $(12,12)$, all of them with coefficients in $\mathbb{Z}[b]$, for which the two identities*

$$\begin{aligned} F(u,v) = (1-u)(1-v)\Big(&S_A(u)(u-b)^2 + S_A(v)(v-b)^2 \\ &+ (u-b)(v-b)K_A(u,v)\Big), \end{aligned} \tag{16}$$

$$\begin{aligned} 27\,F(u,v) = (1-u)(1-v)\Big(&S_{1B}(u)(u-\tfrac{1}{2})^2 + S_{2B}(v)(v-\tfrac{2}{3})^2 \\ &+ (u-\tfrac{1}{2})(v-\tfrac{2}{3})K_B(u,v)\Big) \end{aligned} \tag{17}$$

hold identically in u and v.

Equation (16) is (15) at the diagonal zero $(\mathcal{E}_1, \mathcal{E}_1)$, where $S_2 = S_1 = S_A$ by symmetry, and (17) is (15) at the off-diagonal zero $(\frac{1}{2}, \frac{2}{3})$, the factor 27 clearing the denominators the divisions introduce. The third zero $(\frac{2}{3}, \frac{1}{2})$ needs nothing of its own, because $F(u,v) = F(v,u)$.

Proof sketch. Both are polynomial identities. (17) is a free ring identity: its zero is rational, so no algebraic relation is involved. (16) is an identity of $\mathbb{Z}[b]$ modulo (7), and it is the *only* place in the entire development where $b^2 = 1 - b$ is used. No computation beyond exact algebra. □

Remark 9.5 (how (16) is discharged). The identity (16) is verified by a single linear combination of $b^2 + b - 1 = 0$ with a bivariate multiplier, and that multiplier expands to 420 monomials with 105-bit coefficients. Writing them out doubles the elaboration cost for no gain, so the multiplier is not stored but assembled from the b-parts of the data already in the file — from S_A, K_A and their derivatives in b — and expanded by the prover. The second economy is the emission form: every polynomial in the two modules is written in Horner form, as nested two-term sums of depth at most 26, because flat sums of that many terms — K_A alone has 169 coefficients over $\mathbb{Z}[b]$ — do not elaborate at this size while the nested form does. With both in place the identity was measured in isolation at 80 seconds and 8 GB, and none of the three fallbacks prepared for it was needed. It is the one step whose cost was priced before the rest of the build was attempted.

Completing the square in (15) works exactly as in (13), but without needing S_1 to be a monomial:

$$\begin{aligned} 4S_1(u)W(u,v) = \big(2S_1(u)(u-z_1) + K(u,v)(v-z_2)\big)^2 \\ + \big(4S_1(u)S_2(v) - K(u,v)^2\big)(v-z_2)^2. \end{aligned} \tag{18}$$

So $S_1 \geq 0$, $S_2 \geq 0$ and $4S_1S_2 - K^2 \geq 0$ on a box together give $W \geq 0$ there. The degenerate case costs nothing: if $S_1(u) = 0$ then $4S_1S_2 - K^2 \geq 0$ forces $K(u,\cdot) = 0$ and (15) collapses to $S_2(v)(v - z_2)^2 \geq 0$. All three certificates are therefore shipped as non-strict inequalities. At the zero itself $2S_1(z_1)$, $2S_2(z_2)$ and $4S_1S_2 - K^2$ are the two diagonal entries and the determinant of $\operatorname{Hess} W$, all positive when the zero is non-degenerate; so neither of the two remaining problems inherits z as a zero, which is what makes them accessible to a Bernstein array at all (§8).

9.4. **The partition, and the certificates.** The boxes are cut at $\frac{1}{3} < \frac{11}{20} < \frac{3}{5} < \frac{4}{5} < 1$. Every breakpoint is rational with denominator dividing 60, so u and v are carried at scale 60 and no irrational number ever appears at a box corner; the enclosure of $\mathcal{E}_1$ enters, as in Proposition 4.3, only as a third box variable, on the same width-10^{-6} bracket used for band 1. The conclusion is assembled for $v \leq u$ only, the other half being free from $F(u,v) = F(v,u)$. The box $[\frac{3}{5}, \frac{4}{5}]^2$ carries the zero $(\mathcal{E}_1, \mathcal{E}_1)$ and is handled by (16) and (18); six further cells contain no zero of F and are certified directly; and the case $v \leq \frac{11}{20} \leq u$ is handled by (17) on the box $[\frac{1}{3}, \frac{11}{20}] \times [\frac{11}{20}, 1]$, which carries $(\frac{1}{2}, \frac{2}{3})$ and is the one box of the twelve written with its first variable the smaller.

