GRAPHS ATTAINING $\dim_{\mathrm{m}}(G) = n(G)$ AT ORDERS 12 AND 13, WITH CERTIFIED MULTISET DIMENSIONS OF KING GRIDS

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ABSTRACT. The *multiset dimension* $\dim_{\mathrm{m}}(G)$ of a graph G is the least size of a landmark set $S \subseteq V(G)$ for which the multiset of distances from a vertex to S — the landmark labels forgotten — determines the vertex, and ∞ when no landmark set does. Simanjuntak, Siagian and Vetrík conjectured that the trivial bound $\dim_{\mathrm{m}}(G) \le n(G)$ is never attained. Allikvere refuted the conjecture by an exhaustive scan through order 11, which produced exactly eight extremal graphs, all of order 11; he then asked whether an extremal graph exists for every $n \ge 11$, reporting that his own one-vertex extension search at order 12 found none. We answer that question affirmatively at $n = 12$ and at $n = 13$, by two explicit graphs of 30 and 35 edges and diameter 3 in which the full vertex set is the *only* multiset resolving set. They were found by extending seeds that the earlier search discarded: the order-12 witness descends from an order-11 graph whose own full vertex set does not resolve. Alongside, we certify individual cells of Allikvere's king-grid theorems and the non-monotonicity of the height-four row $\dim_{\mathrm{m}}(P_4 \boxtimes P_n)$, and we bound by 7 four cells that his table leaves as > 6 and that an exhaustive search outside the verified development puts at exactly 7. Every value we certify is a witness together with an exhaustion of all smaller landmark sets, and every statement of the paper is formally verified in Lean 4; the few numbers that come from the unverified search are marked as such where they appear.

1. Introduction

Let G be a graph with vertex set $V(G)$, $n(G) = |V(G)|$, and geodesic distance d. For a set $S = \{s_1, \ldots, s_k\} \subseteq V(G)$ of *landmarks* the *multiset representation* of a vertex x is

$$m\big(x \mid S\big) \;=\; \{\{\, d(x, s_1),\, \ldots,\, d(x, s_k) \,\}\}, \tag{1}$$

the multiset of distances from x to the landmarks. The landmark labels are forgotten: this is the only difference from a classical resolving set, and it is what makes the parameter behave badly. The set S is *m-resolving* when $x \mapsto m\big(x \mid S\big)$ is injective on $V(G)$, and the *multiset dimension* is

$$\dim_{\mathrm{m}}(G) \;=\; \min\big\{\, |S| : S \subseteq V(G) \text{ is } m\text{-resolving} \,\big\} \in \mathbb{N} \cup \{\infty\},$$

with $\dim_{\mathrm{m}}(G) = \infty$ when no landmark set is m-resolving. The parameter is due to Simanjuntak, Siagian and Vetrík [6]. Two conventions travel with it and both are used here. First, $d(x, x) = 0$ is a genuine entry of $m\big(x \mid S\big)$ when $x \in S$; this is the convention of [6], and [1, §4.3] records that it "is essential; it is what forces $\dim_{\mathrm{m}}(G) \ne 2$". Second, $\dim_{\mathrm{m}}(G) = \infty$ occurs for many graphs, K_3 and C_4 among them, so the parameter is genuinely $(\mathbb{N} \cup \{\infty\})$-valued.

Two facts make the parameter awkward, and both are visible on tiny examples. The m-resolving sets do not form an upward-closed family: in K_2 a single landmark resolves, while the full vertex set does not, both vertices receiving $\{\{0, 1\}\}$. And the m-resolving sets do not form a nested family either: [1, §2] exhibits a graph of order 8 in which no 7-element landmark set resolves although $\dim_{\mathrm{m}} = 3$.

The question this paper answers. Because a resolving set has at most $n(G)$ elements, $\dim_{\mathrm{m}}(G) \le n(G)$ whenever $\dim_{\mathrm{m}}(G)$ is finite. Simanjuntak, Siagian and Vetrík conjectured that this bound is never attained.

Conjecture 1.1 ([6, Conjecture 2.1])**.** If G is a graph on n vertices having finite multiset dimension, then $\dim_{\mathrm{m}}(G) \le n - 1$.

The attribution deserves a word, because the two 2026 surveys disagree about it. [1, §1.1] quotes the conjecture from [6], while [3, Problem 2] records it as [7, Conjecture 1]. The primary sources settle it: the statement above is Conjecture 2.1 of [6] verbatim, and [7, Conjecture 1] is the same sentence restated with an explicit citation to [6] for it. So the conjecture is Simanjuntak, Siagian and Vetrík's, and [3] points at a restatement.

The survey of Farhan, Klavžar, Kuziak and Yero, written independently in the same season, puts the same point as an open question: "regarding the trivial upper bound $\dim_{\mathrm{m}}(G) \leq n(G)$, it is not clear whether there is a graph G for which $\dim_{\mathrm{m}}(G) = n(G)$" [3, §3.2]. The parameter was in fact surveyed twice in the same season, in [3] and independently in [4], whose extremal content is Conjecture 1.1 together with a bound for trees and which records no census at any order; the edge analogue, in which distances are measured to edges rather than to vertices, is a different parameter [5].

