THE SHARP ONE-SIDED TAIL BOUND FOR INTEGER-VALUED RANDOM VARIABLES UNDER A FOURTH-MOMENT CONSTRAINT

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ABSTRACT. For a real random variable X with $\mathbf{E}X = 0$, $\mathbf{E}X^2 = 1$ and $\mathbf{E}X^4 \le \kappa$, the exact value of $\sup \mathbf{P}(X \ge t)$ is a classical Chebyshev-type extremal problem, solved in closed form on most of its parameter range by Li, Han, Jiang and Gao. We impose one further hypothesis — that X is *integer-valued* — and determine the resulting map $V_1^{\mathbb{Z}}(t,\kappa)$ completely. For every integer $t \ge 2$ it has exactly two regimes, separated by the threshold $\kappa = t^2 - t + 1$: below the threshold $V_1^{\mathbb{Z}}(t,\kappa) = (\kappa-1)/\big(t^2(t^2-1)\big)$, attained on $\{-1,0,1,t\}$; at and above it $V_1^{\mathbb{Z}}(t,\kappa) = 1/\big(t(t+1)\big)$, attained on $\{-1,0,t\}$, and the fourth-moment constraint is inactive. The threshold is exactly the fourth moment of the two-moment extremal law $\{-1,0,t\}$, an integer polynomial in t; the corresponding boundary of the real-support map is the fourth moment of Cantelli's two-atom law, $t^2 - 1 + t^{-2}$, which is not. Both branches are affine in κ where the real-support value is strictly concave in κ, and both are constant in t on each interval $(n-1, n]$, so $V_1^{\mathbb{Z}}(\cdot,\kappa)$ is a step function where the real-support map is smooth. The mechanism is a pair of dual polynomials, $x(x+1)$ and $x^2(x^2-1)$, that are nonnegative at every integer and negative on the real line; neither is a sum of squares, so neither is admissible in the real-support problem, and the sum-of-squares duality used there cannot reach these bounds. We also record the two-moment case $\sup \mathbf{P}(X \ge t) = 1/\big(t(t+1)\big)$, against Cantelli's $1/(1+t^2)$, which we present as an anticipated value rather than a new one — it is a two-line substitution into a closed form of Boros and Prékopa — and we say precisely why. Every theorem, proposition, lemma and corollary below is formally verified in Lean 4, and we state exactly what lies outside that guarantee.

1. Introduction

The real-support problem. Let X be a real random variable with

$$\mathbf{E}X = 0, \qquad \mathbf{E}X^2 = 1, \qquad \mathbf{E}X^4 \le \kappa, \tag{1}$$

the third moment unconstrained, and put

$$V_1(t,\kappa) \;=\; \sup\big\{\mathbf{P}(X \ge t) \;:\; X \text{ satisfies (1)}\big\}.$$

This is the *kurtosis map.* Without the fourth-moment constraint the answer is Cantelli's inequality [3]: $\sup \mathbf{P}(X \ge t) = 1/(1+t^2)$ for $t > 0$, attained by the two-atom law putting mass $1/(1+t^2)$ at t and $t^2/(1+t^2)$ at $-1/t$. With the constraint the map was determined by Li, Han, Jiang and Gao [11], who exhibit four regimes; the two we use as controls below are their Regime I, the "Cantelli tongue" $V_1(t,\kappa) = 1/(1+t^2)$, and their Regime II,

$$V_1(t,\kappa) = \frac{\kappa - 1}{(t^2-1)^2 + \kappa - 1} \qquad \text{for } t \ge c(\kappa), \qquad c(\kappa) = \frac{\sqrt{\kappa+3} + \sqrt{\kappa-1}}{2}. \tag{2}$$

Their method, and the method of the tractable literature on Chebyshev-type bounds generally [1, 21], is convex duality: one searches for a polynomial q with

$$q(x) \ge 0 \;\text{ for all } x \in \mathbb{R}, \qquad q(x) \ge 1 \;\text{ for all } x \ge t, \tag{3}$$

and minimises the pairing of its coefficients with the moment vector. For a quartic, "$q \ge 0$ on $\mathbb{R}$" *is* a sum-of-squares condition, and that is what makes the problem a semidefinite program; on a general basic closed semialgebraic support the same role is played by Putinar's Positivstellensatz [17].

The question of this paper. We add one hypothesis to (1):

$$\mathbf{P}(X \in \mathbb{Z}) = 1.$$

Write $V_1^{\mathbb{Z}}(t, \kappa)$ for the supremum of $\mathbf{P}(X \geq t)$ over this smaller class, and $V_2^{\mathbb{Z}}(t)$ for the supremum over the two-moment class $\mathbf{E}X = 0$, $\mathbf{E}X^2 = 1$ with no fourth-moment hypothesis at all.

Question 1.1. Determine $V_1^{\mathbb{Z}}$ and $V_2^{\mathbb{Z}}$.

Question 1.1 has two halves and they have different provenance. Its two-moment half is answered by a substitution into a published closed form of Boros and Prékopa, which we carry out in Section 7; we present (4) as anticipated for exactly that reason. The four-moment half is what we pose here as a new question, and no source we could find asks it: it is a crossing of the moment class of [11] with the support restriction of the discrete moment problem, and we found no instance of that crossing being made. A self-posed question deserves a sharper account of the literature than a published one, so we record what was searched, on 2026-08-29. A local 85 GB arXiv corpus of roughly 881,000 papers (coverage through the announcement window of 2026-08-28) was swept in seven full-text passes. An independent same-line probe of that corpus, for co-occurrences of a Cantelli / one-sided-Chebyshev phrase with an integer-support phrase, returned 7 lines, none of them a sharpening of a tail bound, against 2,385 lines containing "integer-valued random variable" alone. A 31-paper hand-read intersection of the Cantelli set with the integer-support set found that all 31 merely apply classical Cantelli to a variable that happens to be integer-valued. At paper level, "nonnegative on the integers" returned 1 paper, "moment problem on a lattice" and "moment problem on the integers" returned none. The literal constants $t^2(t^2-1)$, $t^2 - t + 1$ and $x^2(x^2-1)$ never co-occur with a Cantelli or one-sided-Chebyshev paper anywhere in that corpus. Eleven arXiv API queries, nine OpenAlex queries, eleven Semantic Scholar bulk queries, about thirteen Stack Exchange API queries and about ten web queries were run in addition; the arXiv query `abs:"kurtosis" AND abs:"sharp bound"` returned nothing.

The strongest single piece of evidence is negative and comes from the source itself. The paper [11] is under two months old at the time of writing, its purpose is to survey and complete this exact moment class, and it carries a purpose-built prior-work table organised by which moment functionals define the class. Support is not an axis of that table. A search of all 3,400 lines of its LaTeX source finds the words *lattice*, *integer-valued*, *integer valued* and *discrete support* zero times, and $\mathbb{Z}$ only inside its macro file. On the same date it had zero forward citations in OpenAlex and in Semantic Scholar.