certified statement	multidegree in (u,v)	subdivision	terms
$F \geq 0$ on $[\frac{1}{3}, \frac{11}{20}]^2$	$(14,14)$	none	450
$F \geq 0$ on $[\frac{11}{20}, \frac{3}{5}]^2$	$(14,14)$	none	450
$F \geq 0$ on $[\frac{3}{5}, \frac{4}{5}] \times [\frac{11}{20}, \frac{3}{5}]$	$(14,14)$	one, in u	900
$F \geq 0$ on $[\frac{4}{5}, 1] \times [\frac{11}{20}, \frac{3}{5}]$	$(14,14)$	none	420
$F \geq 0$ on $[\frac{4}{5}, 1] \times [\frac{3}{5}, \frac{4}{5}]$	$(14,14)$	none	420
$F \geq 0$ on $[\frac{4}{5}, 1]^2$	$(14,14)$	none	392
$S_A \geq 0$ on $[\frac{3}{5}, \frac{4}{5}]$	$(11,0)$	none	24
$4S_A(u)S_A(v) - K_A^2 \geq 0$ on $[\frac{3}{5}, \frac{4}{5}]^2$	$(24,24)$	none	1875
$S_{1B} \geq 0$ on $[\frac{1}{3}, \frac{11}{20}]$	$(11,0)$	none	24
$S_{2B} \geq 0$ on $[\frac{11}{20}, 1]$	$(0,11)$	none	24
$4S_{1B}(u)S_{2B}(v) - K_B^2 \geq 0$ on $[\frac{1}{3}, \frac{11}{20}] \times [\frac{11}{20}, 1]$	$(24,24)$	none	1875
$Q - \varepsilon \geq 0$ on $[\frac{1}{3}, 1]$	$(8,0)$	one, in u	36

TABLE 1. The twelve Bernstein arrays; the eleven above the second rule are the partition, the last is the cofactor of Proposition 9.1(iv). Each carries in addition a third box variable for the enclosure of $\mathcal{E}_1$, of degree 1 except in the two discriminants, where it is 2. "Terms" counts the coefficients actually present in the homogenised array, summed over the leaves of the subdivision tree; per leaf it is at most $(n_1+1)(n_2+1)(n_3+1)$ — 450 at $(14,14,1)$ — and falls below that when coefficients vanish, as they do in the three cells with $u \geq \frac{4}{5}$. The eleven sum to 6854 and all twelve to 6890.

Proposition 9.6 (the certificates)**.** *Each of the twelve integer Bernstein arrays described in Table 1 is coefficientwise nonnegative. Consequently*

$$P_2(u) + P_2(v) - P_2(uv) \;\geq\; 0 \qquad \text{for all } u, v \in [1/3, 1].$$

Proof sketch. As in Proposition 4.3, what is asserted is the Boolean value of the coefficientwise test on explicit integer term lists — twelve of them here — evaluated by compiled code, and the passage from those Booleans to the inequalities is the soundness statement of §2.3. Ten of the twelve are accepted at the root; two need a

single subdivision each, and §9.6 records that neither subdivision can be dropped. The eleven arrays of the partition carry 6854 Bernstein coefficients in all and the twelfth, for the cofactor of Proposition 9.1(iv), carries 36 more, so 6890 altogether, at integers of at most 546 bits (Table 1). Given the twelve Booleans, the conclusion is assembled by cases on the position of (u, v): six cells are immediate, and the two cores are (16) and (17) fed into (18), whose hypotheses are the three S-certificates and the two discriminant certificates. The two discriminants are handed to the checker as the *expressions* $4S_1S_2 - K\cdot K$, so the bidegree-$(24, 24)$ arrays are produced by the checker's own arithmetic and no identity of that size appears anywhere in the development. □