Allikvere [1] settled this by an exhaustive scan of all 1 018 690 328 connected graphs of orders 2 through 11: exactly eight of them satisfy $\dim_{\mathrm{m}}(G) = n(G)$, and all eight have order 11. Conjecture 1.1 is therefore false. What his scan could not reach is the next order, and [1, §5.2] says so explicitly, together with the one-vertex extension experiment he ran instead:

> "We checked all $8(2^{11} - 1) = 16\,376$ nonempty neighbourhood extensions of the eight labelled seed graphs by one new vertex . . . : none of the resulting DDI graphs of order 12 has $\dim_{\mathrm{m}} = 12$ (the best value attained is $11 = n - 1$, by an extension of G_2). A full scan of the $\sim 1.6 \times 10^{11}$ connected graphs of order 12 is beyond the scope of this paper. We leave the natural questions open."

Question 1.2 ([1, Problem 5.1])**.** Does a graph of order n with $\dim_{\mathrm{m}}(G) = n$ exist for infinitely many n? More strongly, does one exist for every $n \geq 11$?

Our main result answers the second half of Question 1.2 at the first two orders where it was open.

Theorem. *There is a connected graph of order* 12 *with* $\dim_{\mathrm{m}}(G) = 12$ *and a connected graph of order* 13 *with* $\dim_{\mathrm{m}}(G) = 13$*; in each of them the full vertex set is the unique m-resolving set. Both have diameter* 3*.*

The two graphs are printed in Section 3 as Theorem 3.1. Neither is found by the search of [1, §5.2], and the reason is a choice of seeds rather than a shortage of computer time. That search extended the eight graphs with $\dim_{\mathrm{m}} = 11$. The ancillary data of [1] also lists the 3 776 order-11 graphs with $\dim_{\mathrm{m}} = 10$ and 122 DDI graphs with $\dim_{\mathrm{m}} \in \{8, 9\}$, and those are the productive seeds: the order-12 witness below is a one-vertex extension of an order-11 graph whose full vertex set is not m-resolving at all — so it fails the very property ("DDI") that the earlier search filtered for. The whole computation is a core-minute once the seeds are chosen; the mechanical alternative, a full order-12 scan, is the $\sim 1.6 \times 10^{11}$ graphs that [1, §5.2] declines.

What else is here. The rest of the paper is a certified reconstruction of the published landscape around that result, in the following sense: every value we certify, new or reproduced, is presented as a *witness* (one landmark set of the stated size, checked to m-resolve) together with an *exhaustion* (no landmark set of smaller size m-resolves), and both halves are machine checked. The one place where only the witness half is in reach is Proposition 5.3, which therefore states bounds and not values; the matching lower bounds are quoted separately, in Remark 5.4, as computations. Section 2 fixes the certificate format and proves the one structural lemma we use, on twins. Section 3 is the new result. Section 4 re-derives the published order-11 data and the small-order end of the census. Section 5 does the same for king grids $P_h \boxtimes P_n$ — the strong product of two paths, the graph in which a chess king moves — where [1] settles the Hakanen–Yero question $\dim_{\mathrm{m}}(P_n \boxtimes P_n) = 4$ for $n \geq 5$ and the height-three case $\dim_{\mathrm{m}}(P_3 \boxtimes P_n) = n$ for $n \geq 6$, and where his computed table for heights 4 to 7 is exact only up to the value 6. Raising that cap by one bounds four of the cells he prints as > 6 by 7; an exhaustive search outside the verified development puts all four at exactly

7 and shows that the non-monotonicity he observed in the height-four row survives one level up. Section 6 lists the negative controls, and Section 7 describes the formal verification.

2. The multiset dimension as a certificate

Throughout, a graph on n vertices is identified with its vertex set $\{0, 1, \ldots, n-1\}$ and a distance function $D\colon V \times V \to \mathbb{N}$. All statements of this section are proved for an arbitrary such D, with no reference to how distances are computed; in the applications D is the geodesic distance, obtained by one breadth-first search per source. Definitions (1) and the following carry over verbatim to an arbitrary D, and we write $\dim_{\mathrm{m}}(D)$ for the resulting parameter.

Proposition 2.1 (certificate format)**.** *Let $D\colon V \times V \to \mathbb{N}$ with $|V| = n$, and let S, T range over subsets of V.*

(i) $\dim_{\mathrm{m}}(D) = \inf\{|S| : S$ is m-resolving$\}$ taken in $\mathbb{N} \cup \{\infty\}$, the infimum of the empty set being ∞; no case distinction between the finite and the infinite case is needed anywhere.

(ii) $\dim_{\mathrm{m}}(D) = \infty$ if and only if no $S \subseteq V$ is m-resolving.

(iii) If S is m-resolving, $|S| = k$, and no T with $|T| < k$ is m-resolving, then $\dim_{\mathrm{m}}(D) = k$. If merely S is m-resolving with $|S| = k$, then $\dim_{\mathrm{m}}(D) \le k$; if merely no T with $|T| < k$ is m-resolving, then $\dim_{\mathrm{m}}(D) \ge k$.

