MAXIMAL GREEN SEQUENCES FOR THE CHEKHOV–SHAPIRO $\mathcal{A}_n$-QUIVER AND FOR ITS DOUBLE, AND A DECK INVOLUTION IN THE DOUBLED DONALDSON–THOMAS TRANSFORMATION

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Abstract. Let $\mathcal{A}_n$ be the Chekhov–Shapiro quiver attached to the symplectic groupoid of unipotent upper-triangular $n \times n$ matrices; it has $n(n-1)/2$ vertices, organised into $\lfloor n/2 \rfloor$ main cycles. For even $n = 2m$, Choi has proved that the word $\omega_0 = (\tau_1\tau_2 \cdots \tau_m)^m$ in the cycle mutations τ_l is a cluster Donaldson–Thomas transformation of $\mathcal{A}_n$, and conjectures, on the evidence of $n = 6, 8, 10$, that it is in addition a *maximal green sequence* — that every one of its $n^2(n-2)/2$ mutations is performed at a green vertex. We verify this at $n = 12, 14, 16, 18, 20, 22, 24$, the largest case being 6336 mutations on 276 vertices, and we record what happens between the m copies of $w = \tau_1 \cdots \tau_m$, about which the source proves nothing: after r rounds the red columns of the C-matrix are exactly the vertices of the innermost r main cycles, every entry of that C-matrix lies in $\{-1, 0, 1\}$, and it has exactly $n(n-1)/2$ negative entries. The same source attaches to $\mathcal{A}_n$ a doubled quiver $\mathcal{A}_n^{\mathrm{dbl}}$ on $n(n-1)$ vertices, defined for every n, and asserts that the analogous word $s_0 = (s_1 \cdots s_m)^m$ acts on it as a cluster Donaldson–Thomas transformation, the detailed verification being omitted. We supply that verification for every n with $3 \le n \le 16$, and it does not come out as asserted: the C-matrix of s_0 is not $-I$ but $-P_g$, the negative of the permutation matrix of the deck involution g exchanging $U_{i,j}$ and $\widetilde{U}_{i,j}$. So under the definition the source itself gives, s_0 is a reddening sequence but not a cluster Donaldson–Thomas transformation, while s_0 followed by g is one, and the two Casimir identities displayed in support of the assertion are exchanged. We prove in addition what the source does not claim: s_0 is a maximal green sequence for $\mathcal{A}_n^{\mathrm{dbl}}$ at every n with $3 \le n \le 16$, in both parities — including the odd n, where $\mathcal{A}_n$ itself admits no reddening sequence at all — and one round of $s_1 \cdots s_m$ reddens one level of the double, from the inside out. Every statement below is machine-checked in Lean 4, apart from the clearly labelled remarks and a few elementary counting steps, each identified as such.

1. Introduction

The quiver. Chekhov and Shapiro attached to A. Bondal's symplectic groupoid of unipotent upper-triangular $n \times n$ matrices a quiver $\mathcal{A}_n$ [3], obtained from the Fock–Goncharov SL_n-quiver on a triangle by forgetting the vertices $Z_{0,i,n-i}$, amalgamating $Z_{i,0,n-i}$ with $Z_{n-i,i,0}$, and unfreezing every remaining vertex; the construction is carried out in [2, §3.1], from which we take the quiver. It has $n(n-1)/2$ vertices. The source's Remark 3.1.5 records that $\mathcal{A}_3$ is the Markov quiver, hence arises from a triangulation of the once-punctured torus, and that $\mathcal{A}_4$ arises from a triangulation of the twice-punctured torus.

The vertices are organised [2, §3.2] into $\lfloor n/2 \rfloor$ *main cycles*: cycle l has length $N_l = n/2$ when $l = n/2$, and $N_l = n$ otherwise, its vertices are $X_{l,j}$ with $j \in \mathbb{Z}/N_l$, and $X_{l,j}$ carries the index $n(l-1) + j$. Proposition 3.3.4 of [2] records the complete adjacency in these coordinates; we reproduce it in Section 2.

Associated with a chordless cycle of a quiver is a *cycle mutation* τ_J [2, eq. (3.4)], a word of $2|J|-2$ mutations and one relabelling, where J is the vertex set of the cycle. Write τ_l for the cycle mutation over the l-th main cycle. For $n = 2m$ even, [2, Theorem 7.3.1] proves that

$$\omega_0 \ = \ (\tau_1\tau_2 \cdots \tau_{m-1}\tau_m)^m$$

is a *cluster Donaldson–Thomas transformation* of $\mathcal{A}_n$: its C-matrix is exactly $-I$ and the underlying quiver is preserved [2, Definition 7.1.3]. In particular ω_0 is a reddening sequence. For odd n nothing of the kind exists: [2, Theorem 7.3.7] proves that $\mathcal{A}_n$ admits no reddening sequence at all when n is odd, and hence no maximal green sequence either, since by sign-coherence a seed with no green vertex has every c-vector nonpositive. The even case is the whole story.