The nearest published relative we located is Huber [5], whose variable is integer-valued and whose constants use that integrality; but his extra hypothesis is a non-increasing or a unimodal probability mass function, and his two bounds are Markov- and Chebyshev-shaped: $\mathbf{P}(X \leq a) \leq \mathbf{E}X/(2a-1)$ for a non-increasing mass function on $\{0, 1, 2, \dots\}$ and integer $a \geq 1$, and $\mathbf{P}(|W - \mathbf{E}W| \geq a) \leq \big(\mathrm{Var}(W) + 1/12\big)/\big(2(a - 1/2)^2\big)$ for a unimodal one on $\mathbb{Z}$. No fourth moment appears; the words *fourth moment* and *kurtosis* occur zero times in his source.

Results. Throughout, t is a threshold and $\kappa \geq 1$ a fourth-moment budget; $\kappa \geq 1$ is forced, since $\mathbf{E}X^4 \geq (\mathbf{E}X^2)^2 = 1$. All suprema below are attained, so we write them as maxima.

The two-moment answer is

$$V_2^{\mathbb{Z}}(t) = \frac{1}{t(t+1)} \qquad \text{for every integer } t \geq 1 \tag{4}$$

(Theorem 3.1), against Cantelli's $1/(1+t^2)$; the two agree at $t = 1$ and the lattice constant is strictly smaller for every $t \geq 2$ (Proposition 3.2). We do *not* present (4) as new, and Section 7 says why.

The new content is the four-moment map. For every integer $t \geq 2$ it has exactly two regimes, separated by the threshold $\kappa = t^2 - t + 1$:

$$V_1^{\mathbb{Z}}(t,\kappa) = \begin{cases} \dfrac{\kappa - 1}{t^2(t^2-1)}, & 1 \leq \kappa \leq t^2 - t + 1 \quad \text{(binding)}, \\ \dfrac{1}{t(t+1)}, & \kappa \geq t^2 - t + 1 \quad \text{(saturated)}, \end{cases} \tag{5}$$

the two branches agreeing at the threshold, so that the map is continuous in κ (Theorems 4.1 and 4.2, Proposition 4.3). The extremal law is $\{-1, 0, 1, t\}$ in the binding regime and $\{-1, 0, t\}$ in the saturated one. At $t = 1$ the binding window is empty and $V_1^{\mathbb{Z}}(1,\kappa) = 1/2$ for every $\kappa \geq 1$.

Three features of (5) are worth naming.

The threshold is arithmetic. The number $t^2 - t + 1$ is exactly the fourth moment of the three-atom law $\{-1, 0, t\}$ that solves the two-moment problem. The fourth-moment constraint switches off precisely when that law becomes feasible, and the switch-off point is an integer polynomial in t. The real-support boundary has the same description and a different shape: the Cantelli tongue of [11] closes at $\kappa = \kappa_c(t) = t^2 - 1 + t^{-2}$, which is likewise the fourth moment of the two-moment extremal law there, and which is not a polynomial because that law's non-tail atom sits at $-1/t$ rather than at an integer. Solved for the threshold instead, the lattice boundary is $t = \big(1 + \sqrt{4\kappa - 3}\big)/2$, one square root, against the two square roots of $c(\kappa)$ in (2).

Both branches are affine in κ. On the binding regime $V_1^{\mathbb{Z}}$ is linear in κ and on the saturated one it is constant, where the real-support Regime II value (2) is a strictly concave rational function of κ climbing towards 1. The two maps agree only at $\kappa = 1$, where both vanish.

In t the lattice map is a step function. An integer-valued X satisfies $\mathbf{P}(X \geq t) = \mathbf{P}(X \geq \lceil t \rceil)$, so $V_1^{\mathbb{Z}}(\cdot,\kappa)$ is constant on each interval $(n-1, n]$ and (5) determines it on all of $(0,\infty) \times [1,\infty)$ (Section 5). The real-support map is smooth in t on each of its regimes, so the gap between the two is widest at the *left* end of each step: at $(t,\kappa) = (3/2, 3)$ the lattice value is $1/6$ while the value of [11] is its Regime I value $1/(1+t^2) = 4/13$, a factor of 1.85, against a factor of 1.09 at $(2,3)$.

Why the standard machinery does not reach this. Over $\mathbb{Z}$ the dual conditions (3) weaken to

$$q(n) \geq 0 \ \text{ for all } n \in \mathbb{Z}, \qquad q(n) \geq 1 \ \text{ for all integers } n \geq t, \tag{6}$$

and $\mathbb{Z}$ is not a semialgebraic set, so Putinar's theorem — the engine underneath [1, 21, 11] — has nothing to say about (6). The gap is not academic. The two certificates that prove (4) and (5) are

$$q_2(x) = \frac{x(x+1)}{t(t+1)}, \qquad q_4(x) = \frac{x^2(x^2-1)}{t^2(t^2-1)},$$

and each is nonnegative at every integer, because $n(n+1)$ is a product of two consecutive integers and $n^2(n^2-1) = (n-1)n^2(n+1)$ is a square times another such product; but $q_2(-1/2) < 0$ and $q_4(1/2) < 0$, so neither is a sum of squares and neither is a feasible dual for the real-support problem (Proposition 2.4). That single fact is the whole integrality gap in the value.

The mechanism can be said in one more sentence. The polynomial q_2 is negative exactly on $(-1, 0)$; Cantelli's extremal law at t puts its non-tail mass at $-1/t$, which lies in $(-1, 0)$ for every $t > 1$ and is an integer only at $t = 1$. *The certificate is negative exactly where the real-support extremiser lives*, and that also explains why the two constants coincide at $t = 1$ and separate for every $t \geq 2$.

Organisation. Section 2 fixes the classes and proves lattice weak duality. Section 3 does the two-moment case, Section 4 the four-moment map, Section 5 the extension to real thresholds, Section 6 the comparison with the real-support problem and the negative controls. Section 7 states what in this paper is anticipated by the discrete moment problem literature and what is not, together with the provenance questions we could not close. Section 8 lists open questions and Section 9 describes the formal verification.

2. The classes, and weak duality on the lattice

We work with Borel probability measures on $\mathbb{R}$ rather than with measures on $\mathbb{Z}$, so that the lattice class sits inside the real-support class of [11] literally and not up to an identification; this matters in Section 6, and it is what makes the statement about non-integer thresholds in Section 5 say something.

Definition 2.1. Let μ be a Borel probability measure on $\mathbb{R}$.