(iv) S is m-resolving if and only if the n representations $m\bigl(v \mid S\bigr)$, $v \in V$, listed in vertex order, are pairwise distinct.

(v) Each of "S is m-resolving", "no landmark set of size exactly j is m-resolving", "no landmark set of size less than k is m-resolving" and "no landmark set at all is m-resolving" is decidable, the last two by running over $\sum_{j<k} \binom{n}{j}$ and 2^n sets respectively.

Proof. Everything is formal. For (i) and (ii), $\dim_{\mathrm{m}}$ is the infimum of the image of the set of m-resolving sets under $S \mapsto |S|$ inside $\mathbb{N} \cup \{\infty\}$, and the infimum of the empty set is the top element ∞; (iii) is antisymmetry of $\le$ applied to the two halves. For (iv), injectivity of $v \mapsto m\bigl(v \mid S\bigr)$ on a finite vertex set is distinctness of the listed values; in the implementation each representation is compared as a sorted list, which is the same multiset and turns a permutation test into a structural one. (v) is finiteness of the search space. ☐

The point of (iii) and (iv) is bookkeeping: a claim $\dim_{\mathrm{m}}(G) = k$ is always split into a witness and an exhaustion, and (iv) is what makes the exhaustion affordable, since it builds each representation once instead of once per pair of vertices.

The single structural fact we use is a criterion for $\dim_{\mathrm{m}}(G) = \infty$ that involves no search. In the form we verified it against, it is [6, Lemmas 3.2 and 3.3]. Write $u \sim v$ when $u = v$ or $N(u) \setminus \{v\} = N(v) \setminus \{u\}$, an equivalence relation, and let v^* be the class of v. Then $|v^*| \ge 3$ forces $\dim_{\mathrm{m}}(G) = \infty$, and in a graph of finite multiset dimension a class with $|v^*| = 2$ meets every m-resolving set in exactly one vertex. The survey [3, §3.1] states the first half as "if a graph G has three vertices that are pairwise true twins (or pairwise false twins), then its multiset dimension is infinite" and attributes it to Saenpholphat [9, Theorem 3], a 2009 paper in Thai that we have not seen; the citation here is at second hand through [3]. The version below is stated on the distance function, so that a mixture of true and false twins is allowed, and it is proved for every n with no enumeration.

Definition 2.2. Vertices $u \ne v$ are *distance twins* for D when $D(u,u) = D(v,v)$, $D(u,v) = D(v,u)$, and $D(u,x) = D(v,x)$ for every $x \notin \{u,v\}$. True twins ($N[u] = N[v]$) and false twins ($N(u) = N(v)$) are distance twins for the geodesic distance.

Proposition 2.3 (cf. [6, Lemmas 3.2 and 3.3] and [3, §3.1])**.** *Let u, v be distance twins for D.*

(i) If S contains both of u, v or neither, then $m\bigl(u \mid S\bigr) = m\bigl(v \mid S\bigr)$. Consequently an m-resolving set contains exactly one of u and v.

(ii) If u, v, w are pairwise distance twins, then no landmark set is m-resolving, so $\dim_{\mathrm{m}}(D) = \infty$.

In particular $\dim_{\mathrm{m}}(K_3) = \infty$ and $\dim_{\mathrm{m}}(P_2 \boxtimes P_2) = \dim_{\mathrm{m}}(K_4) = \infty$.

Proof. Split S into $S \cap \{u, v\}$ and $S \setminus \{u, v\}$. On the second part the two representations agree entrywise, by the third clause of Definition 2.2. On the first part, if S contains neither vertex there is nothing to compare, and if it contains both then u contributes $\{\{D(u,u), D(u,v)\}\}$ where v contributes $\{\{D(v,u), D(v,v)\}\}$, and these two multisets coincide by the first two clauses — the diagonal entries are equal and the two cross entries are equal, so the pair of values is the same, only transposed. This proves (i). For (ii), each of the three pairs forces the landmark set to contain exactly one of its two members, and the three parity conditions "$u \in S \not\leftrightarrow v \in S$", "$u \in S \not\leftrightarrow w \in S$", "$v \in S \not\leftrightarrow w \in S$" cannot hold simultaneously. For the two examples, all three vertices of K_3 and all four of K_4 are pairwise distance twins. □

Remark 2.4. Part (i) is not itself an obstruction: a single twin pair never forces $\dim_{\mathrm{m}} = \infty$, it only forces the landmark set to contain exactly one of the pair. Nor is the triple hypothesis of part (ii) necessary: C_4 has $\dim_{\mathrm{m}} = \infty$ but only two twin pairs, its diagonals, and no three pairwise twins. See Section 6.

Graphs as data. Graphs quoted from the literature are given by their `graph6` codes [10], the format used by `nauty` and by [1], and are decoded before use; the decoder is pinned against printed data in Proposition 4.2. For king grids we follow the vertex convention of [1]: a vertex of $P_h \boxtimes P_n$ is a pair (x, y) with column $x < n$ and row $y < h$, and $P_h \boxtimes P_n$ is the graph on those hn cells in which two distinct cells are adjacent when they differ by at most one in each coordinate, so that the distance is the Chebyshev metric $\max(|\Delta x|, |\Delta y|)$.