9.5. **Superadditivity, the minimum, and the band.**

Theorem 9.7 (multiplicative superadditivity, second band)**.** *The band-*2 *polynomial* P_2 *of* (9) *satisfies*

$$P_2(uv) \;\le\; P_2(u) + P_2(v) \qquad \text{for all } u, v \in [1/3, 1].$$

Consequently, for every $d \ge 1$ *and every* $x \in [-1, 1]^d$ *with* $E = \prod_i x_i \ge 0$ *and* $E^2 \ge 1/3$*,*

$$E\, P_2(E^2) \;\le\; G_4^{(2)}(x).$$

Proof sketch. The inequality is Proposition 9.6. The consequence is Proposition 9.1(i), which converts $G_4^{(2)}$ into $E\sum_i P_2(x_i^2)$, combined with Theorem 3.2 applied to $u_i = x_i^2 \in [0, 1]$, whose product is $E^2 \ge 1/3$. Theorem 3.2 is used exactly as it stands: it is a statement about an arbitrary P over an arbitrary finite index set, and the only new input it receives here is the inequality above. The only exhaustive computation is the twelve Booleans of Proposition 9.6. □

Theorem 9.8 (the minimum, and dimension-independence, second band)**.** *Let* $d \ge 1$ *and let* $E \in (0, 1]$ *satisfy* $E^2 \ge 1/3$*. Then*

$$\min\Big\{ G_4^{(2)}(x) \;:\; x \in [-1, 1]^d,\ \prod_i x_i = E \Big\} \;=\; E\, P_2(E^2),$$

the minimum being attained at $x = (E, 1, \dots, 1)$*. Consequently*

$$G_4^{(2)}|_{S_E} > 0 \iff P_2(E^2) > 0,$$

and for any two dimensions $d_1, d_2 \ge 1$*,* $G_4^{(2)}|_{S_E} > 0$ *in dimension* d_1 *if and only if* $G_4^{(2)}|_{S_E} > 0$ *in dimension* d_2*.*

Proof sketch. Word for word the proof of Theorem 5.2, with Theorem 9.7 in place of Theorem 5.1: the lower bound is that theorem, attainment is the witness $(E, 1, \dots, 1)$ on which $\sum_i P_2(x_i^2) = P_2(E^2)$ because $P_2(1) = 0$, and the equivalence follows since $E > 0$. The only exhaustive computation is the twelve Booleans of Proposition 9.6. □

Proposition 9.9 (where P_2 is positive)**.** *Writing* $z = E^2$*, and with* $\mathcal{E}_1 = (\sqrt{5}-1)/2$*,* $\mathcal{E}_1^2 = 1 - \mathcal{E}_1$*:*

$$P_2(z) > 0 \text{ on } \big(\tfrac{1}{3}, \mathcal{E}_1^2\big),\ \big(\tfrac{1}{2}, \mathcal{E}_1\big) \text{ and } \big(\tfrac{2}{3}, 1\big);$$
$$P_2(z) < 0 \text{ on } \big(\mathcal{E}_1^2, \tfrac{1}{2}\big) \text{ and } \big(\mathcal{E}_1, \tfrac{2}{3}\big).$$

Proof sketch. The six roots of Proposition 9.1(ii)–(iii) cut $[1/3, 1]$ into five intervals, and on each the sign of $(1-u)(2u-1)(3u-1)(3u-2)(u-b)(u-(1-b))$ is read off factor by factor once the enclosure $0.618033 \le \mathcal{E}_1 \le 0.618034$ is available; the remaining factor, Q, is positive throughout by Proposition 9.1(iv). That last input is the only computation. □

Together, Theorem 9.8 and Proposition 9.9 say that on the range $E \ge 1/\sqrt{3}$ the set of energies with $G_4^{(2)}|_{S_E} > 0$ is exactly

$$\begin{aligned}&\big(\mathcal{E}_2, \mathcal{E}_1\big) \;\cup\; \big(\cos(\pi/4), \sqrt{\mathcal{E}_1}\big) \;\cup\; \big(\sqrt{2/3}, 1\big)\\ &\qquad = \;(0.57735\ldots, 0.61803\ldots) \cup (0.70710\ldots, 0.78615\ldots) \cup (0.81649\ldots, 1),\end{aligned}$$

the same set in every dimension $d \ge 1$, and that the two complementary intervals $(\mathcal{E}_1, \cos(\pi/4))$ and $(\sqrt{\mathcal{E}_1}, \sqrt{2/3})$ carry no such positivity, again in every dimension. The first of the three is the open second band. Comparison with Proposition 5.3 is worthwhile: on each of the four intervals into which $\mathcal{E}_1^2$, $\frac{1}{2}$ and $\mathcal{E}_1$ cut $(\frac{1}{3}, \frac{2}{3})$ exactly one of the two coefficient vectors makes its band polynomial positive, and the two alternate, while on $(\frac{2}{3}, 1)$ both do. Neither conjugate operator dominates the other, which is consistent with $\boldsymbol{\mu}_4(D)$ being defined by the existence of *some* $\mathbb{A}$.

