THE DENOMINATOR 19 IS FORCED: THE CONDITIONAL CONSTANT $\frac{1}{2}+\frac{1}{3}+\frac{2}{19}$ FOR INTEGER POLYNOMIALS WITH MINIMAL INTEGRALS

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ABSTRACT. For a positive integer N put $\ell_N = \mathrm{lcm}(1,2,\dots,N)$ and let $S_N = \{P \in \mathbb{Z}[x] : \deg P < N,\ \int_0^1 P(x)\,dx = 1/\ell_N\}$ be D. Bazzanella's set of integer polynomials with the smallest possible positive integral on $[0,1]$; it is the set on which the Gelfond–Schnirelman–Nair elementary lower bound for ψ is an equality. Sanna asked how large $\delta_Q = \liminf_N \deg(Q_N)/N$ can be over sequences of explicit polynomials Q_N each dividing some member of S_N, and Bazzanella and Sanna have recently observed that $Q_N = x^{\lfloor N/2\rfloor}(1-x)^{\lfloor N/3\rfloor}(1-x-x^2)^{\lfloor N/19\rfloor}$ would give $\delta_Q = \frac{1}{2}+\frac{1}{3}+\frac{2}{19} = \frac{107}{114} = 0.9385964912\ldots$, verifying the required divisibility by a coefficient search for $N \le 1000$. We prove a Bézout criterion, in both directions, for the whole three-factor family $x^a(1-x)^b(1-x-x^2)^c$ — divisibility of some member of S_N is equivalent to a gcd of $N-a-b-2c$ explicit integers being 1 — and use it to settle four things. The divisibility holds for every $N \le 2000$, twice the published range. The denominator 19 is the least one that can occur: for $13 \le d \le 18$ the analogous sequence fails already at $N = 13, 15, 15, 18, 18, 18$, so no member of that one-parameter family can produce a constant larger than $107/114$. Each of the three exponents is maximal. Finally, and unconditionally in N: no prime p with $\deg Q < p \le N < p^2$ divides the integral $\ell_N \int_0^1 Q(x)x^{p-1-\deg Q}dx$, so an obstructing prime $p \le N$ is at most $\deg Q_N \approx 0.9386\,N$. Every theorem below is machine-checked in Lean 4; §7 says exactly what is checked and on what computation each claim rests.

1. INTRODUCTION

For a positive integer N write

$$\ell_N := \mathrm{lcm}(1,2,\dots,N), \qquad I(P) := \int_0^1 P(x)\,dx.$$

If $P \in \mathbb{Z}[x]$ has $\deg P < N$ then $\ell_N I(P) \in \mathbb{Z}$, so $1/\ell_N$ is the smallest positive value $I(P)$ can take. D. Bazzanella [2] introduced the set of polynomials attaining it,

$$S_N := \Big\{ P \in \mathbb{Z}[x] \ :\ \deg P < N, \quad I(P) = \frac{1}{\ell_N} \Big\}. \tag{1}$$

The set matters because it is where the elementary Gelfond–Schnirelman–Nair lower bound for the Chebyshev function is an equality: for every $P \in S_N$ one has

$$\psi(N) = \sum_{p^m \le N} \log p = \log \ell_N = -\log I(P),$$

and bounding $I(P)$ from above through $\|P\|_{[0,1]}$ gives a lower bound for $\psi(N)$. The method was proposed in 1936 by Gelfond and Schnirelman (Gel'fond's comments in Chebyshev's *Collected Works* [6, pp. 285–288] are where it is recorded; see Pritsker [7]) and was rediscovered and developed further in 1982 by Nair [4, 5]. (The problem should not be confused with the integer Chebyshev problem, where one minimises the uniform norm on $[0,1]$ instead of the integral; see Pritsker [8] for that circle of questions.)

It is natural to ask how much structure a member of S_N can be required to carry, and in particular how large a prescribed factor it can be forced to have. Following [1], call a sequence $\{Q_N\}_{N\ge1}$ of explicit polynomials *admissible* if for every positive integer N there is a $P \in S_N$ with $Q_N \mid P$ in $\mathbb{Z}[x]$,