- $\mu \in \mathcal{P}_2^{\mathbb{Z}}$ if x^2 is μ-integrable, $\int x\,d\mu = 0$, $\int x^2\,d\mu = 1$, and $\mu(\mathbb{Z}) = 1$;
- for $\kappa \in \mathbb{R}$, $\mu \in \mathcal{P}^{\mathbb{Z}}(\kappa)$ if x^4 is μ-integrable, $\int x\,d\mu = 0$, $\int x^2\,d\mu = 1$, $\int x^4\,d\mu \le \kappa$, and $\mu(\mathbb{Z}) = 1$.

The third moment is unconstrained in both. Put

$$V_2^{\mathbb{Z}}(t) = \sup_{\mu\in\mathcal{P}_2^{\mathbb{Z}}} \mu\big([t,\infty)\big), \qquad V_1^{\mathbb{Z}}(t,\kappa) = \sup_{\mu\in\mathcal{P}^{\mathbb{Z}}(\kappa)} \mu\big([t,\infty)\big).$$

Integrability is part of the definition on purpose. It is not implied by the moment equations as literally stated in a measure-theoretic development — an integral of a non-integrable function is conventionally assigned the value 0, and a heavy-tailed law would then satisfy "$\int x^4\,d\mu \le \kappa$" vacuously. The two-moment class $\mathcal{P}_2^{\mathbb{Z}}$ deliberately does *not* assume a finite fourth moment; assuming one would be a genuine restriction and would make the comparison with Cantelli's inequality in Proposition 3.2 dishonest.

Since $\mathcal{P}^{\mathbb{Z}}(\kappa)$ asks for strictly more than $\mathcal{P}_2^{\mathbb{Z}}$, we have $\mathcal{P}^{\mathbb{Z}}(\kappa) \subseteq \mathcal{P}_2^{\mathbb{Z}}$ for every κ, hence

$$V_1^{\mathbb{Z}}(t,\kappa) \le V_2^{\mathbb{Z}}(t) \qquad \text{for all } t,\kappa. \tag{7}$$

This is what lets the saturated regime of (5) inherit its upper bound from the two-moment answer.

The dual side is where the lattice hypothesis is spent.

Proposition 2.2 (Lattice weak duality)**.** *Let $l_0, l_1, l_2, l_4, t \in \mathbb{R}$.*

1. *Let $q(x) = l_0 + l_1 x + l_2 x^2$ satisfy $q(n) \ge 0$ for every $n \in \mathbb{Z}$ and $q(n) \ge 1$ for every $n \in \mathbb{Z}$ with $n \ge t$. Then $\mu([t,\infty)) \le l_0 + l_2$ for every $\mu \in \mathcal{P}_2^{\mathbb{Z}}$.*
2. *Let $q(x) = l_0 + l_1 x + l_2 x^2 + l_4 x^4$ satisfy the same two conditions and $l_4 \ge 0$. Then $\mu([t,\infty)) \le l_0 + l_2 + l_4\kappa$ for every $\mu \in \mathcal{P}^{\mathbb{Z}}(\kappa)$.*

Proof sketch. The argument is the classical one with two pointwise inequalities replaced by μ-almost-sure ones. Since $\mu(\mathbb{Z}) = 1$, the hypothesis $q \ge 0$ on $\mathbb{Z}$ gives $q \ge 0$ μ-a.e., and the hypothesis $q \ge 1$ on $\mathbb{Z} \cap [t,\infty)$ gives $q \ge 1$ almost everywhere for the restriction of μ to $[t,\infty)$. Hence

$$\mu\big([t,\infty)\big) = \int_{[t,\infty)} 1\,d\mu \le \int_{[t,\infty)} q\,d\mu \le \int q\,d\mu,$$

and the right-hand integral is $l_0 + l_1 \cdot 0 + l_2 \cdot 1$, respectively $l_0 + l_1 \cdot 0 + l_2 \cdot 1 + l_4 \int x^4 d\mu$. In the quartic case $l_4 \ge 0$ turns the *inequality* $\int x^4 d\mu \le \kappa$ into the bound $l_4\kappa$. Integrability of q is supplied by Definition 2.1 together with $|x|^k \le 1 + x^2$ for $k \le 2$ and $|x|^k \le 1 + x^4$ for $k \le 4$. The cubic coefficient is absent from the quartic on purpose: the third moment is unconstrained in the class, so a certificate using it would not bound the class. No search is involved. □

The two conditions in Proposition 2.2 quantify over $\mathbb{Z}$, not over $\mathbb{R}$. That is the entire difference from the real-support theory, and the following two elementary facts are what it buys.

Lemma 2.3. *For every $n \in \mathbb{Z}$ one has $n(n+1) \ge 0$ and $n^2(n^2-1) \ge 0$. Moreover $x \mapsto x(x+1)$ and $x \mapsto x^2(x^2-1)$ are nondecreasing on $[1,\infty)$.*

Proof sketch. $n(n+1)$ is a product of two consecutive integers, so one factor is ≥ 0 and the other ≥ 0, or both are ≤ 0. For the second, either $n = 0$, or $n^2 \ge 1$ and both factors are nonnegative. Monotonicity on $[1,\infty)$ is the derivative computation, or directly $x(x+1) - y(y+1) = (x-y)(x+y+1) \ge 0$ and $x^2(x^2-1) - y^2(y^2-1) = (x^2-y^2)(x^2+y^2-1) \ge 0$ for $1 \le y \le x$. No search. □

Proposition 2.4. *Neither $x(x+1)$ nor $x^2(x^2-1)$ is nonnegative on $\mathbb{R}$: the first takes the value $-1/4$ at $x=-1/2$ and the second the value $-3/16$ at $x=1/2$. Consequently neither is a sum of squares of real polynomials, and neither is a feasible dual certificate for the real-support problem, although both are nonnegative at every integer by Lemma 2.3.*

Proof. Two evaluations, and a sum of squares of real polynomials is nonnegative at every real point. ☐

3. The two-moment constant

Theorem 3.1. *For every integer $t \geq 1$, the maximum of $\mathbf{P}(X \geq t)$ over integer-valued X with $\mathbf{E}X = 0$ and $\mathbf{E}X^2 = 1$ is attained and equals*

$$V_2^{\mathbb{Z}}(t) = \frac{1}{t(t+1)}.$$

More generally, if $T \geq 1$ is an integer and p satisfies $T(T+1)p = 1$, then p is the maximum of $\mu([T,\infty))$ over $\mu \in \mathcal{P}_2^{\mathbb{Z}}$.