Theorem 9.10 (the second band, in every dimension)**.** *Let $d \ge 1$ and let $x \in [-1, 1]^d$ have $E = \prod_i x_i > 0$.*

(i) *If $\frac{1}{3} < E^2 < \mathcal{E}_1^2$ — that is, if E lies in the open second band $(\mathcal{E}_2, \mathcal{E}_1)$ — then $G_4^{(2)}(x) > 0$.*
(ii) *If $\frac{1}{2} < E^2 < \mathcal{E}_1$ — that is, if $E \in (\cos(\pi/4), \sqrt{\mathcal{E}_1})$ — then $G_4^{(2)}(x) > 0$.*
(iii) *If $\frac{2}{3} < E^2 < 1$ — that is, if $E \in (\sqrt{2/3}, 1)$ — then $G_4^{(2)}(x) > 0$.*

In all three cases the bound $G_4^{(2)}(x) \ge E\,P_2(E^2) > 0$ holds with the constant of Theorem 9.8, which is attained.

Proof sketch. Each is Theorem 9.7 together with the corresponding case of Proposition 9.9; the hypothesis $E^2 \ge 1/3$ holds in all three, using the rational enclosure of $\sqrt{5}$ in case (ii). The only exhaustive computation is the twelve Booleans of Proposition 9.6. □

Case (i) with $d = 3$ is [1, Conjecture 1.15] for the second band, at the level to which his own equivalence reduces it, and with his own two-dimensional coefficients (8); the theorem settles it simultaneously for every dimension. Together with Theorem 6.1 it settles the conjecture for $n = 1$ and for $n = 2$; for $n \ge 3$ the source's coefficients are found numerically, and there is no exact statement to certify.

Corollary 9.11. *Assume the equivalence of* [1, §1, after (1.10)] *recalled in §1. Then, with $\mathbb{A} = A_4 + \rho_8^{(2)} A_8 + \rho_{12}^{(2)} A_{12} + \rho_{28}^{(2)} A_{28}$ and any $d \ge 1$, every energy of the open second band $(\mathcal{E}_2, \mathcal{E}_1)$, every energy of $(\cos(\pi/4), \sqrt{\mathcal{E}_1})$ and every energy of $(\sqrt{2/3}, 1)$ lies in $\boldsymbol{\mu}_4(D)$.*

As with Corollary 6.2, only part of this is new, and for the same reason. That $\pm(\cos(\pi/4), 1) \subset \boldsymbol{\mu}_{A_4}(D)$ for every $d \ge 1$ is [2, Lemma 4.7], and both of the upper intervals are subintervals of it; what Theorem 9.10(ii) and (iii) add there is the polynomial statement for this linear combination, with an attained constant, and not the membership. The second band is disjoint from [2]'s interval, and it is the

new membership statement. The operator-theoretic half of the corollary is again [1]'s functional calculus, and is not part of the formal development.

Corollary 9.12 (the two-dimensional band, closed)**.** *At $d = 2$, $G_4^{(2)} > 0$ on the whole of S_E for every energy E of the open second band.*

This is Theorem 9.10(i) at $d = 2$, but it is worth stating separately because that case has a history. The companion manuscript [12] certifies a strict positive lower bound for the same polynomial and the same coefficients on $29/50 \le E \le 309/500$, which is 93.40% of $(\mathcal{E}_2, \mathcal{E}_1)$; its two-variable folded certificate stops short of the left threshold at an attained interior zero, and its right endpoint was an economy call. The theorem above covers the band without a gap, and does so in every dimension at once. The formal development records two energies, one in each of the intervals [12] left out — $E = 289/500 = 0.578$ below and $E = 61801/100000 = 0.61801$ above — as explicit two-dimensional instances.

9.6. **Controls.**

Proposition 9.13 (controls, second band)**.**

(i) (too large a band, on the left) *For every $d \ge 1$, the point $(57/100, 1, \dots, 1)$ of $S_{57/100}$ has $G_4^{(2)} < 0$, and $57/100 < \mathcal{E}_2$.*

(ii) (too large a band, on the right) *For every $d \ge 1$, the point $(62/100, 1, \dots, 1)$ of $S_{62/100}$ has $G_4^{(2)} < 0$, and $62/100 > \mathcal{E}_1$.*

(iii) (the region hypothesis bites) $P_2(u) + P_2(v) < P_2(uv)$ *at $u = v = 3/50$; so the region $[1/3, 1]$ of Theorem 9.7 cannot be replaced by $(0, 1]$.*

(iv) (the partition is load-bearing) *The array for F on the whole box $[1/3, 1]^2$* rejects*; so does the array of the cell $[\frac{3}{5}, \frac{4}{5}] \times [\frac{11}{20}, \frac{3}{5}]$ without its subdivision, and so does the cofactor array without its subdivision.*

(v) (the enclosure is a hypothesis) *The core-A discriminant array still accepts when the enclosure of $\mathcal{E}_1$ is widened from width 10^{-6} to width 10^{-3}, and rejects at width 10^{-2}. Core A cannot be enlarged from $[\frac{3}{5}, \frac{4}{5}]^2$ to $[\frac{11}{20}, \frac{4}{5}]^2$: that array rejects.*

(vi) (the estimate is sharp) *For $d \ge 1$ and E as in Theorem 9.8 one has $G_4^{(2)}(E, 1, \dots, 1) = E\,P_2(E^2)$ exactly, so no factor greater than 1 may be inserted in Theorem 9.7.*

(vii) (non-vacuity in dimension 3) *In $d = 3$ the point $(289/500, 1, 1)$ satisfies the hypotheses of Theorem 9.10(i), and $G_4^{(2)} > 0$ there.*

Items (iv) and (v) are Boolean values of the same certificate pipeline as Proposition 9.6 on different boxes; what the development proves about them is that the Boolean is `false` (respectively `true`), and nothing more. The counts of negative Bernstein coefficients that go with the rejections — 128 for the root box, 46 for the undivided cell, 2 for the undivided cofactor, 29 for the widened enclosure, 84 for the enlarged core — were produced by an independent reimplementation of the same checker, verified slot for slot against it on one certificate, and are reported as measurements only. The remaining items are exact algebra and point evaluations.

Item (iii) deserves one more sentence, because it is not sharpness. The true threshold below which multiplicative superadditivity of P_2 fails was measured, outside the formal development, to lie between $611/10^4$ and $612/10^4$ — far below $1/3$. So $[1/3, 1]$ is not the largest region on which Theorem 9.7 is true; it is as far as the

band needs, and it is where the box list is six cells. Extending it downwards would enlarge the partition considerably and would buy nothing, since Proposition 9.9 already describes the positivity set on the whole of $E^2 \geq 1/3$, and item (i) exhibits an energy below $\mathcal{E}_2$ at which the positivity fails.

9.7. **What remains.** Three things. First, the bands $n \geq 3$: there the source's index set is larger — Σ has six elements at $n = 3$ — and both the interpolation nodes and the resulting coefficients are printed only as decimals, to four to six places [1, Tables 1 and 2], exact solutions being available, in its own words, only for very small κ and n. Of band 3's data only the left endpoint $\mathcal{E}_3$ is given in closed form, and that form is not a quadratic surd [1, (6.3)]. A gate for band 3 would therefore have to *derive* exact coefficients instead of reproducing printed ones, and would have to derive its nodes first; that is a harder problem, and it is why band 3 is not attempted here. Second, the $\boldsymbol{\mu}^-$ side, unchanged from §7: a *sub*additivity criterion would settle the maximum problem the same way and make the two-sided dimension-dependence explicit rather than exhibited at one energy. Third, the obstruction of §8 is still described rather than proved. That a Bernstein array cannot accept a box with an attained zero in its interior, or one at a corner where the mixed second derivative is negative, is a statement about the certificate machinery and not about these polynomials; formalising it would need a lemma relating the homogenised term list to the classical coefficient array, and would turn "we tried and it cannot work" into a theorem, here and in the companion manuscript at once.