and put

$$\delta_Q := \liminf_{N\to\infty} \frac{\deg(Q_N)}{N}. \tag{2}$$

Sanna [3, Question] asked how large δ_Q can be, and whether it can be arbitrarily close to, or equal to, 1. Three values are known: $\delta_Q = 1/2$ for $Q_N = x^{\lfloor N/2\rfloor}$ (D. Bazzanella [2]), $\delta_Q = 2/3$ for $Q_N = (x(1-x))^{\lfloor N/3\rfloor}$ (Sanna [3]), and $\delta_Q = 5/6$ for $Q_N = (x^3(1-x)^2)^{\lfloor N/6\rfloor}$, which is Theorem 1.4 of Bazzanella and Sanna [1]; Theorem 1.5 of [1] shows 5/6 to be optimal within the two-factor shape $x^{a_N}(1-x)^{b_N}$, in the sense that $\liminf_N (a_N + b_N)/N \le 5/6$ for every admissible sequence of that shape. Each of the three sequences is admissible in the strong sense required by (2): Theorems 1.2, 1.3 and 1.4 of [1] each assert, for *every* positive integer N, that infinitely many $P \in S_N$ are divisible by the polynomial in question. Theorem 1.3 is Sanna's [3, Theorem 1.4]; Theorem 1.2 is deduced in [1] from D. Bazzanella's [2, Theorem 1] together with the remarks following [3, Theorem 1.3], where Bazzanella's result is quoted in the form $K(N) = \min\{p^m - 1 : p \text{ prime},\ m \ge 1,\ p^m > N/2\}$ for the largest exponent of x (or of $1-x$) available at N; Theorem 1.4 is the main theorem of [1] itself.

The sequence this paper is about. To go past 5/6 one must leave the two-factor shape, and [1] proposes doing so with the Fibonacci polynomial $1 - x - x^2$. Its equation (1.5) is

$$Q_N(x) := x^{\lfloor N/2\rfloor}\,(1-x)^{\lfloor N/3\rfloor}\,(1-x-x^2)^{\lfloor N/19\rfloor}, \tag{3}$$

and the sentence following it reads, verbatim: *"a computation shows that for every positive integer $N \le 1000$ there exists a polynomial $P \in S_N$ such that $Q_N(x)$ divides $P(x)$ in $\mathbb{Z}[x]$ (see the code in the Appendix)"*, and then, with its displayed formula run into the text: *"If the previous property indeed holds for every positive integer N, then $\delta_Q = 1/2 + 1/3 + 2/19 = 0.938\ldots$, which would improve the value of δ_Q given by the sequence of Theorem 1.4."* The exact value of that constant is

$$\frac{1}{2} + \frac{1}{3} + \frac{2}{19} = \frac{107}{114} = 0.9385964912\ldots \tag{4}$$

The computation reported in [1] is a Mathematica `FindInstance` search, printed in its Appendix, over the $m+1$ integer coefficients of the cofactor, where $m = n - \lfloor n/2\rfloor - \lfloor n/3\rfloor - 2\lfloor n/19\rfloor - 1$, run for $n = 3, \ldots, 1000$. Nothing else is asserted about (3): the value (4) is stated as conditional on a property verified on a finite window, and [1] proves nothing about that property.

What is settled here. Let us say that $Q \in \mathbb{Z}[x]$ with $\deg Q < N$ is *N-admissible* if some $P \in S_N$ is divisible by Q in $\mathbb{Z}[x]$, so that a sequence is admissible exactly when Q_N is N-admissible for every N. Our starting point (§2) is that for the three-factor family

$$Q_{a,b,c}(x) := x^a(1-x)^b(1-x-x^2)^c, \qquad D := \deg Q_{a,b,c} = a + b + 2c,$$

N-admissibility is not a search but a gcd: with $m = N - 1 - D$ and $\beta_i := \ell_N \int_0^1 Q_{a,b,c}(x)x^i\,dx \in \mathbb{Z}$, the polynomial $Q_{a,b,c}$ is N-admissible if and only if $\gcd(\beta_0, \ldots, \beta_m) = 1$ (Proposition 2.2). This is the Bézout argument of Lemma 2.4 of [1] with its binomial evaluation removed; [1] states it only for $c = 0$, where the integrals collapse to reciprocals of a single binomial coefficient, and for (3) its Appendix falls back on a coefficient search instead. We prove both directions, and the converse direction is what makes the negative results below genuine refutations rather than failures of a test.

With that criterion, four things follow.

- **The window.** Q_N is N-admissible for every $N \le 2000$ (Theorem 3.1), twice the range checked in [1]; a sub-window, $N \le 200$, is verified without any appeal to compiled evaluation.
- **The denominator 19 is forced** (Theorem 4.1). Replacing 19 in (3) by d gives $\delta(d) = \frac{1}{2} + \frac{1}{3} + \frac{2}{d}$, which is < 1 exactly for $d \ge 13$; for each d with $13 \le d \le 18$ the resulting sequence is *not* admissible, failing already at $N = 13, 15, 15, 18, 18, 18$ respectively. So no member of the one-parameter family of [1] can produce a value larger than 107/114, and $d = 19$ is the only denominator at which that value could be attained.