3. Order 12 and order 13

Theorem 3.1. *Let G_{12} be the graph of order 12 with `graph6` code `K?B@eYw\f[gz` and G_{13} the graph of order 13 with `graph6` code ``L?`CRJWzNlm_A}``. Then:*

(i) G_{12} is connected, has 30 edges, degree sequence $(3,3,4,4,4,5,5,5,6,6,7,8)$ and diameter 3, and
$$\dim_{\mathrm{m}}(G_{12}) \;=\; 12 \;=\; n(G_{12});$$
(ii) G_{13} is connected, has 35 edges, degree sequence $(3,4,4,4,4,5,5,5,6,6,7,8,9)$ and diameter 3, and
$$\dim_{\mathrm{m}}(G_{13}) \;=\; 13 \;=\; n(G_{13}).$$

In each case the full vertex set is m-resolving and no proper subset is, so it is the unique m-resolving set. Consequently the answer to Question 1.2 is affirmative at $n = 12$ and at $n = 13$.

Proof, and what was computed. Each of the two statements is a single decidable proposition, and both halves of it were evaluated exhaustively. For G_{12}: the full vertex set is m-resolving, and every one of the $2^{12} - 1 = 4095$ landmark sets of size at most 11 — the empty set included — is not, so by Proposition 2.1(iii) the value is 12. For G_{13} the same computation runs over the $2^{13} - 1 = 8191$ landmark sets of size at most 12. The auxiliary data (connectivity, edge count, degree sequence, diameter) is read off the decoded adjacency matrix and its breadth-first distance matrix. No part of either claim is an extrapolation from the other, and neither depends on how the graph was found. □

The search. The two graphs were produced by a one-vertex extension search, run in C before anything was verified, and it is worth recording exactly what was enumerated, because the outcome turns on the choice of seeds rather than on the size of the search.

(i) *Reproduction first.* The experiment of [1, §5.2] was rerun on its own terms: extending the eight order-11 graphs with $\dim_{\mathrm{m}} = 11$ by one vertex in all $8(2^{11} - 1) = 16\,376$ ways gives no graph of order 12 with $\dim_{\mathrm{m}} = 12$, the best finite value attained being $11 = n - 1$. This matches [1, §5.2] exactly, including the "best is $n - 1$" clause.

(ii) *Order* 12. The ancillary scan output of [1] lists, besides those eight, the 3 776 order-11 graphs with $\dim_{\mathrm{m}} = 10$ and, among the graphs its own filter kept, the 29 with $\dim_{\mathrm{m}} = 9$ and the 93 with $\dim_{\mathrm{m}} = 8$ — all of the latter two DDI. Extending all 3 906 of them by one vertex in every way produces 7 995 582 labelled graphs of order 12, of which 246 161 survive the preliminary filter of [1, §4.2] — step (3) of its scanner, which discards a graph when its full vertex set does not m-resolve and no $(n-1)$-subset does, a step that can only remove graphs with $\dim_{\mathrm{m}} \le n-2$ or $\dim_{\mathrm{m}} = \infty$; exactly one of the surviving extensions has $\dim_{\mathrm{m}} = 12$, and 3 691 have $\dim_{\mathrm{m}} = 11$.

(iii) *Order* 13. Repeating the step on the 3 692 order-12 graphs with $\dim_{\mathrm{m}} \ge 11$ produces 15 118 740 labelled graphs of order 13, exactly one of which has $\dim_{\mathrm{m}} = 13$.

(iv) *Where it stops.* The same step on the 2 520 order-13 graphs with $\dim_{\mathrm{m}} \ge 12$ builds 20 641 320 labelled graphs of order 14 and produces none with $\dim_{\mathrm{m}} = 14$; the best finite value reached is $13 = n-1$.

Deleting the last vertex of G_{12} gives the order-11 graph `J?B@eYw\f[_`, which has $\dim_{\mathrm{m}} = 10$ and whose full vertex set is *not* m-resolving; G_{12} is its extension by a vertex joined to $\{1, 5, 6, 7, 9, 10\}$. This is precisely why the search of [1, §5.2] could not have found G_{12}: it kept only seeds whose full vertex set resolves. Deleting the last vertex of G_{13} gives an order-12 graph with $\dim_{\mathrm{m}} = 11$, namely `K?`CRJWzNlm_`, and deleting one more gives `J?`CRJWzNl_`, which is the graph called G_2 in [1, Table 1]; so the order-13 witness is a two-step extension of one of the eight published counterexamples.

Remark 3.2 (what is not claimed). Theorem 3.1 is an existence statement and nothing more. How many graphs of order 12 or 13 attain $\dim_{\mathrm{m}}(G) = n(G)$ is not determined here; the enumeration above is over one-vertex extensions of selected seeds, not over all graphs of those orders. In particular the count "exactly eight" at order 11 remains the result of [1, Theorem 1.1] and its billion-graph scan, which we do not reproduce. The case $n = 14$ of Question 1.2 remains open: the extension recipe is exhausted there, and widening the seed set to the order-13 graphs with $\dim_{\mathrm{m}} = n-2$ is the obvious next step.

Remark 3.3 (diameter). All eight order-11 examples of [1, Table 1] have diameter 3, and so do G_{12} and G_{13}. These are two further data points for [1, Problem 5.2]: "Characterize the graphs with $\dim_{\mathrm{m}}(G) = n(G)$, or find an infinite family. All eight known examples have diameter 3 and girth 3; is diameter 3 necessary apart from finitely many exceptions?" We offer no structural explanation.

4. The published census, re-derived

Everything in this section is a check on data already in the literature, carried out so that Theorem 3.1 sits on ground that was verified rather than assumed. None of it is claimed as new.

Proposition 4.1. *Each of the eight graphs of* [1, Table 1]*, listed below by* ***graph6*** *code, is connected of order* 11 *and satisfies* $\dim_{\mathrm{m}}(G) = 11 = n(G)$*; in each the full vertex set is the unique m-resolving set. In particular Conjecture 1.1 is false, and a graph with* $\dim_{\mathrm{m}}(G) = n(G)$ *exists, which answers the question quoted from* [3, §3.2] *in Section 1.*

G_1: `J?BDf?[hvq_` G_2: `J?`CRJWzNl_` $G_3$: `J?AF?zUyfh_` $G_4$: `J?B@eRhuT_`
G_5: `J?bB`rcb^h_` $G_6$: `J?ABcZirTz?` $G_7$: `J?`e`rcJ^h_` $G_8$: `J?ABFDyvfm_`

Moreover G_1 *has* 23 *edges, degree sequence* $(2, 3, 3, 3, 4, 4, 4, 5, 5, 6, 7)$ *and diameter* 3.

Proof. For each code, the full vertex set is checked to be m-resolving and each of the $2^{11} - 1 = 2047$ landmark sets of size at most 10 is checked not to be; the value follows from Proposition 2.1(iii). The refutation of Conjecture 1.1 uses G_1 alone: it is connected, its multiset dimension is finite, and it exceeds $n - 1 = 10$. □

The count and the exhaustiveness in [1, Theorem 1.1] — that these are the *only* graphs of order at most 11 with $\dim_{\mathrm{m}}(G) = n(G)$ — rest on a scan of 1 018 690 328 connected graphs and are not reproduced. What is reproduced is the decoding, so that the graphs being checked are demonstrably the published ones.

Proposition 4.2. *The* `graph6` *decoding of the code listed for* G_1 *has exactly the* 23 *edges printed in* [1, §3]*, and its distance-degree sequences* $\big(a_0, a_1, a_2, a_3\big)(v)$*, where* $a_i(v)$ *is the number of vertices at distance exactly* i *from* v*, are, in vertex order,*

$$\begin{array}{llll} (1,5,5,0) & (1,3,6,1) & (1,4,5,1) & (1,3,7,0) \\ (1,2,7,1) & (1,4,6,0) & (1,4,4,2) & (1,5,4,1) \\ (1,3,5,2) & (1,6,4,0) & (1,7,3,0) & \end{array}$$

agreeing entry for entry with the table in the proof of [1, Proposition 3.1]*, which prints the triples* (a_1, a_2, a_3)*.*

Proof. Direct comparison of the decoded adjacency matrix with the printed edge list, and of the distance matrix with the printed table. □

The negative half of [1, Theorem 1.1] — that all eight have order exactly 11, so no graph of smaller order attains the bound — is out of reach in the same way, but its bottom rows are not.

Proposition 4.3. *Every connected graph on at most* 5 *vertices has* $\dim_{\mathrm{m}}(G) \le 3$ *or* $\dim_{\mathrm{m}}(G) = \infty$*; at orders* 3 *and* 4 *the finite values are all* ≤ 1*. Consequently no connected graph of order* 4 *or* 5 *satisfies* $\dim_{\mathrm{m}}(G) = n(G)$*. The bound* 3 *at order* 5 *is attained, so it is not an overestimate.*

Proof. Exhaustive over labelled graphs: for $n \le 5$ every one of the $2^{\binom{n}{2}}$ edge sets is generated, the connected ones are kept (1024 edge sets at $n = 5$, of which 728 are connected), and for each the assertion "no landmark set m-resolves, or some landmark set of size $\le k$ does" is decided by running over all 2^n subsets. Working with labelled graphs rather than isomorphism classes costs nothing for a maximum, since every value that occurs, occurs on a labelled graph. Tightness at order 5 is one exhibited edge set for which no landmark set of size ≤ 2 resolves. The statement about $\dim_{\mathrm{m}}(G) = n(G)$ follows because $3 < 5$ and $1 < 4$. □

These are the rows $n = 3, 4, 5$ of the maximum column of [1, Table 2], and they agree with it. The same scan, run outside the formal development in C, reaches $n = 7$ and gives maximum finite values $1, 1, 1, 3, 4, 5$ at $n = 2, \ldots, 7$ — the published column at every order in reach — with connected labelled graph counts $1, 4, 38, 728, 26\,704, 1\,866\,256$, which is the sequence [11, A001187] exactly and serves as a positive control on the enumeration.