Proof sketch. Primal. Let $p = 1/\big(t(t+1)\big)$ and take the law

$$\mu = \frac{1}{t+1}\,\delta_{-1} + \Big(1 - \frac{1}{t}\Big)\delta_0 + \frac{1}{t(t+1)}\,\delta_t.$$

The three masses are nonnegative exactly because $t \geq 1$, and they sum to 1 because $\frac{1}{t+1} + \frac{1}{t(t+1)} = \frac{1}{t}$. Its first moment is $-\frac{1}{t+1} + t \cdot \frac{1}{t(t+1)} = 0$ and its second is $\frac{1}{t+1} + \frac{t^2}{t(t+1)} = 1$; all three atoms are integers, and the only one at or above t is t itself, so $\mu([t,\infty)) = 1/\big(t(t+1)\big)$.

Dual. Apply Proposition 2.2(1) to $q(x) = p\,x(x+1)$, i.e. $l_0 = 0$, $l_1 = l_2 = p$. Nonnegativity at the integers is Lemma 2.3. For an integer $n \geq t$, monotonicity of $x(x+1)$ on $[1,\infty)$ gives $n(n+1) \geq t(t+1)$, hence $q(n) \geq p\,t(t+1) = 1$. The dual value is $l_0 + l_2 = p$, which matches the primal value, so both are optimal.

Nothing here is a search: the primal is one explicit law and the dual is one explicit quadratic, both valid uniformly in t. ☐

Proposition 3.2. *For every integer $t \geq 2$,*

$$\frac{1}{t(t+1)} < \frac{1}{1+t^2},$$

and at $t = 1$ the two sides are equal, both being $1/2$.

Proof. $t(t+1) - (1+t^2) = t - 1$, which is positive for $t \geq 2$ and zero at $t = 1$; both denominators are positive. ☐

So the lattice restriction buys nothing at the first threshold and something at every later one. Numerically, $V_2^{\mathbb{Z}}(2) = 1/6$ against Cantelli's $1/5$, and $V_2^{\mathbb{Z}}(3) = 1/12$ against $1/10$. The reason for the coincidence at $t = 1$ was given in Section 1: Cantelli's extremal law puts its non-tail mass at $-1/t$, which is an integer precisely when $t = 1$.

4. The four-moment map

Theorem 4.1 (binding regime)**.** *Let $t \geq 2$ be an integer and let $1 \leq \kappa \leq t^2 - t + 1$. Then the maximum of $\mathbf{P}(X \geq t)$ over integer-valued X with $\mathbf{E}X = 0$, $\mathbf{E}X^2 = 1$ and $\mathbf{E}X^4 \leq \kappa$ is attained and equals*

$$V_1^{\mathbb{Z}}(t,\kappa) = \frac{\kappa - 1}{t^2(t^2-1)}.$$

More generally, if $T \geq 2$ is an integer and $p \geq 0$ satisfies $T^2(T^2-1)p = \kappa - 1$ and $T(T+1)p \leq 1$, then p is the maximum of $\mu([T,\infty))$ over $\mu \in \mathcal{P}^{\mathbb{Z}}(\kappa)$.

Proof sketch. Write $D = t^2(t^2-1)$ and $p = (\kappa-1)/D$, so $p \geq 0$ since $\kappa \geq 1$ and $t \geq 2$.

Primal. Take the four-atom law on $\{-1, 0, 1, t\}$ with masses

$$w_{-1} = \frac{1 - t(t-1)p}{2}, \qquad w_0 = (t^2-1)p, \qquad w_{+1} = \frac{1 - t(t+1)p}{2}, \qquad w_t = p.$$

That the masses sum to 1, that the first moment vanishes and that the second moment is 1 are polynomial identities in t and p, valid for every p; only the fourth moment uses the defining relation, and it gives $1 + t^2(t^2-1)p = \kappa$ exactly. Nonnegativity is where the regime window comes from: $w_0 \geq 0$ and $w_{-1} \geq w_{+1}$ are automatic, and

$$w_{+1} \geq 0 \iff t(t+1)p \leq 1 \iff \frac{\kappa - 1}{t(t-1)} \leq 1 \iff \kappa \leq t^2 - t + 1.$$

The mass at $+1$ is what runs out. All four atoms are integers, the only one at or above t is t, so the upper-tail mass is p.

Dual. Apply Proposition 2.2(2) with $l_0 = l_1 = 0$, $l_2 = -1/D$, $l_4 = 1/D$, that is, with

$$q(x) = \frac{x^2(x^2-1)}{D}.$$

Here $l_4 = 1/D > 0$, as required. Nonnegativity at the integers is Lemma 2.3; for an integer $n \geq t$ monotonicity of $x^2(x^2-1)$ on $[1, \infty)$ gives $n^2(n^2-1) \geq t^2(t^2-1) = D$, hence $q(n) \geq 1$. The dual value is $l_0 + l_2 + l_4\kappa = (\kappa-1)/D$, which matches the primal value.

Again there is no search: one explicit law and one explicit quartic, both valid uniformly in t and κ throughout the stated window. □

Theorem 4.2 (saturated regime)**.** *Let $t \geq 1$ be an integer and let $\kappa \geq t^2 - t + 1$. Then the fourth-moment constraint is inactive: the maximum of $\mathbf{P}(X \geq t)$ over integer-valued X with $\mathbf{E}X = 0$, $\mathbf{E}X^2 = 1$ and $\mathbf{E}X^4 \leq \kappa$ is attained and equals*

$$V_1^{\mathbb{Z}}(t, \kappa) = \frac{1}{t(t+1)} = V_2^{\mathbb{Z}}(t).$$

The threshold $t^2 - t + 1$ is exactly the fourth moment of the extremal law of Theorem 3.1.

Proof sketch. Primal. The law μ of Theorem 3.1 on $\{-1, 0, t\}$ has

$$\int x^4 \, d\mu = \frac{1}{t+1} + \frac{t^4}{t(t+1)} = \frac{1+t^3}{1+t} = t^2 - t + 1,$$

so it lies in $\mathcal{P}^{\mathbb{Z}}(\kappa)$ exactly when $\kappa \geq t^2 - t + 1$, and there its upper-tail mass is $1/\big(t(t+1)\big)$.

Dual. No new certificate is needed: by (7) the four-moment class is contained in the two-moment class, so the upper bound of Theorem 3.1 applies verbatim. No search. □

Proposition 4.3. *For every integer $t \geq 2$,*

$$\frac{(t^2 - t + 1) - 1}{t^2(t^2-1)} = \frac{1}{t(t+1)},$$

so the two branches of (5) *agree at $\kappa = t^2 - t + 1$ and $V_1^{\mathbb{Z}}(t, \cdot)$ is continuous on* $[1, \infty)$.

Proof. $t^2 - t = t(t-1)$ and $t^2(t^2-1) = t^2(t-1)(t+1)$, so the left side is $t(t-1)/\big(t^2(t-1)(t+1)\big) = 1/\big(t(t+1)\big)$. □

Together, Theorems 4.1 and 4.2 and Proposition 4.3 give (5) for every integer $t \geq 2$ and every $\kappa \geq 1$. At $t = 1$ the binding window $1 \leq \kappa \leq t^2 - t + 1 = 1$ degenerates to the single point $\kappa = 1$ and Theorem 4.2 covers all of $\kappa \geq 1$, giving $V_1^{\mathbb{Z}}(1, \kappa) = 1/2$.

Remark 4.4 (the shape of the two maps). Both branches of (5) are affine in κ. On the binding regime the slope is the constant $1/\big(t^2(t^2-1)\big)$ and on the saturated regime it is 0; the map climbs linearly and then stops. The real-support Regime II value (2) is instead strictly concave in κ and increases to 1 as $\kappa \to \infty$. The two agree only at $\kappa = 1$, where both are 0. The following table records the two values at six points; the right-hand column is (2) evaluated by hand, and at each of these six points $t \geq c(\kappa)$, so Regime II is the applicable branch of [11].

(t,κ)	$V_1^{\mathbb{Z}}(t,\kappa)$	$V_1(t,\kappa)$ [11], Regime II
$(2,2)$	$1/12$	$1/10$
$(2,5/2)$	$1/8$	$1/7$
$(2,11/4)$	$7/48$	$7/43$
$(2,3)$	$1/6$	$2/11$
$(3,4)$	$1/24$	$3/67$
$(3,7)$	$1/12$	$3/35$

The first five lattice entries are instances of Theorem 4.1 and the last of Theorem 4.2; $(2,3)$ and $(3,7)$ are the threshold cells $\kappa = t^2 - t + 1$ and are covered by both.

5. Non-integer thresholds: the map is a step function in t

Theorem 5.1. *Let μ be a Borel probability measure on $\mathbb{R}$ with $\mu(\mathbb{Z}) = 1$. Then $\mu([t,\infty)) = \mu([\lceil t\rceil,\infty))$ for every $t \in \mathbb{R}$. Consequently, for all κ and all real t,*

$$\big\{\mu([t,\infty)) : \mu \in \mathcal{P}^{\mathbb{Z}}(\kappa)\big\} = \big\{\mu([\lceil t\rceil,\infty)) : \mu \in \mathcal{P}^{\mathbb{Z}}(\kappa)\big\},$$

and likewise for $\mathcal{P}_2^{\mathbb{Z}}$; in particular the two sets of attainable tail probabilities at t and at s coincide whenever $\lceil t\rceil = \lceil s\rceil$. Hence $V_1^{\mathbb{Z}}(t,\kappa) = V_1^{\mathbb{Z}}(\lceil t\rceil,\kappa)$ and $V_2^{\mathbb{Z}}(t) = V_2^{\mathbb{Z}}(\lceil t\rceil)$, and the closed forms of Sections 3 and 4, read with $\lceil t\rceil$ in place of t, determine $V_1^{\mathbb{Z}}$ on all of $(0,\infty)\times[1,\infty)$.

Proof sketch. The interval $[t,\lceil t\rceil)$ contains no integer, so $[t,\infty)$ and $[\lceil t\rceil,\infty)$ differ by a μ-null set; the two indicator functions therefore agree μ-almost everywhere, and the measures of the two sets agree. The statement about the attainable sets follows because the class $\mathcal{P}^{\mathbb{Z}}(\kappa)$ does not mention t: the same μ witnesses both sides. No search. $\square$

So $V_1^{\mathbb{Z}}(\cdot,\kappa)$ is constant on each interval $(n-1,n]$, $n \in \mathbb{Z}$, whereas the map of [11] is smooth in t on the interior of each of its regimes. The gap between the two is therefore widest at the left end of each step, and the difference is not marginal.

Corollary 5.2. *$V_1^{\mathbb{Z}}(3/2,3) = 1/6$, and more generally $V_1^{\mathbb{Z}}(t,\kappa) = V_1^{\mathbb{Z}}(2,\kappa)$ for every $t \in (1,2]$ and every κ. In particular no integer-valued X with $\mathbf{E}X = 0$, $\mathbf{E}X^2 = 1$ and $\mathbf{E}X^4 \leq 3$ has $\mathbf{P}(X \geq 3/2) = 4/13$.*

Proof. Every $t \in (1,2]$ has $\lceil t\rceil = 2$, so Theorem 5.1 reduces the claim to Theorem 4.1 at $(2,3)$, which gives $1/6$; and $4/13 > 1/6$. $\square$

At $(t,\kappa) = (3/2,3)$ the value of [11] is its Regime I value $1/(1+t^2) = 4/13$. The Regime I window is stated there as $b(\kappa) \leq t \leq c(\kappa)$ with

$$b(\kappa) = \tfrac{1}{2}\big(\sqrt{\kappa+3} - \sqrt{\kappa-1}\big) = \frac{1}{c(\kappa)},$$

and, equivalently in the source's own words, as $\kappa \geq \kappa_c(t) = t^2 - 1 + t^{-2}$. Both readings put $(3/2,3)$ in Regime I: $b(3) = (\sqrt{6}-\sqrt{2})/2 \approx 0.5176 \leq 3/2 \leq c(3) = (\sqrt{6}+\sqrt{2})/2 \approx 1.9319$, and the rational form reads $3 \geq \kappa_c(3/2) = 61/36$. The lattice bound is thus a factor $(4/13)/(1/6) = 24/13 \approx 1.85$ sharper at $(3/2,3)$, against $(2/11)/(1/6) = 12/11 \approx 1.09$ at $(2,3)$. Note that this cell exercises a different branch of [11] from every comparison in Remark 4.4.

6. Comparison with the real-support problem, and negative controls

The statement "the lattice value is strictly smaller" is only meaningful once the two classes are known to be nested, so we record the containment and then exhibit the gap on both sides at one point.

Theorem 6.1. *Every $\mu \in \mathcal{P}^{\mathbb{Z}}(\kappa)$ satisfies* (1)*, so the set of upper-tail probabilities attainable by integer-valued laws is contained in the set attainable by real-valued laws, at the same (t, κ). At $(t, \kappa) = (2, 13/4)$ the containment is strict, and both sides are exhibited:*

(1) *the two-atom law $\frac{4}{5}\delta_{-1/2} + \frac{1}{5}\delta_2$ — Cantelli's extremal law at $t = 2$ — has mean 0, second moment 1, third moment $3/2$ and fourth moment $13/4$, and its upper-tail mass at 2 is $1/5$; so $1/5$ is attained in the real-support class;*
(2) *no integer-valued law in the same class attains $1/5$, the exact lattice maximum there being $V_1^{\mathbb{Z}}(2, 13/4) = 1/6$.*

The gap is $1/5 - 1/6 = 1/30$; relative to the real-support value the lattice bound is smaller by a sixth, about 17%.