- **Every exponent is maximal** (Theorem 4.3): raising any one of $\lfloor N/2\rfloor$, $\lfloor N/3\rfloor$, $\lfloor N/19\rfloor$ by 1 destroys N-admissibility, at $N = 5$, $N = 5$ and $N = 13$ respectively, while Q_5 and Q_{13} themselves are admissible at those N.
- **Large primes never obstruct** (Theorem 5.1 and Corollary 5.2), unconditionally and uniformly in N: for a prime p with $D < p \le N < p^2$ the single index $i = p-1-D$ already gives $p \nmid \beta_i$. Hence a prime $p \le N$ dividing all of $\beta_0, \dots, \beta_m$ satisfies $p \le D$; for the sequence (3) this says that an obstructing prime $p \le N$ is at most $\deg Q_N = \lfloor N/2\rfloor + \lfloor N/3\rfloor + 2\lfloor N/19\rfloor$, so the top $1 - 107/114 = 7/114 \approx 6.14\%$ of the prime range is free for every N.

Sections 6 and 8 record what is known beyond that: an exact-arithmetic computation, outside the formal development, confirming the property for every $N \le 10^4$ and at $N = 12000, 20000, 30000$, and a search over 1.3 million exponent rules that finds no shape beating 107/114 within its horizon. None of this proves the property of (3), and we do not claim it: the constant (4) remains conditional, exactly as [1] leaves it. What changes is that the condition is now pinned down — it is a statement about primes below $0.9386\,N$ — and that the denominator 19 is no longer an unexplained choice.

1.1. **The text used, and what was searched.** We read [1] from the LaTeX source of its arXiv e-print v1, and we compiled that source, so that the numbers of the results of [1] quoted below are the numbers its own LaTeX produces. On 3 September 2026 the abstract page listed v1 as the only version, in `math.NT` alone, with no journal reference; the arXiv API returned the same metadata, a listing query over `math.NT` submissions of the preceding sixty days returned nothing on this problem, and OpenAlex recorded no citing works. A search of a local full-text mirror of the arXiv for the phrases *minimal integrals*, *integer polynomials with small integrals* and *Gelfond–Schnirelman* (in both the *Schnirelman* and the *Shnirelman* transliterations, the latter being the one [1] uses) returned, besides [1] itself, only papers on the elementary prime-counting method and false positives in which *minimal integrals* means something else — Darboux integrals of a partial differential equation, integrals of motion, minimal integrals of a holonomy groupoid, or the minimum of a family of integrals. All fifteen references of [1] were read, and the two closest, [2] and [3], concern factors $x^a(1-x)^b$ only.

2. The criterion

Throughout, a polynomial $P = \sum_j P_j x^j \in \mathbb{Z}[x]$ is identified with its coefficient vector, and deg is the usual degree. Fix $a, b, c \in \mathbb{N}$ and write

$$Q = Q_{a,b,c} = x^a(1-x)^b(1-x-x^2)^c, \qquad D = \deg Q = a + b + 2c.$$

The leading coefficient of Q is $(-1)^{b+c} = \pm 1$; in particular Q is never zero and $\deg Q$ is exactly $a + b + 2c$.

Definition 2.1. Let $N > D$ and put $m := N - 1 - D$. For $0 \le i \le m$ set

$$\beta_i = \beta_i(N, a, b, c) := \ell_N \int_0^1 Q(x)\, x^i\, dx = \sum_{j=0}^{D} Q_j\, \frac{\ell_N}{i + j + 1}.$$

Each β_i is an integer: for $0 \le i \le m$ and $0 \le j \le D$ one has $1 \le i + j + 1 \le m + D + 1 = N$, so $i + j + 1$ divides ℓ_N. The numbers β_i are the only input the criterion needs.

Proposition 2.2 (the criterion)**.** *Let $a, b, c \in \mathbb{N}$ and $N > D = a + b + 2c$, and let $\beta_0, \dots, \beta_m$ be as in Definition* 2.1*, $m = N - 1 - D$. Then $Q_{a,b,c}$ is N-admissible — that is, there exists $P \in \mathbb{Z}[x]$ with $\deg P < N$, $Q_{a,b,c} \mid P$ in $\mathbb{Z}[x]$ and $I(P) = 1/\ell_N$ — if and only if*

$$\gcd(\beta_0, \beta_1, \dots, \beta_m) = 1.$$

Proof sketch. A polynomial P with $\deg P < N$ and $Q \mid P$ is exactly a product $P = Qq$ with $q = \sum_{i=0}^{m} y_i x^i \in \mathbb{Z}[x]$, since a nonzero such P has $\deg q = \deg P - D \le N - 1 - D = m$ (and members of S_N are nonzero). Integrating termwise,

$$\ell_N\, I(Qq) = \sum_{i=0}^{m} y_i\, \ell_N \int_0^1 Q(x) x^i\, dx = \sum_{i=0}^{m} y_i \beta_i,$$

so $P = Qq$ lies in S_N if and only if $\sum_i y_i\beta_i = 1$. A linear Diophantine equation $\sum_i y_i\beta_i = 1$ has an integer solution if and only if $\gcd(\beta_0, \dots, \beta_m) = 1$, which gives both directions at once: Bézout produces the cofactor from the gcd, and conversely any cofactor exhibits 1 as an integer combination of the β_i, so that their gcd divides 1. □

Remark 2.3. This is the mechanism of Lemma 2.4 of [1], which proves the corresponding equivalence for $Q = x^a(1-x)^b$ by expanding the cofactor in powers of $1-x$ and evaluating the resulting Beta integrals (its Lemma 2.1), so that its coefficients are the explicit integers $\ell_N/\big((a+b+n+1)\binom{a+b+n}{a}\big)$, and then applying Bézout (its Lemma 2.3). That evaluation is unavailable for $c > 0$, which is why the Appendix of [1] searches for the cofactor's coefficients instead. The Bézout step, however, does not need it, and Proposition 2.2 is what makes every computation in this paper cheap: deciding N-admissibility costs one gcd of $m+1$ integers, and $m+1 = N - \deg Q_N \geq N - \frac{107}{114}N \approx 0.0614\,N$ for the sequence (3). All numbers of results of [1] used here — Lemmas 2.1, 2.3, 2.4 and Theorems 1.2–1.5 — are those of its arXiv e-print v1 and may change in a published version.

Remark 2.4. In the formal development the integral $I(P)$ of a polynomial of degree $< n$ is handled through the rational number $\sum_{j<n} P_j/(j+1)$, and a separate theorem identifies that sum with the interval integral $\int_0^1 P(x)\,dx$ over $\mathbb{R}$. So S_N in (1) is the set of [1], not a re-definition of it; see §7.

3. The window

Theorem 3.1. *Let Q_N be the sequence* (3)*. For every N with $1 \leq N \leq 2000$ there exists $P \in \mathbb{Z}[x]$ with*

$$\deg P < N, \qquad Q_N \mid P \text{ in } \mathbb{Z}[x], \qquad \int_0^1 P(x)\,dx = \frac{1}{\ell_N};$$

that is, Q_N is N-admissible for every $N \leq 2000$. The same conclusion for the sub-windows $N \leq 200$ and $N \leq 1000$ is obtained with strictly less computational input, as explained in §7*.*

Proof sketch. By Proposition 2.2 it is enough, for each of the 2000 values of N in turn, to check that the integers $\beta_0, \dots, \beta_{m_N}$ attached to the exponent triple $(\lfloor N/2\rfloor, \lfloor N/3\rfloor, \lfloor N/19\rfloor)$ have gcd 1. They are formed exactly, and no exception occurs. The computation is exhaustive over the stated range and asserts nothing outside it. □

The range $N \leq 1000$ is exactly the range of the computation reported in [1]; that part of Theorem 3.1 is a confirmation of published data, obtained by a different route (a gcd rather than a coefficient search) and, for $N \leq 200$, by the proof kernel itself. The range $1001 \leq N \leq 2000$ is new. We stress that this is a window and not a proof: nothing in Theorem 3.1 bears on $N > 2000$.

4. The denominator 19, and the maximality of the exponents

The choice of 19 in (3) is presented in [1] without comment. It is forced. For $d \geq 1$ put

$$Q_N^{(d)}(x) := x^{\lfloor N/2\rfloor}(1-x)^{\lfloor N/3\rfloor}(1-x-x^2)^{\lfloor N/d\rfloor}, \qquad \delta(d) := \frac{1}{2} + \frac{1}{3} + \frac{2}{d},$$

so that $\delta(d)$ is the value (2) would take, and $\delta(d) < 1$ exactly for $d \geq 13$.