The question. Being a reddening sequence says only where the word ends. A *maximal green sequence* [1, 8] says something at every step: each mutation is performed at a green vertex of the seed it acts on, and the seed reached at the end has no green vertex left. The Conclusion of [2] states, verbatim:

> *Maximal green sequences*: The cluster DT-transformation ω_0 is a maximal green sequence for $n = 6, 8, 10$. Based on this evidence, we conjecture that ω_0 is a maximal green sequence for all even n.

The phrase "maximal green sequence" occurs once in [2], in that sentence, and is not defined there. The case $n = 4$ appears separately, as the worked Example 7.1.7, where $\omega_0 = \tau_2\tau_1\tau_2\tau_1$ is exhibited as a *reddening* sequence; greenness is not asserted for it. So $n = 12$ is the first open case, and the evidence for the conjecture, as published, is three values.

Because $C^{\omega_0} = -I$ is already a theorem for every even n, and because the columns $-e_i$ of $-I$ are red, the maximality clause is free (Lemma 2.1). The conjecture therefore reduces, for each fixed n, to a single finite assertion:

> every one of the $n^2(n-2)/2$ mutations of ω_0 is at a green vertex.

That is a walk down an explicit word of integer matrix operations, and it is what we run.

The double. [2] attaches to $\mathcal{A}_n$ a second quiver: glue a flipped second copy of $\mathcal{A}_n$ to $\mathcal{A}_n$ along the amalgamated variables, and unfold each vertex $X_{i,j}$ into two vertices $U_{i,j}$ and $\widetilde{U}_{i,j}$. The result is the *doubled $\mathcal{A}_n$-quiver* $\mathcal{A}_n^{\mathrm{dbl}}$, "a 2 : 1 quiver covering of the standard $\mathcal{A}_n$-quiver" in the source's words; it has $n(n-1)$ vertices, and it is defined for every n, odd as well as even. Its main cycles, set out in Section 7, lie two over each main cycle of $\mathcal{A}_n$ — except that for even n the two over the innermost one merge into a single cycle — and carry cycle mutations f_i over the copy through the $U_{i,j}$ and $\widetilde{f}_i$ over the copy through the $\widetilde{U}_{i,j}$. With $s_i := f_i\widetilde{f}_i$ for $1 \le i < m$ and $s_m := f_m$ for even n, $f_m\widetilde{f_m}f_m$ for odd n, Remark 7.3.5 of [2] asserts:

> For arbitrary n, let $m = \lfloor n/2 \rfloor$. The cluster transformation $s_0 := (s_1 \cdots s_m)^m$ acts as the cluster DT-transformation on the doubled $\mathcal{A}_n$-quiver.

The remark sketches an argument through two Casimir identities and closes: "We omit the detailed verification as the proof is very analogous to Theorem 7.3.1." Supplying that verification is the second half of this paper, and the answer is not the asserted one: the C-matrix of s_0 is the negative of the permutation matrix of the deck involution (Theorem 8.1). The greenness question can be asked on the double as well, and there — unlike on $\mathcal{A}_n$ — it has an answer for odd n too.

Results. Throughout, $m = \lfloor n/2 \rfloor$. In Sections 3–6, where the quiver is $\mathcal{A}_n$, $n = 2m$ is even and $N = n(n-1)/2$ is the number of vertices; in Sections 7–10 the quiver is $\mathcal{A}_n^{\mathrm{dbl}}$, on $n(n-1)$ vertices, and n is arbitrary.

- *Seven new cases of the conjecture* (Theorems 4.1 and 4.2). ω_0 is a maximal green sequence for $\mathcal{A}_n$ at $n = 12, 14, 16, 18, 20, 22, 24$: all 720, 1176, 1792, 2592, 3600, 4840, 6336 of its mutations are at green vertices, the exchange matrix returns unchanged, and the final C-matrix is exactly $-I$. These are the seven even values immediately past the data reported in [2], and each is an exhaustive computation over the whole word.
- *The round invariant* (Theorem 5.1). Write C_r for the C-matrix after r copies of $w = \tau_1 \cdots \tau_m$, so $C_0 = I$ and $C_m = -I$. For $n = 4, 6, 8, 10, 12, 14, 16$ and every r with $1 \le r \le m$: the nonpositive columns of C_r are exactly the vertices of the innermost r main cycles, the nonnegative columns are exactly the rest, every entry of C_r lies in $\{-1, 0, 1\}$, and C_r has exactly N negative entries. One round reddens the innermost still-green cycle and touches the colour of nothing else.

- *A maximal green sequence on the double, in both parities* (Theorems 8.5 and 8.6). s_0 is a maximal green sequence for $\mathcal{A}_n^{\mathrm{dbl}}$ at every n with $3 \le n \le 16$: all $n(n-1)^2$ of its mutations are at green vertices, the exchange matrix returns, and the seed it ends at has no green vertex, so it cannot be extended. For odd n this stands beside [2, Theorem 7.3.7], by which $\mathcal{A}_n$ itself has no reddening sequence at all. