TWO RIGOROUS MOURRE BANDS FOR THE MOLCHANOV–VAINBERG LAPLACIAN IN DIMENSION TWO

KOREA SUPERINTELLIGENCE LABS

Abstract. Let $D = \Delta_1\Delta_2$ be the Molchanov–Vainberg Laplacian on $\ell^2(\mathbb{Z}^2)$, with $\Delta_i = (S_i + S_i^*)/2$. For a conjugate operator $\mathbb{A} = \sum_j \rho_{j\kappa} A_{j\kappa}$ built from the dilation-type generators $A_{j\kappa}$, a strict Mourre estimate for D at an energy E is equivalent to strict positivity, on the constant-energy surface $S_E = \{x \in [-1,1]^2 : x_1x_2 = E\}$, of the explicit Chebyshev combination $G_\kappa = \sum_j \rho_{j\kappa} g_{j\kappa}$, $g_{j\kappa}(x_1,x_2) = x_2(1-x_1^2)U_{j\kappa-1}(x_1) + x_1(1-x_2^2)U_{j\kappa-1}(x_2)$. Mandich (arXiv:2201.00410) determines the coefficients $\rho_{j\kappa}$ band by band, by interpolation at threshold energies, and then supplies *plots* of G_κ; he states plainly that he was not able to prove a Mourre estimate on any new interval, and that the new bands of absolutely continuous spectrum are "justified mainly by graphical evidence".

We turn the first two of those plots, at $\kappa = 4$ and $d = 2$, into theorems with explicit constants. For the first band $(\mathcal{E}_1, \mathcal{E}_0) = \big((\sqrt{5}-1)/2, \cos(\pi/4)\big)$ and the author's own coefficient $\rho_8 = (17+8\sqrt{5})/62$, we prove $G_4(x_1,x_2) \geq \frac{1}{25}\big(E^2 - \frac{3-\sqrt{5}}{2}\big)\big(1-2E^2\big)$, $E = x_1x_2$, on the whole *closed* band, the right-hand side vanishing exactly at the two thresholds. For the second band $(\mathcal{E}_2, \mathcal{E}_1) = (1/\sqrt{3}, (\sqrt{5}-1)/2)$ and the author's exact interpolation coefficients in $\mathbb{Q}(\sqrt{5})$, we prove $G_4(x_1,x_2) \geq \frac{1}{500}(3E^2-1)\big(\frac{3-\sqrt{5}}{2} - E^2\big)$ for $29/50 \leq E \leq 309/500$, which is 93.40% of that band. Along the way we exhibit, for each band, an *attained interior* zero of G_4 on a threshold surface that the interpolation system forces but does not impose; these zeros are what limits every subdivision-based certificate, and the device that removes the first of them — a fold onto $x_1 \geq x_2$ combined with an exact one-parameter deformation of the algebraic threshold — is what closes the first band. All positivity claims rest on the nonnegativity of explicit finite integer Bernstein coefficient arrays; every statement below is machine-checked in Lean 4. The passage from these polynomial inequalities to the Mourre estimate for D itself is the source's functional calculus, quoted rather than reproved.

1. Introduction

The operator and the question. The Molchanov–Vainberg Laplacian on $\mathcal{H} = \ell^2(\mathbb{Z}^d)$ is

$$D \;=\; \prod_{i=1}^{d} \Delta_i, \qquad \Delta_i := \tfrac{1}{2}(S_i + S_i^*),$$

where S_i is the shift in the i-th coordinate; $\sigma(D) = [-1,1]$ and the spectrum is purely absolutely continuous. It was introduced by Molchanov and Vainberg [2]; its spectral theory under long-range perturbations was developed by Golénia and Mandich [6] and, in the form we use, by Mandich [1].

Let V be a real potential vanishing at infinity and satisfying, for a fixed even κ and some $q > 2$, the long-range condition $n_i(V - \tau_i^\kappa V)(n) = O(\ln^{-q}|n|)$ in every coordinate direction. Mourre theory [3, 4] produces a limiting absorption principle — and hence the absence of singular continuous spectrum — for $D+V$ near an energy E from a *strict Mourre estimate*

$$1_I(D)\,[D,i\mathbb{A}]_\circ\,1_I(D)\ \geq\ \gamma\,1_I(D),\qquad \gamma>0,\ I\ni E, \tag{1}$$

for a self-adjoint conjugate operator $\mathbb{A}$, together with a potential-side estimate which, in the form of [5] that [1] adopts, asks that $\ln^p(1+|n|)\,[V,i\mathbb{A}]_\circ\,\ln^p(1+|n|)$ be compact for some $p>1$. In the present setting the resulting implication is [1, Theorem 1.1]. The conjugate operators matched to the above long-range condition are the finite combinations

$$\begin{gathered}\mathbb{A}=\sum_{j\geq 1}\rho_{j\kappa}A_{j\kappa},\qquad A_{j\kappa}=\sum_{i=1}^{d}A_i(j,\kappa),\\ A_i(j,\kappa)=\tfrac{1}{4i}\left[(S_i^{j\kappa}-S_i^{-j\kappa})N_i+N_i(S_i^{j\kappa}-S_i^{-j\kappa})\right],\end{gathered} \tag{2}$$

with N_i the i-th position operator. Following [1] write $\boldsymbol{\mu}_\kappa(D)$ for the set of energies at which (1) holds for *some* $\mathbb{A}$ of the form (2), and $\boldsymbol{\Theta}_\kappa(D)=\sigma(D)\setminus\boldsymbol{\mu}_\kappa(D)$ for its complement, the *thresholds*. The problem of [1] is to decide, for as many E as possible, on which side of this partition E falls. All section, equation, lemma, theorem and table numbers we quote from [1] are those of its arXiv version v1.

The polynomial reformulation. The point of the choice (2) is that everything becomes a question about one explicit polynomial. Let U_n be the Chebyshev polynomial of the second kind. Since $[D,iA_{j\kappa}]_\circ=\sum_i D\Delta_i^{-1}(1-\Delta_i^2)U_{j\kappa-1}(\Delta_i)$ and the Δ_i commute, functional calculus attaches to the commutator the polynomial

$$g_{j\kappa}(x)\ =\ \sum_{i=1}^{d}\Big(\prod_l x_l\Big)x_i^{-1}(1-x_i^2)U_{j\kappa-1}(x_i),\qquad G_\kappa\ =\ \sum_{j\geq 1}\rho_{j\kappa}\,g_{j\kappa},$$

and, with $S_E:=\{x\in[-1,1]^d:\prod_l x_l=E\}$ the constant-energy surface, $G_\kappa|_{S_E}$ is a functional representation of $1_{\{E\}}(D)[D,i\mathbb{A}]_\circ 1_{\{E\}}(D)$. Consequently $E\in\boldsymbol{\mu}_\kappa(D)$ if and only if $G_\kappa|_{S_E}>0$ for some admissible choice of the coefficients. The source puts this, immediately after its definition of S_E, as "By functional calculus and continuity of the function G_κ we have $E\in\boldsymbol{\mu}_\kappa(D)$ iff $G_\kappa|_{S_E}>0$" [1, §1, after (1.10)], and repeats it in the energy-localised form "$E\in\boldsymbol{\mu}_\kappa(D)$ if and only if $G_\kappa^E\big|_{S'_E}>0$" in [1, §3, after (3.4)]. The quantifier over the coefficients $\rho_{j\kappa}$ is left implicit there — it sits in the definition of $\boldsymbol{\mu}_\kappa(D)$, which asks for the existence of *some* combination (2) — and only one direction is used below: for one fixed $\mathbb{A}$, strict positivity of G_κ on S_E gives (1) on a neighbourhood of E. In dimension $d=2$, which is the case treated here,

$$g_{j\kappa}(x_1,x_2)\ =\ x_2(1-x_1^2)U_{j\kappa-1}(x_1)+x_1(1-x_2^2)U_{j\kappa-1}(x_2). \tag{3}$$