Theorem 4.1. *None of the following polynomials divides any member of the corresponding S_N:*

$$x^6(1-x)^4(1-x-x^2) \text{ in } S_{13}, \quad x^7(1-x)^5(1-x-x^2) \text{ in } S_{15}, \quad x^9(1-x)^6(1-x-x^2) \text{ in } S_{18}.$$

Consequently, for each d with $13 \leq d \leq 18$ the sequence $\{Q_N^{(d)}\}$ is not admissible: it fails at $N = 13$ for $d = 13$, at $N = 15$ for $d \in \{14, 15\}$, and at $N = 18$ for $d \in \{16, 17, 18\}$. Hence $d = 19$ is the least denominator with $\delta(d) < 1$ for which $\{Q_N^{(d)}\}$ can be admissible: among those d, no admissible member of the family $\{Q_N^{(d)}\}$ has δ larger than

$$\delta(19) = \frac{107}{114} = 0.9385964912\ldots,$$

and that value can be attained only at $d = 19$, whether it is attained being exactly the open property of (3). (*The remaining denominators $d \le 12$, where $\delta(d) \ge 1$, are excluded by the degree count of Remark* 4.2.) *Moreover the failure at $N = 18$ is caused by the extra factor and not by S_{18}: the polynomial $x^9(1-x)^6 = Q_{18}^{(19)}$ does divide a member of S_{18}.*

Proof sketch. The three non-divisibility statements are the three cases $(a, b, c) = (6, 4, 1)$ at $N = 13$, $(7, 5, 1)$ at $N = 15$ and $(9, 6, 1)$ at $N = 18$ of Proposition 2.2. In all three $m = N - 1 - D = 0$, so the gcd to be computed is that of the single integer β_0; formed exactly it equals 9, 3 and 8 respectively, and the converse half of the criterion turns that into the assertion that *no* $P \in S_N$ is divisible by the polynomial in question. The deduction is then arithmetic: $\lfloor 13/13 \rfloor = 1$ gives $Q_{13}^{(13)} = x^6(1-x)^4(1-x-x^2)$; $\lfloor 15/14 \rfloor = \lfloor 15/15 \rfloor = 1$ give $Q_{15}^{(14)} = Q_{15}^{(15)} = x^7(1-x)^5(1-x-x^2)$; and $\lfloor 18/16 \rfloor = \lfloor 18/17 \rfloor = \lfloor 18/18 \rfloor = 1$ give $Q_{18}^{(16)} = Q_{18}^{(17)} = Q_{18}^{(18)} = x^9(1-x)^6(1-x-x^2)$. Since δ is decreasing in d and $\delta(d) < 1$ only for $d \ge 13$, the six refuted values exhaust the range $13 \le d \le 18$, and $d = 19$ is the least remaining candidate. The positive statement at $N = 18$ is the case $(9, 6, 0)$ of the criterion, where $(\beta_0, \beta_1, \beta_2) = (153, 90, 55)$ has gcd 1. □

Remark 4.2. For $d \le 12$ one has $\delta(d) \ge 1$ and the sequence fails for a trivial reason, which we record for completeness (this is an elementary degree count, outside the formal development): at $N = 6d$ one gets $\deg Q_{6d}^{(d)} = 3d + 2d + 2 \cdot 6 = 5d + 12 \ge 6d = N$, while a nonzero P with $\deg P < N$ divisible by $Q_{6d}^{(d)}$ would need $\deg Q_{6d}^{(d)} < N$; and members of S_N are nonzero. So the interesting range really is $d \ge 13$, and Theorem 4.1 settles all of it below 19.

Theorem 4.3. *Each of the three exponents in* (3) *is maximal at an explicit N:*

(i) *no $P \in S_5$ is divisible by $x^3(1-x)$, where $3 = \lfloor 5/2 \rfloor + 1$ and $1 = \lfloor 5/3 \rfloor$;*
(ii) *no $P \in S_5$ is divisible by $x^2(1-x)^2$, where $2 = \lfloor 5/2 \rfloor$ and $2 = \lfloor 5/3 \rfloor + 1$;*
(iii) *no $P \in S_{13}$ is divisible by $x^6(1-x)^4(1-x-x^2)$, where $6 = \lfloor 13/2 \rfloor$, $4 = \lfloor 13/3 \rfloor$ and $1 = \lfloor 13/19 \rfloor + 1$.*

By contrast $Q_5 = x^2(1-x)$ and $Q_{13} = x^6(1-x)^4$ are 5- and 13-admissible (Theorem 3.1). *So raising any single exponent of* (3) *by one destroys the property, at $N = 5$, $N = 5$ and $N = 13$ respectively.*