5. King grids

[1, Theorem 1.2] proves $\dim_{\mathrm{m}}(P_n \boxtimes P_n) = 4$ for every $n \ge 5$, answering a question of Hakanen and Yero [8], who had computed $\dim_{\mathrm{m}}(P_4 \boxtimes P_4) = 6$ and $\dim_{\mathrm{m}}(P_5 \boxtimes P_5) = \dim_{\mathrm{m}}(P_6 \boxtimes P_6) = 4$ and proved $3 \le \dim_{\mathrm{m}}(P_n \boxtimes P_n) \le 4$ for $n \ge 7$. [1, Theorem 1.3] proves $\dim_{\mathrm{m}}(P_3 \boxtimes P_n) = n$ for $n \ge 6$, which is the height-three case of the rectangular problem that opens the problem list of [3] — "Determine $\dim_{\mathrm{m}}(P_n \boxtimes P_m)$, $n, m \ge 3$" [3, Problem 1], posed there after [8]. We certify the individual cells.

Proposition 5.1. *With landmarks written as cells* (x, y)*, column then row:*

grid	$\dim_{\mathrm{m}}$	*witness*
$P_3 \boxtimes P_3$	∞	—
$P_3 \boxtimes P_4$	5	$(0,0)(0,1)(1,0)(1,2)(3,0)$
$P_3 \boxtimes P_5$	6	$(0,0)(0,1)(1,0)(1,2)(3,0)(4,0)$
$P_3 \boxtimes P_6$	6	$(0,0)(1,0)(2,0)(3,0)(3,2)(5,0)$
$P_3 \boxtimes P_7$	7	$(0,0)(1,0)(1,2)(3,0)(4,0)(4,2)(6,0)$
$P_3 \boxtimes P_8$	8	$(0,0)(1,0)(1,2)(3,0)(4,0)(4,2)(6,0)(7,0)$
$P_4 \boxtimes P_4$	6	$(0,0)(0,1)(0,2)(1,1)(2,3)(3,2)$
$P_5 \boxtimes P_5$	4	$(0,0)(0,1)(0,4)(4,0)$
$P_6 \boxtimes P_6$	4	$(0,0)(0,4)(1,4)(5,5)$
$P_7 \boxtimes P_7$	4	$(0,0)(0,1)(0,6)(6,0)$
$P_8 \boxtimes P_8$	4	$(0,0)(0,5)(1,5)(7,7)$
$P_9 \boxtimes P_9$	4	$(0,0)(0,1)(0,8)(8,0)$

Proof. Each finite value is the witness in the table, checked to be m-resolving, together with the exhaustion of every landmark set of smaller size, layer by layer over the $\binom{hn}{j}$ sets of each size j below the value. At $P_7 \boxtimes P_7$ that is $\binom{49}{0} + \cdots + \binom{49}{3} = 19\,650$ sets and at $P_9 \boxtimes P_9$ it is $\binom{81}{0} + \cdots + \binom{81}{3} = 88\,642$ sets. For $P_3 \boxtimes P_3$, all $2^9 = 512$ landmark sets are checked to fail, so the value is ∞ by Proposition 2.1(ii). □

The values in Proposition 5.1 are those of [1, Theorems 1.2 and 1.3] and, for $P_4 \boxtimes P_4$, $P_5 \boxtimes P_5$ and $P_6 \boxtimes P_6$, of [8] as quoted in [1, §1.2]; they were independently recomputed here before being certified, and agree with the published ancillary tables cell for cell. The infinite theorems themselves are not formalized: what is certified is each listed cell.

The height-four row. For heights 4 to 7, [1, §19] reports a computed landscape from an exhaustive search capped at 6: exact whenever the value is at most 6, and recorded as > 6 otherwise. It singles out the height-four row:

> "The height-4 row is not monotone in n_2: sizes 5 and 6 and values > 6 interleave ($n_2 = 11$: 5; $n_2 = 12$: 6; $n_2 = 13$: >6; $n_2 = 14$: 6). Determining $\dim_{\mathrm{m}}(P_4 \boxtimes P_n)$ exactly appears to require new ideas and is left open."

Non-monotonicity is the reason no argument by monotonicity in the board length can settle the row, so it is worth having as a theorem rather than as a table entry.

Proposition 5.2. $\dim_{\mathrm{m}}(P_4 \boxtimes P_{10}) = 6$*,* $\dim_{\mathrm{m}}(P_4 \boxtimes P_{11}) = 5$ *and* $\dim_{\mathrm{m}}(P_4 \boxtimes P_{12}) = 6$*. In particular*

$$\dim_{\mathrm{m}}(P_4 \boxtimes P_{11}) \;<\; \dim_{\mathrm{m}}(P_4 \boxtimes P_{10}) \quad \textit{and} \quad \dim_{\mathrm{m}}(P_4 \boxtimes P_{11}) \;<\; \dim_{\mathrm{m}}(P_4 \boxtimes P_{12}),$$