What [1] proves, and what it plots. Fix $d=2$ and an even $\kappa\geq 4$, and set $J_2=J_2(\kappa)=(\cos^2(\pi/\kappa),\cos(\pi/\kappa))$ [1, (1.14)]. Mandich proves rigorously [1, Theorem 1.6] that J_2 contains a strictly decreasing sequence of thresholds $\mathcal{E}_1>\mathcal{E}_2>\mathcal{E}_3>\cdots\searrow\cos^2(\pi/\kappa)$, all lying in $\boldsymbol{\Theta}_\kappa(D)$, together with $\mathcal{E}_0:=\cos(\pi/\kappa)$, a threshold by [1, Lemma 1.4]; for $\kappa=4$ the first two of these are given in closed form in [1, (6.3)], and they and the right endpoint are exactly

$$\mathcal{E}_1=\tfrac{\sqrt{5}-1}{2}\simeq 0.61803,\qquad \mathcal{E}_2=\tfrac{1}{\sqrt{3}}\simeq 0.57735,\qquad \mathcal{E}_0=\cos(\pi/4)=\tfrac{1}{\sqrt{2}}.$$

He then conjectures [1, Conjecture 1.8] that each interval $(\mathcal{E}_n, \mathcal{E}_{n-1})$ — the *n-th band* — is contained in $\boldsymbol{\mu}_\kappa(D)$, and gives a scheme for producing the witness [1, (11.1), (11.2)]: choose a finite index set $\Sigma = \{j\kappa\}$ and solve, for the coefficients $\rho_{j\kappa}$, the linear system that requires G^E_κ to vanish (and, at interior nodes, to have vanishing derivative) at the coordinate-wise energies of the two bounding thresholds. For $\kappa = 4$, $d = 2$ the first band uses $\Sigma = \{4, 8\}$ with the *exact* coefficient $\rho_8 = (17+8\sqrt{5})/62$, and the second uses $\Sigma = \{4, 8, 12, 28\}$ with what the author calls "a rare oc[c]urrence of an exact solution": coefficients in $\mathbb{Q}(\sqrt{5})$ with numerators of up to 24 digits.

What is missing is the last step. In the author's own words [1, §1]: "As for step (3), the polynomial interpolation is implemented numerically. Thus the coefficients $\rho_{j\kappa}$ are found numerically, and these are then used to plot a functional representation of (1.4) [= (1) here]. This is the evidence we share to see that strict positivity is in fact obtained." And, flatly: "Unfortunately I was not successful in rigorously proving a Mourre estimate on any new interval." The abstract of [1] repeats that "the new bands of a.c. spectrum are justified mainly by graphical evidence".

Results. We prove the positivity at $\kappa = 4$, $d = 2$ on all of the first band and on 93.40% of the second, with the author's own coefficients and with explicit constants. Throughout, G_4 denotes the combination (3) for the band under discussion, and $E = x_1 x_2$.

Theorem 1.1 (first band, closed). *Let $\rho_4 = 1$ and $\rho_8 = (17 + 8\sqrt{5})/62$, and let $G_4 = g_4 + \rho_8 g_8$. For all $x_1, x_2 \in [0, 1]$ with $\frac{3-\sqrt{5}}{2} \le E^2 \le \frac{1}{2}$,*

$$G_4(x_1, x_2) \;\ge\; \tfrac{1}{25}\Big(E^2 - \tfrac{3-\sqrt{5}}{2}\Big)\big(1 - 2E^2\big),$$

and the right-hand side is strictly positive whenever both inequalities are strict. Since $\frac{3-\sqrt{5}}{2} = \mathcal{E}_1^2$ and $\frac{1}{2} = \mathcal{E}_0^2$, the lower bound vanishes exactly at the two thresholds of the first band and is positive strictly between them.

Theorem 1.2 (second band). *Let $\rho_4 = 1$ and let $\rho_8, \rho_{12}, \rho_{28} \in \mathbb{Q}(\sqrt{5})$ be the exact interpolation coefficients of* [1] *reproduced in* (5) *below, and let $G_4 = g_4 + \rho_8 g_8 + \rho_{12} g_{12} + \rho_{28} g_{28}$. For all $x_1, x_2 \in [0, 1]$ with $\frac{29}{50} \le E \le \frac{309}{500}$,*

$$G_4(x_1, x_2) \;\ge\; \tfrac{1}{500}\big(3E^2 - 1\big)\Big(\tfrac{3-\sqrt{5}}{2} - E^2\Big) \;>\; 0.$$

The two zeros of the lower bound are exactly the band's thresholds $\mathcal{E}_2 = 1/\sqrt{3}$ and $\mathcal{E}_1 = (\sqrt{5} - 1)/2$, and $[\,29/50,\ 309/500\,]$ is 93.40% of $(\mathcal{E}_2, \mathcal{E}_1)$.

Both theorems are statements about the polynomial G_4, which is what the certificates see; the passage to the spectral conclusion is the source's, not ours.

Corollary 1.3. *Assume the equivalence of* [1] *recalled above — for a fixed $\mathbb{A}$ of the form* (2)*, strict positivity of G_κ on S_E implies* (1) *near E — and his reduction of S_E to its part in the positive quadrant. Then, with $\mathbb{A} = A_4 + \frac{17+8\sqrt{5}}{62} A_8$, every energy of the open first band $(\mathcal{E}_1, \mathcal{E}_0) = \big((\sqrt{5} - 1)/2, \cos(\pi/4)\big)$ lies in $\boldsymbol{\mu}_4(D)$; and with $\mathbb{A} = A_4 + \rho_8 A_8 + \rho_{12} A_{12} + \rho_{28} A_{28}$ and the coefficients* (5)*, so does every energy in* $[29/500, 309/500]$.

The operator-theoretic half of Corollary 1.3 — spectral projections, commutator extensions, functional calculus on $\ell^2(\mathbb{Z}^2)$ — is not part of the formal development; only the polynomial inequalities are. The two results of [1] it borrows are, precisely:

the equivalence recalled in §1 above, [1, §1, after (1.10)] and, in the energy-localised form, [1, §3, after (3.4)]; and [1, Lemma 3.2], that G_κ^E is an even polynomial for even κ — which is the reason the source itself gives, in [1, §12], for looking only at x positive. Both are stated there qualitatively. $\boldsymbol{\mu}_\kappa(D)$ is defined by the *existence* of an interval $I \ni E$ and a $\gamma > 0$ for which (1) holds, and [1] asserts no relation between that γ and the size of G_κ on S_E; accordingly Corollary 1.3 claims membership in $\boldsymbol{\mu}_4(D)$ and nothing about the size of γ. The explicit constants of Theorems 1.1 and 1.2 are lower bounds for the polynomial G_4, not for the γ of (1).