Proof sketch. Parts (i), (ii) and (iii) are the cases $(3, 1, 0)$ and $(2, 2, 0)$ at $N = 5$ and $(6, 4, 1)$ at $N = 13$ of Proposition 2.2; in each of them $m = 0$ and the single integer β_0 is 3, 2 and 9 respectively, so the gcd exceeds 1. Part (iii) is the same computation as the $d = 13$ case of Theorem 4.1. The positive statements are the cases $(2, 1, 0)$ at $N = 5$ and $(6, 4, 0)$ at $N = 13$, where $(\beta_0, \beta_1) = (5, 3)$ and $(\beta_0, \beta_1, \beta_2) = (156, 91, 56)$ have gcd 1. □

Remark 4.4. Parts (i) and (ii) are explicit instances of a published phenomenon: Theorems 1.2 and 1.3 of [1], going back to D. Bazzanella [2] and Sanna [3] respectively, assert the optimality of the exponents $\lfloor N/2 \rfloor$ and $\lfloor N/3 \rfloor$ in $x^{\lfloor N/2 \rfloor}$ and $(x(1-x))^{\lfloor N/3 \rfloor}$: each states that "there exist infinitely many positive integers N" for which the once-raised exponent ($x^{\lfloor N/2 \rfloor + 1}$, resp. $(x(1-x))^{\lfloor N/3 \rfloor + 1}$) divides no polynomial in S_N. What is new in (i) and (ii) is only the explicit value $N = 5$, and the presence of the other factors. Part (iii) concerns the exponent of $1 - x - x^2$, about which [1] says nothing.

5. Large primes never obstruct

Everything so far has been a finite computation. This section is not: it is a statement about all N at once, and it locates the whole remaining difficulty of (3) below $0.9386\,N$.

Theorem 5.1. *Let $a, b, c \in \mathbb{N}$, $Q = Q_{a,b,c}$, $D = \deg Q = a + b + 2c$, and let N and a prime p satisfy*

$$D < p \le N < p^2.$$

Put $i := p - 1 - D$. Then $p \nmid \beta_i$, where $\beta_i = \ell_N \int_0^1 Q(x) x^i \, dx$ is as in Definition 2.1.

Proof sketch. This is a proof, not a computation. Write $\beta_i = \sum_{j=0}^{D} Q_j\, \ell_N/(i+j+1)$. With $i = p-1-D$ the denominators $i+j+1$ run over the interval $[p-D,\, p]$ as j runs over $[0, D]$; that interval has length $D < p$ and its right endpoint is p, so p divides $i+j+1$ exactly when $j = D$. Since $p \le N < p^2$, the exponent of p in $\ell_N = \operatorname{lcm}(1, \dots, N)$ is exactly 1; consequently $\ell_N/(i+j+1)$ is divisible by p for every $j < D$, while ℓ_N/p is not. Terms with $j > D$ vanish because $\deg Q = D$. The surviving term is $Q_D \cdot \ell_N/p$ with $Q_D = (-1)^{b+c} = \pm 1$ the leading coefficient of Q, so $p \nmid \beta_i$. □

Corollary 5.2. (i) *Let $a, b, c \in \mathbb{N}$, $D = a+b+2c < N$, and let p be a prime with $p \le N < p^2$. If p divides $\gcd(\beta_0, \dots, \beta_m)$ then $p \le D$.*

(ii) *Let $N \ge 2$ and let p be a prime with $p \le N$ dividing the gcd attached to the sequence* (3)*, that is $p \mid \gcd(\beta_0, \dots, \beta_{m_N})$ with $(a, b, c) = (\lfloor N/2 \rfloor, \lfloor N/3 \rfloor, \lfloor N/19 \rfloor)$. Then*

$$p \;\le\; \deg Q_N = \Big\lfloor \frac{N}{2} \Big\rfloor + \Big\lfloor \frac{N}{3} \Big\rfloor + 2\Big\lfloor \frac{N}{19} \Big\rfloor.$$

Proof sketch. (i) is Theorem 5.1 read by contraposition: if $p > D$ then the index $i = p-1-D$ lies in $[0, m]$, because $0 \le p-1-D$ and $p-1-D \le N-1-D = m$, and $p \nmid \beta_i$, whereas a common prime factor of $\beta_0, \dots, \beta_m$ divides β_i. For (ii) the side condition $N < p^2$ is automatic: a prime p violating the conclusion would satisfy $p > \deg Q_N \ge \lfloor N/2 \rfloor$, whence $p^2 \ge (\lfloor N/2 \rfloor + 1)^2 > 2\lfloor N/2 \rfloor + 1 \ge N$ for $N \ge 2$. □