Proof sketch. The containment is immediate from Definition 2.1, since the lattice class is the real-support class with one extra hypothesis. The moments of the two-atom law in (1) are four arithmetic identities. For (2), $13/4 \geq 3 = t^2 - t + 1$ at $t = 2$, so Theorem 4.2 gives the maximum $1/6 < 1/5$. Note that the non-tail atom of Cantelli's law sits at $-1/2$, which is exactly where the certificate $x(x+1)$ of Section 3 is negative, and is not an integer. No search. □

Remark 6.2. The point $(2, 13/4)$ is not arbitrary: $c(13/4) = (\sqrt{25/4} + \sqrt{9/4})/2 = 2$ exactly, so $t = 2$ sits on the Regime I/Regime II boundary of [11] and the two branches there both give $1/5$, which is also Cantelli's constant. This is what makes Cantelli's extremal law a legitimate witness at a *finite* κ. The identity is an arithmetic check on the formulas quoted in (2) and is not part of the formal development.

More such separations hold at the cells of Remark 4.4.

Proposition 6.3. *None of the following real-support optima is attained by an integer-valued law: Cantelli's $1/5$ at $t = 2$ in the two-moment class; the value $1/10$ of* [11] *at $(t, \kappa) = (2, 2)$; the value $2/11$ at $(2, 3)$; the value $3/67$ at $(3, 4)$; and the value $4/13$ at $(3/2, 3)$.*

Proof. Each is strictly larger than the corresponding lattice maximum, computed in Theorem 3.1, Theorem 4.1 or Corollary 5.2: $1/6 < 1/5$, $1/12 < 1/10$, $1/6 < 2/11$, $1/24 < 3/67$ and $1/6 < 4/13$. □

Finally we record the checks that the closed forms are not accidentally true and that their hypotheses carry content. Each is a claim that a formula applied *outside* its own window returns a value that is provably not the maximum.

Proposition 6.4 (negative controls)**.** (1) *At $(t, \kappa) = (2, 2)$, whose value is $1/12$, neither $1/10$ nor $1/13$ is the maximum; and $1/12$ is attained, so the class is non-empty.*
(2) *The hypothesis $\kappa \leq t^2 - t + 1$ of Theorem 4.1 is not decoration: at $(2, 4)$, one unit above the threshold, the binding formula evaluates to $1/4$, which is not the maximum — that maximum is $1/6$ by Theorem 4.2.*
(3) *The hypothesis $t \geq 2$ is not decoration: at $t = 1$ the denominator $t^2(t^2 - 1)$ of the binding formula vanishes, the expression evaluates to 0, and 0 is not the maximum — that maximum is $1/2$.*
(4) *The hypothesis $\kappa \geq 1$ is not decoration: at $(2, 1/2)$ the binding formula evaluates to $-1/24$, which is not the maximum, since every attainable value is a probability and hence nonnegative.*

Proof. Each item compares two rational numbers against a maximum already computed, together with the arithmetic evaluations stated. □

Item (4) uses only nonnegativity of the value set, which is what the formal development contains; that for $\kappa < 1$ the class is in fact *empty*, by $\mathbf{E}X^4 \geq (\mathbf{E}X^2)^2$, is a remark we make here and do not use.

We also record the quantitative statement behind the window in which the extremal laws were first located, which is used nowhere in the proofs above.

Proposition 6.5. *Let X be integer-valued with $\mathbf{E}X = 0$, $\mathbf{E}X^2 = 1$ and $\mathbf{E}X^4 \leq \kappa$. Then for every $N > 0$,*

$$\mathbf{P}\big(|X| \geq N\big) \leq \frac{\kappa}{N^4}.$$

For instance at $\kappa = 3$ no such law puts more than $1/27$ of its mass at distance 3 or more from the origin, against an optimal tail mass of $1/6$ at $(t, \kappa) = (2, 3)$.

Proof. Markov's inequality applied to x^4: $N^4\mathbf{P}(|X| \geq N) \leq \mathbf{E}\big[X^4\mathbf{1}_{|X|\geq N}\big] \leq \mathbf{E}X^4 \leq \kappa$. □

We stress what Proposition 6.5 does *not* do. It bounds the mass far from the origin, not the number of atoms, so it does not by itself justify a finite enumeration over laws with at most four atoms in a window; bounding the number of atoms of an extremal law in a finite-dimensional moment problem is a separate classical matter, of the type settled by Richter [19] and Rogosinski [20], and is not carried out here. None of the results of this paper needs such a justification, because each carries its own dual certificate valid over the whole class. See Section 9.

7. What is anticipated, and what is not

The certificate *class* used here — a polynomial deliberately nonnegative only on the lattice — is not new, and we do not claim it. It is the whole method of Infusino, Kuna, Lebowitz and Speer [6], whose abstract describes their approach as "finding an optimal polynomial of degree n which is nonnegative on $\mathbb{N}_0$ (and which depends on the moments), and requiring that its expectation be nonnegative", generalising the cases $n = 1$ and $n = 2$, "the Percus–Yamada condition". Their degree-2 family is written there as $T_k(x) = (x-k)(x-k-1)$, $k \in \mathbb{N}_0$, which is exactly the shape of $x(x+1)$ used here, shifted; their degree-3 family is $S_k(x) = x(x-k)(x-k-1)$. For the degree-two condition itself they point to Percus and Yamada [14, 22]; for the moment problem on an infinite discrete set to Karlin and Studden [7], Chapter VII, and for the specific choice $\mathcal{X} = \mathbb{N}_0$ to its Section 8. The class is also folklore: it is the subject of a MathOverflow thread [13].

What we did not find in the literature is that class used as the dual of a *tail-probability* optimisation. The problem of [6] is a realizability problem — which moment sequences come from a measure on $\mathbb{N}_0$ — and the words "Chebyshev" and "Cantelli" occur zero times in its LaTeX source.

We therefore separate two claims sharply.

The two-moment value is presented as anticipated, not as new. We could not locate a published statement of $V_2^{\mathbb{Z}}(t) = 1/\big(t(t+1)\big)$ in the form (4), and the search behind that negative is described in Section 1. We nevertheless present Theorem 3.1 as a warm-up and a control, not as a new result, and the reason is concrete rather than rhetorical: both halves of it are a substitution away from published work.