Changes from version 1 of this paper. Nothing stated in version 1 is withdrawn: every theorem, proposition and corollary of that version stands here with its statement unchanged, and every attribution and every scope restriction it carries is unchanged too. What is new is the whole of §9, together with the data it needs (§2.2), items (v)–(viii) of the results list, and the corresponding sentences of the abstract, the introduction and the provenance paragraph. One section is *replaced*. Version 1 ended with a section, also about band 2, describing that band as an open problem and reporting computations made outside the formal development. Those computations were right as far as they went — the degree 14, the six roots, the kernel's bidegree $(13, 13)$ and its three zeros are all reproduced above — and so was its diagnosis, that what was missing was a decomposition. Its guess at the route was not: it expected three local decompositions glued by a partition along $\{1/3,\ 1/2,\ \mathcal{E}_1,\ 2/3,\ 1\}$, whereas two suffice (the third zero is free by $F(u, v) = F(v, u)$) and the partition that works is cut at $\{1/3,\ 11/20,\ 3/5,\ 4/5,\ 1\}$, so that no box corner is irrational. Finally, the counts in §10 are larger, under version 1's own convention, and exactly two of version 1's sentences were repaired, both because version 1 had only one certificate to speak of: the opening sentence of §2.3, which said that one positivity statement below is reduced to the Bernstein device, and a sentence in the proof of Proposition 4.3, which called that proposition's certificate the only exhaustive computation in the paper and now names the band it is about.

10. Verification

Every theorem, proposition and corollary above — Corollaries 6.2 and 9.11, whose status is described where they are stated, excepted — together with the

soundness of the certificate device of §2.3, has been machine-checked in Lean 4 (toolchain `v4.32.0`) against *Mathlib* [13, 14]; the development contains no `sorry` and declares no axiom by hand. It consists of seven modules and 201 lemmas and theorems, 91 of which are the statements above, counted as in version 1 of this paper, which reported 71 and 36 for its three modules.

The reasoning layer is free of compiled evaluation. For the first band that means: the criterion of §3, the separation, faithfulness and factorisation of Proposition 4.1, the kernel identity and the high half $K \geq 0$ of Proposition 4.2, the algebraic specialisations $\mathcal{E}_1^2 = 1 - \mathcal{E}_1 = (3 - \sqrt{5})/2$ and $(2 + 4\mathcal{E}_1)^2 = 20$, the rational enclosure $2236067977/10^9 \leq \sqrt{5} \leq 2236067978/10^9$ and the interval memberships it yields, the sign analysis of Proposition 5.3, the point evaluations and the exact two-dimensional argument of Theorem 7.1, and the controls 6.3(i), (ii), (iii), (vii). For the second band it means: the three identities that match the coefficients (8) to their $\mathbb{Z}[\mathcal{E}_1]$ forms, the separation and the factorisation of Proposition 9.1(i)–(iii), the two local decompositions of Proposition 9.4, the completion of the square (18) with its degenerate branch, the whole of Proposition 9.2, and the controls 9.13(i), (iii) and (vi). All of these depend on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`.

What is delegated to compiled evaluation, through Lean's `native_decide`, is exactly twenty-three Boolean tests on explicit integer term lists, each carrying one native axiom. Five belong to the first band: the accepting certificate of Proposition 4.3 and the four rejecting or accepting controls of Proposition 6.3(iv) and (v). Eighteen belong to the second: the twelve accepting arrays of Table 1 and the six controls of Proposition 9.13(iv) and (v). Thirteen of the twenty-three are used by a theorem — the first band's single certificate, which is the sole extra trust carried by Propositions 4.3 and 4.4, Theorems 5.1, 5.2 and 6.1 and control 6.3(vi); and all twelve of the second band's, which are the sole extra trust carried by every band-2 statement not in the list above. The split among them is as expected: the eleven arrays of the partition carry the inequality of Proposition 9.6 and Theorems 9.7 and 9.8; the twelfth, the cofactor array, carries Proposition 9.1(iv), Proposition 9.9 and control 9.13(ii); and Theorem 9.10, Corollary 9.12 and control 9.13(vii) rest on all twelve, as does Proposition 9.6 read as a statement about all twelve arrays. The remaining ten native axioms belong to controls that no theorem uses. Those thirteen Booleans are the only exhaustive computations on which any claim above rests.