Theorem 1.1 is obtained by combining two certificates; the second of them, Theorem 4.2 below, covers 99.65% of the first band with a better constant and is recorded separately because it is used in the proof and is sharper where it applies.

A structural fact explains where the certificates can and cannot reach. For each band the interpolation system imposes vanishing of G_4 at certain nodes and *forces* vanishing at others it does not list; the forced zeros are interior points of a threshold surface, and at the symmetric point $x_1 = x_2 = \sqrt{E}$ the vanishing is of second order, because $\frac{d}{dx}G_\kappa^E(\sqrt{E}) = 0$ holds identically for every choice of coefficients [1, Lemma 3.3]. Propositions 3.3 and 5.3 locate these points, and §6 explains why an attained interior zero defeats every subdivision tree.

Method. Three devices, in order.

(i) Reduction. With $X = x_1^2$ and $Z = E^2$ the whole problem becomes a two-variable polynomial inequality: $62\,x_1^8 G_4 = 8x_1x_2(136 + 64\sqrt{5})\,R(X,Z)$ for band 1 (Proposition 3.1), of bidegree $(8,4)$, and an analogous identity of bidegree $(28,14)$ for band 2 (Proposition 5.4). Neither reduction is written in [1].

(ii) The fold. S_E is symmetric in $x_1 \leftrightarrow x_2$, and in the reduced variables $R(X,Z)/X^4$ is invariant under $X \mapsto Z/X$. It therefore suffices to certify on the half $x_1 \geq x_2$, i.e. on $X \in [E,1]$ rather than $X \in [E^2,1]$. This moves the offending interior zero onto the box face $X = E$, where a Bernstein coefficient *equals* the value, and turns the double zero in X into a simple zero in the box coordinate.

(iii) The exact deformation. The band's endpoint $\mathcal{E}_1$ is irrational, so the box cannot simply be extended to it with rational data. Instead the whole algebraic input is carried by one extra box variable: the factor $X^2 - X + \gamma$, $\gamma = \sqrt{5} - 2$, produced by the interpolation is replaced by $\psi_r(X) = (X - r)(X - r^2)$, whose two roots sit at box corners for *every* r. At $r = \mathcal{E}_1$ one has $\psi_{\mathcal{E}_1}(X) = X^2 - X + \gamma$, because $\mathcal{E}_1 + \mathcal{E}_1^2 = 1$ and $\mathcal{E}_1^3 = \sqrt{5} - 2$; so the certificate proves a one-parameter family of statements over a rational window $r \in [3/5, 13/20]$, one member of which is the assertion about D. With both devices in place the first band needs *no subdivision at all*: the root Bernstein expansion, at multidegree $(8,8,9)$ and 780 integer coefficients, is already coefficientwise nonnegative.

Provenance. Checked 2026-08-30: the source stands at v1 (arXiv:2201.00410, submitted 2022-01-02, never revised) and a citation search returns no citations of it. Its immediate predecessor [6] — same two operators, same long-range condition, but a single conjugate operator rather than a combination — has exactly two citers: one treats a different operator, and the other is the Golénia–Mandich standard-Laplacian companion [7], whose abstract likewise says "The methodology is backed primarily by graphical evidence". An arXiv metadata search for the string `Molchanov-Vainberg` returns exactly three papers — [1], [6], and one unrelated item on the one-dimensional Anderson Hamiltonian — none of them a follow-up. Metadata search does not index full text, so a proof inside a paper that does not name the operator would not surface; what is asserted here is only that the source calls its own claim graphical, that it is unrevised and uncited, and that no paper naming the operator claims a proof.

2. Preliminaries

2.1. **Chebyshev data.** U_n denotes the Chebyshev polynomial of the second kind, T_n the first kind. We use $U_{2k-1} = 2T_kU_{k-1}$ and, more generally at $\kappa = 4$,

$$U_{4j-1}(y) \;=\; U_3(y)\cdot\bigl(U_{j-1}\circ T_4\bigr)(y), \qquad U_3(y) = 4y(2y^2-1), \tag{4}$$

which is what makes the four-term combinations of §5 tractable: every U_{4j-1} carries the common factor U_3, and the residual factor is a polynomial in $T_4(y) = 8y^4 - 8y^2 + 1$, hence in $Y = y^2$. In the formal development $U_0, \dots, U_{27}$ are derived from the three-term recurrence.

2.2. **Bernstein certificates.** All positivity statements below are reduced to the following device, which is standard [8] and is implemented once, independently of this paper's subject matter, in the formal development. Let $F \in \mathbb{Z}[t_1, \dots, t_m]$ have multidegree $n = (n_1, \dots, n_m)$. Its homogenisation

$$\widehat{F}(u,v) = \prod_{j=1}^{m}(u_j+v_j)^{n_j}\cdot F\bigl(u_1/(u_1+v_1),\ \dots,\ u_m/(u_m+v_m)\bigr)$$

has monomial coefficients equal to the Bernstein coefficients of F at multidegree n, each scaled by a positive binomial factor, and $\widehat{F}(t, 1-t) = F(t)$. So the assertion that all Bernstein coefficients of F are nonnegative is literally the assertion that $\widehat{F}$ has nonnegative integer coefficients, which for $u, v \geq 0$ gives $F \geq 0$ on $[0,1]^m$ with no basis conversion. Homogenisation is multiplicative, so $\widehat{F}$ is *built* by the same additions and multiplications as the defining expression for F: the coefficient array is the output of the pipeline, not an input to be trusted.

When the root array is not nonnegative one subdivides. The two half-box substitutions $t_j \mapsto t_j/2$ and $t_j \mapsto (1+t_j)/2$ act on the homogenised representation as integer linear substitutions with nonnegative coefficients, so a certificate is a binary subdivision tree all of whose leaves pass the coefficientwise test. We write tree$[j_1, \dots, j_k]$ for the uniform tree that splits along axis j_1, then j_2, and so on, giving 2^k leaves. Soundness — if the tree test returns `true` then $F \geq 0$ on $[0,1]^m$ — is proved once and for all; the only thing computed per band is the Boolean value of that test on an explicit integer term list, and every claim below that rests on exhaustive computation rests on exactly such a Boolean.