The value is a substitution into a published closed form. Boros and Prékopa [2] give the tightest upper bound on $\mathbf{P}(\xi \geq k)$ over integer-valued ξ supported on $[0, n]$ with prescribed first two binomial moments S_1, S_2. We could not obtain their text, so we quote the closed form as restated by Ramachandra and Natarajan [18, v6], whose third branch reads, for $k \geq 1 + 2S_2/S_1$,

$$\mathbf{P}(\xi \geq k) \;\leq\; \frac{(i-1)(i-2S_1) + 2S_2}{(k-i)^2 + (k-i)}, \qquad i = \left\lceil \frac{(k-1)S_1 - 2S_2}{k - S_1} \right\rceil. \tag{8}$$

Put $\xi = X + 1$. Then $S_1 = \mathbf{E}\xi = 1$ and $S_2 = \mathbf{E}\binom{\xi}{2} = \big(\mathbf{E}X^2 + \mathbf{E}X\big)/2 = 1/2$, while $\mathbf{P}(X \geq t) = \mathbf{P}(\xi \geq t+1)$, so $k = t + 1$. The branch condition $k \geq 2$ holds for every $t \geq 1$; for $t \geq 2$, $i = \lceil (t-1)/t \rceil = 1$, the numerator of (8) collapses to $2S_2 = 1$ and its denominator to $t^2 + t$, and

(8) reads $1/\big(t(t+1)\big)$. (At $t=1$, $i=0$ and it reads $3/6=1/2$, again our value.) That is the whole derivation of the constant.

The certificate is published too. Written in the same shifted variable, our dual $x(x+1)$ is $\xi(\xi-1)=T_0(\xi)$, a member of the degree-two Percus–Yamada family $T_k(x)=(x-k)(x-k-1)$ of [6] quoted above. It is also the polynomial that appears inside the proof of Dawson and Sankoff [4]. With $A_1,\dots,A_n$ events, ν the number of them that occur, $S_1=\mathbf{E}\nu$ and $S_2=\mathbf{E}\binom{\nu}{2}$, one has, for every integer $r\ge 1$,

$$\mathbf{P}\Big(\bigcup_i A_i\Big) \;\ge\; \frac{2}{r+1}S_1-\frac{2}{r(r+1)}S_2, \tag{9}$$

because $\mathbf{1}_{\{\nu\ge1\}}-\frac{2\nu}{r+1}+\frac{\nu(\nu-1)}{r(r+1)}$ is 0 at $\nu=0$ and, for $\nu\ge1$, equals $(\nu-r)(\nu-r-1)/\big(r(r+1)\big)=T_r(\nu)/\big(r(r+1)\big)\ge 0$ because ν is an *integer*. Their Theorem 1.1 is (9) at $r=\lfloor 2S_2/S_1\rfloor+1$ when $S_1>0$, rewritten; the expression (9) itself appears in the course of their proof, at their (2.7), with the index shifted by one. The denominator $r(r+1)$ there and the $t(t+1)$ here are the same phenomenon and the same family of polynomials.

What Theorem 3.1 adds is one line, which we do not claim either. The support $[0,n]$ of [2] is preassigned and bounded below, so (8) governs the integer laws with $X\ge-1$; our certificate $x(x+1)/\big(t(t+1)\big)$ is nonnegative at *every* integer, which carries the same constant to all of $\mathbb{Z}$, and its optimality is witnessed by a law that lives inside the window anyway. A reader who regards Theorem 3.1 as folklore will get no argument from us.

What is claimed new. Theorem 4.1 — the binding branch of (5), with its extremal law on $\{-1,0,1,t\}$ and its certificate $x^2(x^2-1)$ — and the threshold $\kappa=t^2-t+1$ of Theorem 4.2, which says at exactly which κ the fourth-moment constraint goes inactive and completes the map. The value in the saturated regime coincides with Theorem 3.1's and is not claimed twice; what is claimed there is the location of the boundary and the fact that the two branches close the map. For the reasons given in Section 1, no published method we are aware of reaches these: $\mathbb{Z}$ is not semialgebraic, both certificates are provably not sums of squares (Proposition 2.4), and the discrete moment problem closed forms described below fix their moments as equalities on a preassigned finite support and return a number, not a formula in (t,κ) with a regime boundary.

The nearest published machinery, and how far it specialises. The discrete moment problem of Kwerel, of Prékopa, and of Boros and Prékopa [9, 16, 2, 15] bounds, by linear programming duality over a *finite, preassigned* integer support, the probability that a prescribed number of events occurs, from the first few moments of that count. As (8) shows, that machinery does reach our two-moment constant, and we say so. It does not reach (5), for two reasons that are about the class rather than the event. Their moment data are *equalities* on a fixed number of moments, where the fourth-moment datum here is the inequality $\mathbf{E}X^4\le\kappa$ and κ is a free parameter to be optimised over; and their closed forms are indexed by the dual basis of one fixed programme, so they return a number at given (n,k,S_1,S_2) rather than a formula in (t,κ) with its regime boundary. We could not find, anywhere, a four-moment closed form of the shape (5) on an integer support, nor the threshold $\kappa=t^2-t+1$.

Provenance questions we could not close. We state these as dated caveats rather than resolve them by guessing. As of 2026-08-29:

(1) **Karlin–Studden** [7] **was not read in print**, only searched. The copy we could reach is a scan whose full text is searchable but not readable, so the evidence below is a full-volume text search rather than a reading. On that search the phrases *lattice*, *integer valued*, *integer-valued*, *integral valued*, *discrete random variable*, *fourth moment* and *first four moments* return nothing; *fourth* and *kurtosis* occur once each, both inside the bibliography; and the only one-sided inequality the book states is the ordinary continuous one, in Chapter XII. Chapter VII, Section 8 — the section [6] points to — is about realizability on $\mathbb{N}_0$, and [6]

adds that the technique used there characterises the realizable cone only up to an unknown parameter. So we do *not* cite [7] for the bound. What Chapter VII does contribute is the mechanism: its index calculus, in Section 2, gives an isolated pair of adjacent lattice points index 2, which is exactly what lets a polynomial nonnegative on $\mathbb{Z}$ carry two consecutive roots.

(2) **Boros–Prékopa 1989** [2] **and Prékopa's monograph** [15] **were not obtained**; both were paywalled at the time of writing. Everything we assert about [2] — including the closed form (8) — is quoted from the restatement in [18] and not from [2] itself, and a reader who needs (8) exactly as its authors wrote it should go to the original. We regard this as settled in the direction that matters: it makes Theorem 3.1 anticipated, which is how we present it. What we could not check inside [2] or [15] is whether they anywhere carry a *fourth*-moment analogue; nothing in the restatement, in the abstract of [2], or in the surveys we could read suggests one.

(3) **Krein–Nudelman** [8] **was not read in the primary either.** One external reading of both books is on record and points away from a tail bound: [11] cites [7] and [8] only for the classical theory of principal representations for Chebyshev systems, that is, for the structure and atom count of extremal measures.