The numbers reported alongside them are measurements of the same term lists: 175 coefficients at multidegree $(4, 4, 6)$ with no subdivision for the first band; for the second, the twelve arrays of Table 1, 6854 Bernstein coefficients over the eleven of the partition and 36 for the cofactor, 6890 in all, integers of at most 546 bits, ten of the twelve accepted without subdivision. The first band's three modules compile in about 45 seconds; the second band's four take about 470 seconds of wall time and 423 seconds of CPU at a peak resident set of 7.7 GB, of which the single identity (16) accounts for about 80 seconds (Remark 9.5).

The material described as lying outside the formal development — the general kernel identity (10), the Hessian and Bernstein computations of §8, the measured threshold $c^* = 0.0944243\ldots$, the ratio measurements after Proposition 6.3, the reported properties of the kernel W_2 (its bidegree, its grid minimum, its three zeros and its two displayed values), the band-2 superadditivity threshold measured

between $611/10^4$ and $612/10^4$, and the counts of negative Bernstein coefficients quoted for the rejecting controls — was produced in a separate implementation and is reported for context only; no theorem uses it. That separate implementation was itself checked against the Lean certificate machinery slot for slot on one certificate, all 24 coefficients agreeing exactly, before any of the larger arrays was attempted. The repository is private; repository reference to be added at submission.

References

[1] M.-A. Mandich, *Thresholds and more bands of A.C. spectrum for the Molchanov–Vainberg Schrödinger operator with a more general long range condition*, arXiv:2201.00410 [math.SP], 2022. All numbering quoted above is that of v1, the only version, and was read off a compilation of the v1 e-print source: Conjecture 1.15 (label `conjecture11_3d`), Theorem 1.1 (`lapy305`), equations (1.4), (1.8)–(1.10), (6.3) (`exact_sols_k4`, the closed forms of $\mathcal{E}_1, \dots, \mathcal{E}_8$ at $\kappa = 4$), (11.1) (`system_interpol`, n odd) and (11.2) (`system_interpol_even`, n even), Tables 1, 2 and 5, and §11 and §12. Band 2's exact coefficients are printed in §12 in an unnumbered display just after Table 2, together with the sentence quoted in §2.2. The arXiv record was checked on 2026-09-03 and still shows v1, submitted 2022-01-02, with no journal reference and no DOI.

[2] S. Golénia and M.-A. Mandich, *Bands of pure absolutely continuous spectrum for lattice Schrödinger operators with a more general long range condition*, J. Math. Phys. **62** (2021), no. 9, 092104. DOI `10.1063/5.0053416`; preprint arXiv:2102.00726 [math.FA], 2021, whose title carries the additional qualifier "Part I". Numbering quoted above is that of the v1 e-print, compiled: the separable form of §1 is its §4, equation (4.4) (source label `MV:k general222`); the dimension monotonicity is Lemma 4.3, the top-band lemma `edgySP` is Lemma 4.7, the computation of $\boldsymbol{\mu}_{A_4}(D_2)$ is Lemma 4.9, and the numerical tables are Tables 15 and 17. In the published version those two tables are numbered XV and XVII, which is how [1] cites them; that version sits behind a paywall and its internal numbering could not be checked directly.

[3] S. Golénia and M.-A. Mandich, *Thresholds and more bands of a.c. spectrum for the discrete Schrödinger operator with a more general long range condition*, arXiv:2201.09545 [math.FA], 2022. The standard-Laplacian companion of [1], carrying the analogous tables for Δ and the same $d = 3$ conjecture at $\kappa = 2$ (its `conjecture11_3d`: the same statement, with the bands $(\mathcal{E}_n + 1, \mathcal{E}_{n-1} + 1)$ and again "exactly" the two-dimensional coefficients); checked 2026-09-03, still v1, no journal reference.

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[13] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, Automated Deduction — CADE 28, Lecture Notes in Computer Science **12699**, Springer, 2021, 625–635. DOI 10.1007/978-3-030-79876-5_37.

[14] The mathlib Community, *The Lean mathematical library*, Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020), 367–381. DOI 10.1145/3372885.3373824. The development above was checked against the Mathlib revision 81a5d25.