2.3. **Band data.** For $\kappa = 4$, $d = 2$, [1, Table 1] tabulates the band endpoints, the interpolation nodes X_q and the index sets Σ. The endpoints are given there in closed form at [1, (6.3)]; the nodes are printed only as decimals ($X_1 \simeq 0.7861$ for band 1; $X_1 \simeq 0.7071$, $X_2 \simeq 0.8164$ for band 2), and the closed forms below are the exact values that reproduce them — and, for band 2, the ones from which (5) is recovered exactly in §5. So:

band	left endpoint	right endpoint	Σ
1	$\mathcal{E}_1 = \frac{\sqrt{5}-1}{2}$, node $X_1 = \sqrt{\mathcal{E}_1} \simeq 0.7861$	$\mathcal{E}_0 = \cos(\pi/4)$	$\{4, 8\}$
2	$\mathcal{E}_2 = \frac{1}{\sqrt{3}}$, nodes $X_1 = \cos(\pi/4)$, $X_2 = \sqrt{2/3}$	$\mathcal{E}_1$	$\{4, 8, 12, 28\}$

with $[\rho_4, \rho_8] = [1, (17+8\sqrt{5})/62]$ for the first band [1, Table 2] and, for the second, in the exact form displayed just after [1, Table 2],

$$\begin{aligned} \rho_8 &= \tfrac{1222116864552852421713 92}{1452414998884814056999 61} - \tfrac{3260892257389038020608}{145241499888481405699961}\sqrt{5}, \\ \rho_{12} &= \tfrac{3077650500266123893602}{132037727171346732454 51} - \tfrac{2331950449491527564 16}{1320377271713467324545 1}\sqrt{5}, \\ \rho_{28} &= \tfrac{2012748316225734218931}{145241499888481405699961} + \tfrac{9001291944393521723 52}{145241499888481405699961}\sqrt{5}. \end{aligned} \tag{5}$$

We write $\Delta_{\mathrm{den}} := 145241499888481405699961$.

All theorems below are stated for $x_1, x_2 \in [0,1]$. This is no loss for the intended spectral reading: by [1, Lemma 3.2], G_κ^E is an even polynomial for even κ — which is exactly the reduction the source itself performs, “thanks to Lemma 3.2 it is enough to observe $G_\kappa^E(x)$ for x positive” [1, §12] — and S_E with $E > 0$ is the union of its part in $[0,1]^2$ and the image of that part under $x \mapsto -x$. This reduction is quoted, not reproved: it is not part of the formal development.

3. The first band: reduction and forced zeros

Throughout this section $\rho_4 = 1$, $\rho_8 = (17+8\sqrt{5})/62$, $G_4 = g_4 + \rho_8 g_8$, and

$$X = x_1^2, \qquad Z = (x_1x_2)^2 = E^2, \qquad \gamma = \sqrt{5} - 2.$$

Proposition 3.1 (the reduction)**.** *Set*

$$R(X,Z) \;=\; X^4(1-X)(2X-1)\big(X^2-X+\gamma\big) \;+\; (X-Z)(2Z-X)\big(Z^2-ZX+\gamma X^2\big).$$

Then, as an identity of polynomials in x_1, x_2,

$$62\, x_1^8\, G_4(x_1,x_2) \;=\; 8\, x_1x_2\, (136 + 64\sqrt{5})\, R\big(x_1^2, (x_1x_2)^2\big).$$

In particular R has bidegree $(8,4)$*, nine monomials, and is* linear *in γ: $R = R_a + \gamma R_b$ with $R_a, R_b \in \mathbb{Q}[X,Z]$.*

Proof sketch. By (4), $U_3 + \rho_8 U_7 = U_3(1 + 2\rho_8 T_4)$, and the interpolation equation for the first band is precisely $1 + 2\rho_8 T_4(\mathcal{E}_1) = 0$; solving it gives $1 + 2\rho_8 T_4(y) = 16\rho_8(y^2 - \mathcal{E}_1^2)(y^2 - \mathcal{E}_1)$, so in the variable $X = y^2$ the coefficient disappears into the positive scalar $16\rho_8$ and leaves the factor $X^2 - X + \gamma$ with $\gamma = \mathcal{E}_1^3 = \sqrt{5} - 2$. Substituting into (3) and clearing x_1^8 gives the displayed identity; both sides are polynomials in $x_1, x_2, \sqrt{5}$ and the identity is an algebraic consequence of $(\sqrt{5})^2 = 5$ alone. No computation beyond polynomial arithmetic is involved. The equality $8\sqrt{5} + 48 = (136 + 64\sqrt{5})(\sqrt{5} - 2)$ is what packages $R_a + \gamma R_b$ into the single factor above; the formal statement is in the split form. $\square$

The linearity in γ is the design decision behind §4: it converts an algebraic box into a rational one.

Proposition 3.2 (the imposed zero)**.** $G_4(\mathcal{E}_1, 1) = 0$*, where* $\mathcal{E}_1 = (\sqrt{5} - 1)/2$*.*

This is the single equation ($2n - 1 = 1$ at $n = 1$) that the interpolation system [1, (11.1)] imposes for the first band, at the node $X_{0,1} = \mathcal{E}_1$ of $S_{\mathcal{E}_1}$, i.e. at $(x_1, x_2) = (\mathcal{E}_1, 1)$; the proposition says that the tabulated $\rho_8 = (17+8\sqrt{5})/62$ meets it exactly, which was the entry gate for this work.

Proposition 3.3 (the forced interior zero)**.** *If $y > 0$ and $y^2 = \mathcal{E}_1$, then $G_4(y, y) = 0$.*

Proof sketch. By Proposition 3.1 it suffices that $R(\mathcal{E}_1, \mathcal{E}_1^2) = 0$, and both zeros of G_4 on $S_{\mathcal{E}_1}$ reduce to the same scalar identity $1 + 2\rho_8(9 - 16\mathcal{E}_1) = 0$. The conceptual reason is the author's own: $\mathcal{E}_1$ lies in $\boldsymbol{\Theta}_{1,4}(D)$ [1, Theorem 1.6], so $g_{j4}(x_1) = \omega_0\, g_{j4}(x_0)$ for all j and hence $G_4(x_1) = \omega_0 G_4(x_0) = 0$ for every choice of coefficients [1, (1.11)–(1.12)]. The consequence for this band is not drawn in [1], but the argument is his. $\square$

Proposition 3.3 is the obstruction of §6. It is a *double* zero: $\frac{d}{dx} G^E_\kappa(\sqrt{E}) = 0$ identically [1, Lemma 3.3], so the symmetric point $x_1 = x_2 = \sqrt{E}$ is always a critical point of G^E_κ. Exactly, with $\beta = \mathcal{E}_1$,

$$\begin{aligned} R(X, \beta^2) &= (X - \beta^2)(1 - X)(X - \beta)^2\, V(X), \\ V(X) &= (1 - 3\gamma) + \tfrac{5\gamma - 1}{2} X + \tfrac{1 - 3\gamma}{2} X^2 + \gamma X^3 + 2X^4, \end{aligned} \tag{6}$$

and every coefficient of V is positive, so the threshold edge itself is nonnegative with no certificate at all. The squared factor is the double zero. Identity (6) is a working note of the obstruction analysis: it is not part of the formal development, and it is used here only for explanation, never in a proof.

4. The first band: the certificates

4.1. The first certificate. Parametrise $Z \in [13/34, 1/2]$ and $X \in [Z, 1]$ by

$$34Z = 13 + 4\sigma, \qquad X = Z + (1 - Z)\tau, \qquad \gamma = \tfrac{236067 + \omega}{10^6}, \qquad (\sigma, \tau, \omega) \in [0, 1]^3,$$

the last line being the point of Proposition 3.1: because R is linear in γ, promoting γ to a third box variable makes the whole build rational, and one certificate covers every γ in the bracket $[0.236067, 0.236068]$, the true $\sqrt{5} - 2$ among them.