so $n \mapsto \dim_{\mathrm{m}}(P_4 \boxtimes P_n)$ *is not monotone.*

Proof. Witnesses: $(0,0)(0,1)(1,2)(3,3)(6,0)(9,3)$ for $P_4 \boxtimes P_{10}$; $(0,0)(1,0)(4,3)(7,0)(10,3)$ for $P_4 \boxtimes P_{11}$; $(0,0)(1,0)(2,0)(5,3)(8,0)(11,3)$ for $P_4 \boxtimes P_{12}$. Exhaustions: every landmark set of size at most 5 in $P_4 \boxtimes P_{10}$ fails, which is $\binom{40}{0} + \cdots + \binom{40}{5} = 760\,099$ sets; likewise every landmark set of size at most 4 in $P_4 \boxtimes P_{11}$ and of size at most 5 in $P_4 \boxtimes P_{12}$, the last being the largest of the three searches. The inequalities are then arithmetic. □

The cells left at > 6. Raising the search cap from 6 to 7 bounds four of the cells that [1, §19] prints as > 6, and — outside the verified development, in Remark 5.4 — decides them. This is that paper's own method with a larger cap and a few minutes of processor time; we claim no new technique, only the cells.

Proposition 5.3. $\dim_{\mathrm{m}}(P_4 \boxtimes P_{13}) \leq 7$*,* $\dim_{\mathrm{m}}(P_4 \boxtimes P_{15}) \leq 7$*,* $\dim_{\mathrm{m}}(P_4 \boxtimes P_{17}) \leq 7$ *and* $\dim_{\mathrm{m}}(P_5 \boxtimes P_{15}) \leq 7$*; in particular all four are finite. Moreover* $\dim_{\mathrm{m}}(P_4 \boxtimes P_{13}) \geq 5$*.*

Proof. The four witnesses, in the coordinates of Section 2, are

$$\begin{array}{ll} P_4 \boxtimes P_{13}: & (0,0)\,(0,1)\,(1,2)\,(3,3)\,(6,0)\,(9,3)\,(12,0) \\ P_4 \boxtimes P_{15}: & (0,0)\,(1,0)\,(2,0)\,(5,3)\,(8,0)\,(11,3)\,(14,0) \\ P_4 \boxtimes P_{17}: & (0,0)\,(1,0)\,(4,3)\,(7,0)\,(10,3)\,(13,0)\,(16,3) \\ P_5 \boxtimes P_{15}: & (0,0)\,(0,1)\,(2,4)\,(5,0)\,(8,3)\,(11,0)\,(14,3). \end{array}$$

Each is checked to be m-resolving, which gives the upper bounds by Proposition 2.1(iii). The lower bound at $P_4 \boxtimes P_{13}$ is the exhaustion of every landmark set of size at most 4, which is $\binom{52}{0} + \cdots + \binom{52}{4} = 294\,204$ sets. ☐

Remark 5.4 (exhaustive search outside the formal development). The matching lower bounds that would turn the four inequalities of Proposition 5.3 into equalities are out of reach of the verified exhaustion: ruling out size 6 at $P_4 \boxtimes P_{13}$ alone is $\binom{52}{6} = 20\,358\,520$ landmark sets. They were obtained instead by an exhaustive search in C, which reproduces every printed cell of the landscape of [1, §19] before reaching these, and which reports

$$\dim_{\mathrm{m}}(P_4 \boxtimes P_{13}) = \dim_{\mathrm{m}}(P_4 \boxtimes P_{15}) = \dim_{\mathrm{m}}(P_4 \boxtimes P_{17}) = \dim_{\mathrm{m}}(P_5 \boxtimes P_{15}) = 7,$$
$$\dim_{\mathrm{m}}(P_4 \boxtimes P_{16}) > 7, \qquad \dim_{\mathrm{m}}(P_4 \boxtimes P_{18}) > 7.$$

These six values are computations, not verified theorems, and are offered as such. Read against the published row, the height-four sequence at $n_2 = 11, \ldots, 18$ becomes $5, 6, 7, 6, 7, {>}7, 7, {>}7$: the interleaving that [1, §19] observes between 5, 6 and > 6 continues one level up, between 7 and > 7. The same search, run past the surveyed range of [1], finds a second non-monotone pair at height six: $\dim_{\mathrm{m}}(P_6 \boxtimes P_{18}) = 6 > 5 = \dim_{\mathrm{m}}(P_6 \boxtimes P_{19})$. Deciding $P_4 \boxtimes P_{16}$ and $P_4 \boxtimes P_{18}$ at 8 would need $\binom{64}{8}$ and $\binom{72}{8}$ landmark sets, and the automorphism group of the board has order 4, so symmetry buys only a factor of four; an argument, which [1, §19] says it does not have, would be worth more.

6. Controls

Every claim in this paper is a value of a parameter obtained by exhaustive search, so the controls have to refute the two ways such a value can be wrong — too large and too small — exercise the two conventions of Section 1, and show that no hypothesis is carried vacuously.