Combining Corollary 5.2 with Proposition 2.2: if Q_N fails to be N-admissible for some $N \ge 2$, then either all the integers $\beta_0, \dots, \beta_{m_N}$ vanish, or they have a common prime factor, and every such prime factor *that is at most* N is at most $\deg Q_N$. (The hypothesis $p \le N$ is genuinely needed in Theorem 5.1 and is kept in the formal statement; nothing here excludes a common prime factor exceeding N.) Since

$$\frac{\deg Q_N}{N} \longrightarrow \frac{107}{114} = 0.9385964912\ldots,$$

the conclusion is that the top $7/114 \approx 6.14\%$ of the prime range below N is free of obstructions, uniformly in N and with no computation involved. In particular, when N is itself prime the index singled out by Theorem 5.1 is $i = N - 1 - \deg Q_N = m_N$, the very last one: the largest prime is handled by the last coefficient.

We emphasise what is *not* proved. Nothing here says anything about primes $p \le \deg Q_N$, and the property behind (4) remains open in general, exactly as [1] leaves it.

6. Computations outside the formal development

The following two items are numerical, were carried out in exact integer arithmetic outside the machine-checked development, and are recorded as measurements rather than as theorems.

Remark 6.1 (the property to $N \le 10^4$). Evaluating the criterion of Proposition 2.2 for the sequence (3) with exact integers shows that Q_N is N-admissible for *every* $N \le 10\,000$ — ten times the range of [1] — and also at the isolated values $N = 12\,000$, $20\,000$ and $30\,000$. Two independently written implementations were used: one forming the coefficient vector of Q_N by convolution and β_i termwise, the other computing β_i from the Beta integral (Lemma 2.1 of [1]) together with the recurrence $(k+1)e_{k+1} = (k-c)e_k + (k-1-2c)e_{k-1}$ for the coefficients e_k of $(1-x-x^2)^c$; they agree on all gcds for $N \le 120$ and on the full vectors $(\beta_i)_i$ at $N = 37, 61, 100, 119, 120$, where the formal definitions were also evaluated and agreed. The gcd is never close to failing: whenever it equals 1 it does so exactly. The complete sweep to $N \le 10^4$ cost 3.5 CPU-hours, and the three isolated values $N = 12\,000$, $20\,000$, $30\,000$ cost 10.4 s, 55.9 s and 380 s; extrapolating from these, a single value at $N = 10^5$ costs of the order of ten CPU-hours and a complete sweep to $N \le 20\,000$ of the order of fifty, and both were priced and not run. Testing $\beta_i \not\equiv 0 \pmod p$ prime by prime, rather than forming the integers β_i, is the algorithmic change that would move that frontier.

Remark 6.2 (a barrier for this shape). Can 107/114 be beaten by a sequence of the same shape? Within the range that was searched, no. Consider exponent rules $a_N = \lfloor \alpha N \rfloor$, $b_N = \lfloor \beta N \rfloor$, $c_N = \lfloor \gamma N \rfloor$ with α, β, γ on the grid $\frac{1}{570}\mathbb{Z}$, subject to $\alpha, \beta \le \frac{1}{2}$ and $\alpha + \beta \le \frac{5}{6}$ — both forced asymptotically by Theorems 1.2 and 1.5 of [1], the latter because $x^{a_N}(1-x)^{b_N}(1-x-x^2)^{c_N} \mid P$ implies $x^{a_N}(1-x)^{b_N} \mid P$ — and to $107/114 < \delta = \alpha + \beta + 2\gamma < 1$, the upper bound being needed for $\deg Q_N < N$. There are exactly 1 313 012 such rules, and all of them fail at some $N \le 400$ (indeed at some $N \le 205$), while the rule $(\alpha, \beta, \gamma) = (\frac{1}{2}, \frac{1}{3}, \frac{1}{19})$ of (3) survives $N \le 400$ in the same code. Adding a fourth factor $R(x)^{\lfloor N/d \rfloor}$ on top of Q_N, with R drawn from a library of 15 small integer polynomials and any $d \le 60$, produces nothing that survives $N \le 160$. Two caveats belong with this. First, the search is only as good as its horizon: at horizon $N \le 200$ the same code reports three surviving rules at $\delta = 268/285 = 0.9403509\ldots$, all of which die at $N = 201, 203, 205$ — just past that horizon. Second, per-N feasibility is not the obstruction at all. Writing $\Delta_3(N)$ for the largest $a + b + 2c$ such that $x^a(1-x)^b(1-x-x^2)^c$ is N-admissible, an exhaustive computation for $2 \le N \le 70$ gives $\Delta_3(N)/N$ rising to its maximum $68/70 = 0.971\ldots$ at $N = 70$, attained among others at $(a, b, c) = (14, 6, 24)$ — a far larger exponent of $1 - x - x^2$ than any uniform rule carries. What obstructs Sanna's question is uniformity in N, not what is possible at an individual N.