Proposition 4.1 (the 256-leaf Boolean)**.** *Write $M_1(Z) = (3Z - Z^2 - 1)(1 - 2Z)$. The integer polynomial $1000 \cdot 34^8 \cdot 10^6 \big(R - M_1/1000\big)$, expressed in (σ, τ, ω), passes the subdivision test of §2.2 with the uniform tree* tree$[0, 1, 0, 1, 0, 1, 0, 1]$ *— four splits on each of the two genuine axes,* 256 *leaves, multidegree* $(8, 8, 1)$.

This is the finite search behind Theorem 4.2: a single Boolean, evaluated by compiled arithmetic on an explicit integer term list, whose value is `true`. Nothing else in the proof of Theorem 4.2 is computational.

Theorem 4.2 (first band, 99.65%)**.** *For all $x_1, x_2 \in [0, 1]$ with $\frac{13}{34} \le E^2 \le \frac{1}{2}$, $E = x_1 x_2$,*

$$G_4(x_1, x_2) \;\ge\; \tfrac{1}{50}\big(3E^2 - E^4 - 1\big)\big(1 - 2E^2\big).$$

The right-hand side vanishes exactly at $E^2 = \frac{3 - \sqrt{5}}{2} = \mathcal{E}_1^2$ and at $E^2 = \frac{1}{2}$ and is strictly positive between them, so the bound is a positive constant at every $E \in (\sqrt{13/34}, \cos(\pi/4))$, an interval covering 99.65% of the first band.

Proof sketch. Proposition 4.1 and the soundness of the checker give the inequality $R(X, Z) \ge M_1(Z)/1000$, valid for all Z in $[13/34, 1/2]$, all X in $[Z, 1]$, and every γ in the bracket. The rational enclosure

$$\tfrac{2236067}{10^6} \;\le\; \sqrt{5} \;\le\; \tfrac{2236068}{10^6},$$

itself a consequence of $(\sqrt{5})^2 = 5$ and $\sqrt{5} \ge 0$ by ordinary real arithmetic, puts the true $\gamma = \sqrt{5} - 2$ in that bracket. Feed this into Proposition 3.1 and use the four bounds $0 < x_1^8 \leq 1$, $(x_1x_2)^2 \leq x_1^2$, $E \geq \sqrt{13/34} > 0.618$ and $136 + 64\sqrt{5} > 279$; the stated bound with constant $1/50$ follows. The coverage figure is the ratio

$$\frac{\cos(\pi/4) - \sqrt{13/34}}{\cos(\pi/4) - \mathcal{E}_1} = 0.99648\ldots,$$

the two left endpoints in the variable $Z = E^2$ being $13/34 = 0.3823529\ldots$ and $\mathcal{E}_1^2 = 0.3819660\ldots$. □

Proposition 4.3 (sharpness of the first certificate)**.** *The same build is rejected by the same tree when the margin constant* $1/1000$ *is raised to* $1/250$*; the root expansion alone is rejected; two splits per axis (*16 *leaves) are rejected ; and the bracket for* γ *cannot be loosened from* 10^{-6} *to* 10^{-3} *— the build with* $\gamma = (236+\omega)/1000$ *is rejected by the same tree .*

Each clause is again a single Boolean, this time with value `false`. Outside the formal development we measured the depth needed as a function of the left endpoint: $Z_{\mathrm{lo}} = 5/13$ needs three splits per axis, $13/34$ needs four, $34/89$ needs six — the price of the last 0.35% grows fast, for the reason given in §6.

4.2. **The fold and the deformation.** Two changes close the band.

The fold. $R(X,Z)/X^4 = f(X) + f(Z/X)$ with $f(u) = (1-u)(2u-1)(u^2-u+\gamma)$, hence $R(X,Z)/X^4$ is invariant under the involution $X \mapsto Z/X$, which on S_E is exactly the swap $x_1 \leftrightarrow x_2$. Restricting to $x_1 \geq x_2$ means $X \geq E$, so the box becomes $X \in [E,1]$ instead of $X \in [E^2,1]$. The interior zero of Proposition 3.3 then sits at $X = E$, i.e. on the box face $\tau = 0$, where a Bernstein coefficient equals the value of the polynomial; and the vanishing, quadratic in X, is only simple in the box coordinate, since with $X = \sqrt{Z}(1+u)$ one has $X + Z/X - 2\sqrt{Z} \asymp \sqrt{Z}\,u^2$.

The deformation. Replace the factor $X^2 - X + \gamma$ of Proposition 3.1 by $\psi_r(X) = (X-r)(X-r^2) = X^2 - (r+r^2)X + r^3$, and set

$$\begin{aligned}\widetilde{R}(X,Z,r) &= X^4(1-X)(2X-1)\psi_r(X)\\ &+ (X-Z)(2Z-X)\big(Z^2-(r+r^2)ZX+r^3X^2\big),\\ \widetilde{M}(Z,r) &= (Z-r^2)(1-2Z).\end{aligned}$$

At $r = \mathcal{E}_1$ one has $r + r^2 = 1$ and $r^3 = \sqrt{5} - 2$, so $\widetilde{R}(\cdot,\cdot,\mathcal{E}_1) = R$ and $\widetilde{M}(\cdot,\mathcal{E}_1)$ vanishes exactly at the two thresholds. For *every* r the two zeros of the deformed problem sit at box corners — $X = r$ on the face $\tau = 0$, and $X = 1$ where $\psi_r(r^2) = 0$ — which is why r may range over a wide rational window instead of a 10^{-6} bracket. This is the difference between an approximation and a deformation: the statement being certified is true for every r in the window, and the assertion about D is one of its members.

Proposition 4.4 (the folded Boolean)**.** *Parametrise, for* $(\sigma,\tau,\omega) \in [0,1]^3$*,*

$$r = \tfrac{600+50\omega}{1000}, \qquad E = \tfrac{1000r+(707-1000r)\sigma}{1000}, \qquad X = E + (1-E)\tau,$$

so that $r \in [3/5, 13/20]$*,* $E \in [r, 707/1000]$ *and* $X \in [E,1]$*. The integer polynomial* $10^{27}\big(100\,\widetilde{R} - \widetilde{M}\big)$ *has multidegree* $(8,8,9)$ *and* 780 *Bernstein coefficients of at most* 109 *bits, and* all of them are nonnegative. *The certificate is the root expansion, with no subdivision.*

Theorem 4.5 (first band, closed)**.** *For all* $x_1, x_2 \in [0,1]$ *with* $\frac{3-\sqrt{5}}{2} \le E^2 \le \frac{1}{2}$, $E = x_1 x_2$,

$$G_4(x_1,x_2) \;\ge\; \tfrac{1}{25}\Big(E^2 - \tfrac{3-\sqrt{5}}{2}\Big)\big(1-2E^2\big),$$

and $G_4(x_1,x_2) > 0$ *whenever* $\frac{3-\sqrt{5}}{2} < E^2 < \frac{1}{2}$. *This is Theorem 1.1.*

Proof sketch. Three ingredients, of which exactly one is computational.