Proposition 6.1. *(i)* A set that fails. *The landmark set* $\{0, 3, 7\}$ *of* G_1 *is not* m*-resolving, while the full vertex set of* G_1 *is.*

(ii) Neighbouring values. $\dim_{\mathrm{m}}(G_1) \neq 10$ *and* $\dim_{\mathrm{m}}(G_1) \neq 12$*; and* $\dim_{\mathrm{m}}(K_4) \neq 4$*, a finite value refuted for a graph whose value is* ∞*.*

(iii) The convention $d(x, x) = 0$. *In* K_2 *the landmark set* $\{0\}$ *is* m*-resolving and the full vertex set is not: both vertices receive* $\{\{0, 1\}\}$*. Hence* $\dim_{\mathrm{m}}(K_2) = 1$ *although* $V(K_2)$ *does not resolve. The two vertices of* K_2 *are distance twins, so a twin* pair *alone never forces* ∞*.*

(iv) The convention $\dim_{\mathrm{m}} = \infty$, twice. $\dim_{\mathrm{m}}(K_3) = \infty$ *by Proposition 2.3 with no search at all;* $\dim_{\mathrm{m}}(C_4) = \infty$ *by exhausting all* 2^4 *landmark sets, in a graph where the hypothesis of Proposition 2.3(ii) provably fails, since two adjacent vertices of* C_4 *are not distance twins. The criterion is therefore sufficient and not necessary.*

(v) Non-vacuity. G_1 *has no distance twin pair whatsoever, so no statement about the extremal graphs is a disguised twin argument; and* C_4 *and* K_2 *show the twin hypothesis is satisfiable.*

(vi) The trap inside Conjecture 1.1. *For the graph of order* 8 *with* `graph6` code `` G?`cmW ``*, from* [1, §2]*, the full vertex set is* m*-resolving and no* 7*-element landmark set is, and yet* $\dim_{\mathrm{m}} = 3$*. "No* $(n-1)$*-element landmark set resolves" is therefore not evidence for* $\dim_{\mathrm{m}} = n$ *— which is the intuition Conjecture 1.1 rested on.*

Proof. (i), (ii), (v) and (vi) are direct evaluations on the decoded graphs; the exhaustion in (vi) runs over the $\binom{8}{7} = 8$ landmark sets of size 7 and, for the value 3, over all 2^8 landmark sets. (iii) and (iv) are evaluations on graphs of order at most 4, and the case of K_3 is Proposition 2.3. ☐

Item (vi) is the reason Theorem 3.1 states both halves. A graph in which the full vertex set is the only candidate that has not yet been ruled out is not thereby extremal, and the order-8 example shows the gap is real.

7. Verification

Every statement in this paper — Propositions 2.1, 2.3, 4.1, 4.2, 4.3, 5.1, 5.2, 5.3, 6.1 and Theorem 3.1 — has been formally verified in Lean 4 against *Mathlib*, in a development that is free of `sorry`; repository reference to be added at submission. The reasoning layer is kernel-clean, depending on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`: this covers the definition of $\dim_{\mathrm{m}}$ as an infimum in $\mathbb{N} \cup \{\infty\}$ and all of Proposition 2.1, the twin lemmas and hence Proposition 2.3 in full for every n (including $\dim_{\mathrm{m}}(K_3) = \dim_{\mathrm{m}}(K_4) = \infty$), and the parts of Proposition 6.1 that concern K_2, K_3, C_4 and K_4. What is delegated to compiled evaluation, through Lean's `native_decide`, is exactly the finite searches, one Boolean each: the eight order-11 certificates and the two decoder checks of Section 4; the two certificates of Theorem 3.1, over 4095 and 8191 landmark sets; the census over all labelled graphs on at most five vertices; each grid and rectangle cell of Section 5, over its own layered exhaustion; and the evaluations in Proposition 6.1. The exhaustive searches that do *not* fit in that budget are named where they occur and are not stated as theorems: the billion-graph scan of [1, Theorem 1.1] behind the count "exactly eight", which we do not reproduce, and the size-6 and size-7 exhaustions behind Remark 5.4. Independently of the formal development, every value certified here was first recomputed by a separate implementation in C (bitmask breadth-first search, multiset representations packed four bits per distance value, subsets enumerated by depth-first search with incremental keys), which reproduces the rectangle landscape of [1, §19] cell for cell, and the small-order rows of its Table 2, before reaching anything new; the two witnesses of Theorem 3.1 were in addition re-checked with the independent audit program shipped in the ancillary package [2], whose distances come from Boolean matrix powers and whose `graph6` parser shares no code with either search.

References

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[6] R. Simanjuntak, P. Siagian and T. Vetrík, *The multiset dimension of graphs*, arXiv:1711.00225v2 (2019). Conjecture 2.1 and Lemmas 3.2 and 3.3 are cited from v2; v1 (1 November 2017) carries the author list R. Simanjuntak, T. Vetrík and P. Bintang Mulia, which is why [3] and [1] name the paper differently.

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[9] V. Saenpholphat, *On multiset dimension in graphs* (in Thai), Proceedings of the 3rd Srinakharinwirot Academic Conference (2009), 193–202, as recorded in [3]; not seen by us.

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[11] OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, `https://oeis.org`; entry A001187 (connected labelled graphs).