(4) **The corpus searched is arXiv-only** and excludes the statistics and quantitative finance categories unless cross-listed to mathematics; the actuarial and risk literature, where moment bounds also live, is a real blind spot, and pre-1992 literature is invisible to it. This limitation is not hypothetical: the corpus search reported in Section 1 did *not* find [2], which is from 1989 and which settles the provenance of Theorem 3.1. We reached it by following citations out of [6] instead. The same failure mode is the honest reason to keep reading item (2) above as open for the four-moment map.

These caveats are about mathematical *content*, not about the references themselves: the bibliographic data of [7], [2], [15], [8], [9], [14] and [22] were verified against Crossref, zbMATH and the reference list of [6], and [4] was read in the primary.

8. Open questions

(1) **The size of the gap on a step, in closed form.** By Theorem 5.1 the ratio of $V_1(t,\kappa)$ to $V_1^{\mathbb{Z}}(t,\kappa)$ is largest at the left end of each interval $(n-1,n]$, and Corollary 5.2 measures it at one point. Its supremum over $t \in (n-1,n]$ should be available in closed form: the numerator is given by the Regime I and Regime II formulas of [11], and the denominator is constant on the interval, so this is a comparison of two closed forms and needs no new certificate.

(2) **More moments.** Adding $\mathbf{E}X^6 \le \lambda$ should produce a sextic certificate nonnegative on $\mathbb{Z}$; by analogy with $x^2(x^2-1)$ the natural guess is a multiple of $x^2(x^2-1)(x^2-4)$, with a three-regime map whose thresholds are the sixth moments of the successive extremal laws.

(3) **Other lattices.** Replacing $\mathbb{Z}$ by $a + h\mathbb{Z}$ for rational a, h is genuinely different rather than a rescaling, because the variance constraint is not scale-free. A numerical refinement on $\frac{1}{2}\mathbb{Z}$ and $\frac{1}{4}\mathbb{Z}$ (see Section 9) shows the value moving monotonically towards the real-support one as $h \to 0$; a closed form in h would explain that interpolation.

(4) **An application.** Integer-valued combinatorial statistics — subgraph counts, discrepancies — are exactly the objects for which one wants anticoncentration, and the constant here is strictly better than the real-support one. But the existing small-ball and Littlewood–Offord line reaches such statistics through arithmetic structure rather than through moments, and we know of no result today that would be improved by substituting (5). We record this as an honest weakness of the motivation.

9. Verification

Every theorem, proposition, lemma and corollary of this paper has been formally verified in Lean 4 (toolchain `v4.32.0`) against *Mathlib* [10, 12]; repository reference to be added at submission. The development consists of seven modules; `#print axioms` was run on every declaration to which a statement of this paper is pinned, and returns for each the single axiom set {`propext`, `Classical.choice`, `Quot.sound`} — no `sorry`, and in particular no compiled evaluation: there is no use of `native_decide`, no external solver, and no stored certificate anywhere in the development. The arithmetic is over $\mathbb{Q}$ and $\mathbb{R}$ throughout. The probability classes are formalised as honest Borel measures on $\mathbb{R}$ with Bochner integrals, integrability being part of Definition 2.1 as discussed there, and the comparison of Theorem 6.1 is machine-checked against a separate formalisation of the real-support class of [11] written by us in a companion study of that paper's map, so that the containment is a theorem about one class rather than an informal identification of two definitions. We are not authors of [11]; it is used here only as a published source of control values, and the certificates and closed forms it reports were re-derived on our side before being quoted.

No result in this paper rests on an exhaustive computation. Every upper bound is delivered by an explicit dual polynomial, valid over the entire class and uniformly in the parameters, via Proposition 2.2; every lower bound is one explicit atomic law. This is worth saying because the closed forms (4) and (5) were *discovered* by a finite search, which we describe for completeness and which is not part of any proof. An exact rational enumerator was written independently of the formal development. Because the linear program defining $V_1^{\mathbb{Z}}$ has four constraints (total mass, first, second and fourth moment, the last with a slack variable), every basic feasible solution has at most four nonzero masses, so it suffices to scan integer 3-subsets of a window solving the first three constraints with $\int x^4 \le \kappa$ slack, and integer 4-subsets solving all four with $\int x^4 = \kappa$. Three runs were made. The first swept $t \in \{1, \ldots, 5\}$ against, for each t, the twelve fixed budgets $\kappa \in \{1, \frac{11}{10}, \frac{3}{2}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4, \frac{9}{2}, 5, 6, 7\}$ together with the six t-dependent ones t^2-t+1, $t^2-t+1\pm\frac{1}{4}$, t^2, t^2+5 and 50, the union taken without repetition and intersected with $\kappa \ge 1$; that leaves $14, 16, 17, 18, 18$ budgets at $t = 1, \ldots, 5$, i.e. 83 cells, and in the window $|x| \le 9$ every one of them agrees with (5). A second run took 72 further cells at 12 real thresholds (integer and non-integer) against 6 values of κ, agreeing with Theorem 5.1 in every cell; a third re-ran six cells at $|x| \le 12$ and at $|x| \le 15$, with identical values and identical extremal laws. As an external control the same enumerator was run on the finer lattices $\frac{1}{2}\mathbb{Z}$ and $\frac{1}{4}\mathbb{Z}$, where the values must climb towards the published real-support ones: at $(2, 3)$ it returns $1/6 < 8/45 < 149/825$ against the limit $2/11$ of (2), and at $(2, 2)$ it returns $1/12 < 4/45 < 16/165$ against $1/10$. The $\frac{1}{4}\mathbb{Z}$ run is 211,876 four-atom systems, about two CPU-minutes. All arithmetic in these runs is exact rational arithmetic.

One small deduction is left to the reader rather than formalised: what the development contains for Proposition 2.4 is the two strict inequalities $\left(-\frac{1}{2}\right)\left(-\frac{1}{2}+1\right) < 0$ and $\left(\frac{1}{2}\right)^2\left(\left(\frac{1}{2}\right)^2 - 1\right) < 0$ — equivalently, that neither $x(x+1)$ nor $x^2(x^2-1)$ is nonnegative on all of $\mathbb{R}$ — together with the two integer nonnegativity facts of Lemma 2.3; the displayed values $-1/4$ and $-3/16$ are the same one-line arithmetic, and the step from "negative somewhere on $\mathbb{R}$" to "not a sum of squares" is the remark made in the proof of Proposition 2.4.

Three further kinds of material lie outside the formal guarantee and are marked where they occur: the numerical values of the real-support map quoted from [11] in Remark 4.4, Remark 6.2, Corollary 5.2 and Proposition 6.3 — the *comparisons* are formal, but the identification of those numbers as the values of [11] rests on reading that paper; the arithmetic of Remark 6.2; and the enumeration runs described in the previous paragraph. The open questions of Section 8 are discussion.

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