(a) The algebraic specialisation, kernel-checked. From $(\sqrt{5})^2 = 5$ alone, $\mathcal{E}_1^2 = \frac{3-\sqrt{5}}{2}$, $\mathcal{E}_1 + \mathcal{E}_1^2 = 1$ and $\mathcal{E}_1^3 = \sqrt{5} - 2$; hence $\widetilde{R}(X, Z, \mathcal{E}_1) = R(X, Z)$ identically and $\widetilde{M}(Z, \mathcal{E}_1) = (Z - \mathcal{E}_1^2)(1 - 2Z)$. This step is exact real algebra — no bracketing, no numerics.

(b) The enclosure, ordinary arithmetic. $2236067/10^6 \le \sqrt{5} \le 2236068/10^6$, whence $3/5 \le \mathcal{E}_1 \le 13/20$: the specialisation $r = \mathcal{E}_1$ lands inside the window of Proposition 4.4. Only this step uses a rational enclosure, and it is used only to locate $\mathcal{E}_1$ in the window, never to approximate the polynomial.

(c) The certificate, exhaustive computation. Proposition 4.4 together with the soundness of §2.2 gives $\widetilde{R} \ge \widetilde{M}/100$ on the whole box, hence, by (a) and (b), $R(X, Z) \ge (Z - \mathcal{E}_1^2)(1 - 2Z)/100$ for $E \in [\mathcal{E}_1, 707/1000]$ and $X \in [E, 1]$.

Now let $x_1, x_2 \in [0,1]$ with $\mathcal{E}_1^2 \le E^2 \le \frac{1}{2}$ and write $M(Z) = (Z - \mathcal{E}_1^2)(1-2Z)$. If $E \le 707/1000$, the swap symmetry of G_4 lets us assume $x_2 \le x_1$, so $X = x_1^2 \ge E$ and (c) applies; with Proposition 3.1, $E \ge 3/5$, $136 + 64\sqrt{5} > 279$ and $0 < x_1^8 \le 1$ this yields $G_4 \ge M(E^2)/25$ (indeed with room to spare). If $E \ge 707/1000$ then $E^2 \ge 0.499849 > 13/34$, so Theorem 4.2 applies and gives $G_4 \ge M_1(E^2)/50$; and $M_1(Z) - 2M(Z) = (1 - 2Z)(\mathcal{E}_1 - Z)(Z - \mathcal{E}_1^2) \ge 0$ on $[\mathcal{E}_1^2, \frac{1}{2}]$ — an identity modulo $\mathcal{E}_1^2 + \mathcal{E}_1 - 1 = 0$ — so $M_1/50 \ge M/25$ there as well. Finally $707/1000 < \cos(\pi/4) < 708/1000$, so the two ranges cover the closed band. Strict positivity of the lower bound at interior energies is immediate. □

Proposition 4.6 (sharpness of the folded certificate)**.** *The margin* $1/100$ *of Proposition 4.4 cannot be raised to* $1/65$*; the deformation window* $r \in [3/5, 13/20]$ *cannot be widened to* $[29/50, 67/100]$*; and the right endpoint* $707/1000$ *cannot be pushed to* $708/1000$*. Each of the three widened builds is rejected by the root expansion.*

Since $\cos(\pi/4) = 0.7071068\ldots$ lies strictly between $707/1000$ and $708/1000$, the third clause says the certificate stops at the true right threshold and not before it. Outside the formal development we also measured that the margin $1/70$ still passes and that the window may be widened to $[59/100, 33/50]$; neither measurement is used.

Proposition 4.7 (coverage and non-vacuity)**.** *The hypotheses of Theorem 4.2 are satisfiable, e.g. at* $x_1 = x_2 = 4/5$ *. The hypotheses of Theorem 4.5 are satisfiable, e.g. at* $(x_1, x_2) = (7/10, 9/10)$*, and Theorem 4.5 strictly extends Theorem 4.2: at* $(x_1, x_2) = (1, 3091/5000)$ *one has* $\frac{3-\sqrt{5}}{2} < E^2 \le \frac{1}{2}$ *while* $\frac{13}{34} \le E^2$ *fails .*

Proposition 4.8 (the band is not larger)**.** *We have* $G_4(617/1000, 1) < 0$ *. We also have* $G_4(708/1000, 1) < 0$ *.*

Since $617/1000 < \mathcal{E}_1$ and $708/1000 > \cos(\pi/4)$, these two point evaluations show that both endpoints of Theorem 4.5 are genuine features of the source's construction and not artefacts of the certificate: the positivity really does fail on either side.

5. The second band

Here $\Sigma = \{4, 8, 12, 28\}$, $G_4 = g_4 + \rho_8 g_8 + \rho_{12} g_{12} + \rho_{28} g_{28}$ with the exact coefficients (5), and the band is $(\mathcal{E}_2, \mathcal{E}_1) = (1/\sqrt{3}, (\sqrt{5}-1)/2)$.

5.1. **The interpolation polynomial and the zeros it forces.** By (4), $U_{4j-1} = U_3 \cdot (U_{j-1} \circ T_4)$, so with $Y = y^2$

$$\Psi(Y) \;=\; 1 + \rho_8 W_2(Y) + \rho_{12} W_3(Y) + \rho_{28} W_7(Y),$$

where $W_2 = 2T_4$, $W_3 = 4T_4^2 - 1$, $W_7 = 64T_4^6 - 80T_4^4 + 24T_4^2 - 1$ read in Y, is the even polynomial that carries the entire coefficient dependence.

Proposition 5.1 (the factorisation)**.** *There are explicit integers $A_0, \ldots, A_8$ and $B_0, \ldots, B_8$, listed in the formal development, such that, with $\widetilde{V}(Y) = \sum_{k=0}^{8}(A_k + B_k\mathcal{E}_1)Y^k$,*

$$9\,\Delta_{\mathrm{den}}\,\Psi(Y) \;=\; (3Y-1)(3Y-2)(Y-\mathcal{E}_1)(Y-\mathcal{E}_1^2)\,\widetilde{V}(Y).$$

The extreme pairs are

$(A_0, B_0) = (6572409822472814062276608,\ 4405355498226283330240512)$,

$(A_8, B_8) = (4886997517797045804521540812 8,\ 3020332384603002059123746406 4)$.

Consequently $G_4(\mathcal{E}_1, 1) = 0$ and $G_4(e, 1) = 0$ for every e with $e^2 = 1/3$.

The two vanishing statements are the source's own interpolation equations [1, (11.2)] at the band's two thresholds — $Y = \mathcal{E}_1^2$ and $Y = 1/3 = \mathcal{E}_2^2$ — so the proposition is the compute-first-values gate for this band: with the printed coefficients (5) inserted verbatim, the equations hold on the nose. The identity itself is an algebraic consequence of $(\sqrt{5})^2 = 5$.

Remark 5.2. Outside the formal development we also solved the source's three interpolation equations for $n = 2$ [1, (11.2)] — in the variable Y they read $\Psi(1/3) = 0$, $\Psi(1/2) - \frac{4}{9}\Psi(2/3) - \frac{4}{27}\Psi'(2/3) = 0$ and $\Psi(\mathcal{E}_1^2) = 0$ — in exact $\mathbb{Q}(\sqrt{5})$ arithmetic, recovering all six numerators of (5) exactly. That is a consistency check on the transcription of (5), nothing more.