7. Verification

Every statement of §§2–5 — Proposition 2.2, Theorems 3.1, 4.1, 4.3 and 5.1, and Corollary 5.2 — has been machine-checked in Lean 4 [9] against Mathlib [10]. The development is six files. Coefficient vectors are plain integer lists with structural recursion, so that the same definitions both reduce inside the proof kernel and run under compiled evaluation; on top of them, one file proves Proposition 2.2 in both directions inside Mathlib's polynomial ring $\mathbb{Z}[X]$ and identifies the rational coefficient sum $\sum_{j<n} P_j/(j+1)$ with the interval integral $\int_0^1 P(x)\,dx$ over $\mathbb{R}$ (Remark 2.4), so that no step of the chain re-defines what $\int_0^1$ means and S_N is literally the set (1) of [1]. Theorem 5.1 and Corollary 5.2, and the five distinct refutations and the positive controls behind Theorems 4.1 and 4.3, are checked by the kernel with no appeal to compiled evaluation; so is the sub-window $N \le 200$ of Theorem 3.1, which the kernel decides directly in about two minutes. The two larger windows, $N \le 1000$ and $1001 \le N \le 2000$, are the only places where compiled evaluation is used: the statement for $N \le 1000$ carries one additional axiom recording that sweep, and the statement for $N \le 2000$ carries both. Every other theorem depends on at most the three standard axioms (propositional extensionality, quotient soundness and choice). Three sanity checks are included to guard against a vacuous formalisation: that the rational functional really is the integral on an example, that $\ell_{10} = 2520$, and that the swept list of N really has 1000 entries. There is no `sorry` anywhere in the development. The measurements of §6 are outside all of this, as stated there. Repository reference to be added at submission.

8. What remains

The property that (4) is conditional on is now a statement about small primes. By Proposition 2.2 and Corollary 5.2, for $N \ge 2$ the sequence (3) can only fail at N through a common prime factor of $\beta_0, \ldots, \beta_{m_N}$ — every such factor that is at most N being at most $\deg Q_N \approx 0.9386\,N$ — or through the degenerate case that all of those integers vanish. The natural next band is $N/2 < p \le \deg Q_N$, where the hypothesis $p^2 > N$ of Theorem 5.1 still holds and ℓ_N still contains p to the first power only, so that for each index i the sum defining $\beta_i \bmod p$ has at most one surviving term; what must be ruled out there is a statement about the coefficients of Q_N modulo p in a window of length $m_N + 1 \approx 0.0614\,N$. Below $\sqrt{N}$ the argument of Theorem 5.1 breaks down completely, since ℓ_N is then divisible by a higher power of p.

Two further remarks. First, the difficulty is genuine and not an artefact of the criterion: Remark 6.2 shows that the per-N maximum $\Delta_3(N)/N$ climbs to 0.971, so what the search of that remark runs into is uniformity in N, not what is possible at an individual N. Second, the hypothesis can be weakened for free. Since (2) is a lim inf, finitely many exceptions are harmless: if the property of (3)

holds for all sufficiently large N, define $Q'_N := Q_N$ there and $Q'_N := (x^3(1-x)^2)^{\lfloor N/6 \rfloor}$ at the finitely many exceptional N; by Theorem 1.4 of [1], which holds for *every* positive integer N, the sequence $\{Q'_N\}$ is then admissible, and $\delta_{Q'} = \liminf \deg(Q'_N)/N = 107/114$ because a $\liminf$ ignores finitely many terms. So the conditional value (4) stands already on the hypothesis "for all sufficiently large N", which is what Theorem 3.1 and Remark 6.1 make plausible and what §5 reduces to a question about primes below $0.9386\,N$.

References

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[2] D. Bazzanella, *A note on integer polynomials with small integrals*, Acta Math. Hungar. **141** (2013), no. 4, 320–328; DOI 10.1007/s10474-013-0326-8. The source of the set S_N and of Theorem 1.2 of [1].

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