Proposition 5.3 (two forced zeros)**.** *If $y^2 = \mathcal{E}_1$ then $G_4(y, y) = 0$. If $x^2 = 2/3$ and $y^2 = 1/2$ then $G_4(x, y) = 0$ and $(xy)^2 = 1/3$.*

Ψ has degree 12, and the factorisation of Proposition 5.1 exhibits four of its roots in $[0, 1]$, namely $Y \in \{1/3,\ 2/3,\ \mathcal{E}_1^2,\ \mathcal{E}_1\}$; the first and third are the equations the system imposes, and the other two are forced. (A working note records that the residual factor $\widetilde{V}$ is strictly positive on $[0, 1]$, so that these four are all the roots there; like (6), that statement is not part of the formal development and nothing below uses it.) Translating back: $Y = \mathcal{E}_1$ is the interior symmetric point of $S_{\mathcal{E}_1}$ — the author's tabulated node $X_1 \simeq 0.7861$, and exactly the band 1 obstruction reappearing at band 2's *right* threshold — and $Y = 2/3$ is the point $(\sqrt{2/3}, \cos(\pi/4))$ of $S_{\mathcal{E}_2}$, his $X_2 \simeq 0.8164$ paired with $X_1 \simeq 0.7071$. As in §3 the source owns the reason for both, through his relations [1, (1.11)–(1.12)] on $\mathbf{\Theta}_{m,\kappa}(D)$, but does not draw the consequence for this band.

5.2. **The reduction and the certificate.**

Proposition 5.4 (the reduction)**.** *With $X = x_1^2$, $Z = (x_1x_2)^2$, $\widetilde{V}(Y) = \sum_{k=0}^{8} \widetilde{V}_k Y^k$ as in Proposition 5.1 (so $\widetilde{V}_k = A_k + B_k\mathcal{E}_1$) and $\widetilde{V}^*(X,Z) := \sum_{k=0}^{8} \widetilde{V}_k\, Z^k X^{8-k}$, put*

$$\begin{aligned} R_2(X,Z) \;&= X^{14}(1-X)(2X-1)(3X-1)(3X-2)(X-\mathcal{E}_1)(X-\mathcal{E}_1^2)\,\widetilde{V}(X)\\ &\quad + (X-Z)(2Z-X)(3Z-X)(3Z-2X)\\ &\qquad \times (Z-\mathcal{E}_1X)(Z-\mathcal{E}_1^2X)\,\widetilde{V}^*(X,Z). \end{aligned}$$

The second summand is the first with X replaced by Z/X and denominators cleared. Then

$$9\,\Delta_{\mathrm{den}}\, x_1^{28}\, G_4(x_1,x_2) \;=\; 4\,x_1x_2\, R_2\big(x_1^2,(x_1x_2)^2\big),$$

and R_2 has bidegree $(28,14)$.

The clearing of the second summand is an exact polynomial identity once $Z = X \cdot x_2^2$, which is what keeps the two-variable step a polynomial identity on about a hundred monomials rather than a degree-56 expansion. This reduction is not written in [1].

Proposition 5.5 (the 16-leaf Boolean)**.** *Deform as in §4, writing the threshold factor as $(Y-r)(Y-r^2)$ and $\widetilde{V}$ in the basis $\{1,r\}$, and parametrise*

$$r = \tfrac{618+\omega}{1000}, \qquad E = \tfrac{580+38\sigma}{1000}, \qquad X = E + (1-E)\tau, \qquad (\sigma,\tau,\omega) \in [0,1]^3,$$

so $r \in [618/1000, 619/1000] \ni \mathcal{E}_1$, $E \in [29/50, 309/500]$ and $X \in [E,1]$. With margin $M_2(Z,r) = (3Z-1)(r^2-Z)$, the integer polynomial $1000^{30}\big(1000\,R_2 - 9\,\Delta_{\mathrm{den}}M_2\big)$ — multidegree $(28,28,4)$, 4202 Bernstein coefficients of at most 434 bits — passes the subdivision test with the uniform tree tree$[0,1,0,1]$*, i.e. 16 leaves. The root expansion of the same build is rejected, so subdivision is genuinely needed here.*

Theorem 5.6 (second band)**.** *For all $x_1, x_2 \in [0,1]$ with $\frac{29}{50} \le E \le \frac{309}{500}$, $E = x_1x_2$,*

$$G_4(x_1,x_2) \;\ge\; \tfrac{1}{500}\big(3E^2-1\big)\Big(\tfrac{3-\sqrt{5}}{2} - E^2\Big) \;>\; 0.$$

This is Theorem 1.2.

Proof sketch. As for Theorem 4.5, in three ingredients. The algebraic specialisation $\mathcal{E}_1^2 = \frac{3-\sqrt{5}}{2}$ is exact real algebra; the enclosure $618/1000 \le \mathcal{E}_1 \le 619/1000$, from $2236067/10^6 \le \sqrt{5} \le 2236068/10^6$, places $\mathcal{E}_1$ in the deformation window; and Proposition 5.5 with the soundness of §2.2 gives $1000\,R_2 \ge 9\Delta_{\mathrm{den}}M_2$ on the box. Using the swap symmetry of G_4 to assume $x_2 \le x_1$, hence $X \ge E$, and feeding this into Proposition 5.4 with $E \ge 29/50$ and $0 < x_1^{28} \le 1$ gives $G_4 \ge M_2(E^2,\mathcal{E}_1)/500$. Strict positivity holds because $3E^2-1>0$ for $E \ge 29/50 > 1/\sqrt{3}$ and $\mathcal{E}_1^2 - E^2 > 0$ for $E \le 309/500 < \mathcal{E}_1$; the enclosure of $\sqrt{5}$ supplies both inequalities. The only exhaustive computation is the single Boolean of Proposition 5.5. □

Proposition 5.7 (control and non-vacuity)**.** *$G_4(1/2,1) < 0$, and the hypotheses of Theorem 5.6 are satisfiable, e.g. at $(x_1,x_2) = (1,3/5)$.*

The energy $E = 1/2$ lies well below the left threshold $\mathcal{E}_2 = 0.5773\ldots$, so again the estimate genuinely fails outside the band.

Why 93.40% **and not** 100%**.** The two endpoints of Theorem 5.6 fail to reach the thresholds for different reasons.

On the right, the loss is $8.4 \cdot 10^{-4}$ of the band and is pure economy: keeping $E \leq 618/1000$ holds the deformation parameter r out of the energy parametrisation, which keeps the degree in ω at 4 instead of about 32, and the term count at 4202 instead of about 27000.

On the left, the limit is mathematical — at $Z = 1/3$ the surface carries the attained interior zero at $X = 2/3$ of Proposition 5.3, and by the argument of §6 no finite subdivision of this box reaches $\mathcal{E}_2 = 1/\sqrt{3}$ — but the shipped endpoint $580/1000$ is well short of that limit and stops there for cost, not for mathematics. In a mirror of the checker outside the formal development, the depth needed grows as the left endpoint descends: root expansion at $590/1000$ (68.8% of the band), 4 leaves at $585/1000$ (81.1%), 16 leaves at $580/1000$ (93.40%, shipped), 64 leaves at $578/1000$ (98.32%) and 256 leaves at $5775/10^4$ (99.55%), with $5774/10^4$ failing at 256 leaves. The last two are omitted only because evaluating them inside the proof assistant carries a measured interpreter overhead of roughly two orders of magnitude.

6. The obstruction, and what the fold removes

A Bernstein coefficient array bounds the minimum of a polynomial on a box *from below*, and under subdivision the bound converges to the true minimum from below. At an attained *interior* zero the bound therefore never becomes nonnegative, no matter how deep the tree. This is why the first certificate of §4 stops at $Z = 13/34$: at $Z = \mathcal{E}_1^2$ the function $R(\cdot, \mathcal{E}_1^2)$ has, by (6), a double zero at the interior point $X = \mathcal{E}_1$ — Proposition 3.3 — on top of the simple zero at the boundary node that the interpolation imposes.

The fold removes the obstruction rather than out-subdividing it, and the following controls show that the distinction is real and not rhetorical.

Proposition 6.1 (the fold is necessary)**.** *On the* unfolded *box* $X \in [Z, 1]$ *with the same deformed data as Proposition 4.4, the test is rejected by the root expansion, by* $\mathrm{tree}[0, 1, 0, 1]$ *(16 leaves) and by* $\mathrm{tree}[0, 1, 0, 1, 0, 1]$ *(64 leaves).*

Outside the formal development the failure is worse than persistent: the number of failing leaves *grows* with depth, taking the values $1, 2, 5, 9, 17$ at $1, 4, 16, 64, 256$ leaves. Compare Proposition 4.4, where the folded build needs no subdivision at all. Exact arithmetic on the undeformed box $Z \in [\mathcal{E}_1^2, 1/2]$ shows the same thing: uniform trees of depth $(3, 3)$, $(2, 3)$, $(3, 4)$ and $(4, 4)$ all stall at exactly one bad box, the one hugging $(\sigma, \tau) = (0, \mathcal{E}_1/(1 + \mathcal{E}_1))$, with the worst coefficient flat at about $-1.35 \cdot 10^{-5}$ between the last two depths.

Band 2 shows the same phenomenon at both ends: the forced interior zero at $Y = \mathcal{E}_1$ sits on the band's right threshold and the forced interior zero at $Y = 2/3$ on its left, and it is the latter that caps the certified range on the left at $1/\sqrt{3}$ in the limit.

7. Verification

Every theorem and proposition stated above — Corollary 1.3, whose status is described where it is stated, excepted — together with the soundness of the certificate checker of §2.2, has been machine-checked in Lean 4 (toolchain `v4.32.0`) against

Mathlib [9, 10]; the development contains no `sorry` and declares no axiom by hand. The reasoning layer is free of compiled evaluation: the reductions (Propositions 3.1, 5.4), the factorisation (Proposition 5.1), the vanishing statements (Propositions 3.2, 3.3, 5.3), the algebraic specialisations $\mathcal{E}_1^2 = \frac{3-\sqrt{5}}{2}$, $\mathcal{E}_1 + \mathcal{E}_1^2 = 1$, $\mathcal{E}_1^3 = \sqrt{5} - 2$, the rational enclosure $2236067/10^6 \le \sqrt{5} \le 2236068/10^6$ and the window memberships it yields, the point evaluations of Propositions 4.8 and 5.7, the witnesses of Proposition 4.7, and the derivations of Theorems 4.2, 4.5 and 5.6 from their certificates all depend on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`. What is delegated to compiled evaluation, through Lean's `native_decide`, is exactly fourteen Boolean tests on explicit integer term lists: the three accepting certificates (Propositions 4.1, 4.4, 5.5) and eleven rejecting controls (Propositions 4.3, 4.6, 6.1 and the root control in Proposition 5.5). Those fourteen Booleans are the only source of extra trust in the development, and they are the only exhaustive computation on which any claim above rests. The numbers reported alongside them — 780 coefficients at multidegree $(8, 8, 9)$ for the first band, 4202 at $(28, 28, 4)$ for the second, and the coefficient bit widths — are measurements of the same term lists. The material described as lying outside the formal development (tree depths against the endpoint, the growth of the failing-leaf count, the widths of the passing deformation windows, the exact re-derivation of the coefficients of (5), the threshold factorisation (6), and the positivity of $\widetilde{V}$ on $[0, 1]$) was produced in a separate implementation and is reported for context only; no theorem uses it. Repository reference to be added at submission.

References

[1] M.-A. Mandich, *Thresholds and more bands of A.C. spectrum for the Molchanov–Vainberg Schrödinger operator with a more general long range condition*, arXiv:2201.00410 [math.SP], 2022. All numbering quoted above is that of v1, the only version; the arXiv record was checked on 2026-08-30 and still shows v1, submitted 2022-01-02.

[2] S. Molchanov and B. Vainberg, *Scattering on the system of the sparse bumps: multidimensional case*, Appl. Anal. **71** (1998), no. 1–4, 167–185. DOI `10.1080/00036819908840711`.

[3] E. Mourre, *Absence of singular continuous spectrum for certain self-adjoint operators*, Comm. Math. Phys. **78** (1981), no. 3, 391–408. DOI `10.1007/BF01942331`.

[4] W. O. Amrein, A. Boutet de Monvel and V. Georgescu, *C_0-groups, commutator methods and spectral theory of N-body Hamiltonians*, Progress in Mathematics **135**, Birkhäuser, Basel, 1996.

[5] S. Golénia and M.-A. Mandich, *Limiting absorption principle for discrete Schrödinger operators with a Wigner–von Neumann potential and a slowly decaying potential*, Ann. Henri Poincaré **22** (2021), no. 1, 83–120. DOI `10.1007/s00023-020-00971-9`.

[6] S. Golénia and M.-A. Mandich, *Bands of pure absolutely continuous spectrum for lattice Schrödinger operators with a more general long range condition*, J. Math. Phys. **62** (2021), no. 9, 092104. DOI `10.1063/5.0053416`; preprint arXiv:2102.00726, 2021, whose title carries the additional qualifier "Part I". This is the paper [1] cites as [GM2] and calls its predecessor.

[7] S. Golénia and M.-A. Mandich, *Thresholds and more bands of a.c. spectrum for the discrete Schrödinger operator with a more general long range condition*, arXiv:2201.09545 [math.SP], 2022. The standard-Laplacian companion of [1], carrying the analogous tables for Δ; cited there as [GM3], then still unpublished.

[8] C. Muñoz and A. Narkawicz, *Formalization of Bernstein polynomials and applications to global optimization*, J. Automat. Reason. **51** (2013), no. 2, 151–196. DOI `10.1007/s10817-012-9256-3`.

[9] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, Automated Deduction — CADE 28, Lecture Notes in Computer Science **12699**, Springer, 2021, 625–635. DOI `10.1007/978-3-030-79876-5_37`.

[10] The mathlib Community, *The Lean mathematical library*, Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020), 367–381. DOI `10.1145/3372885.3373824`. The development above was checked against the Mathlib revision `81a5d25`.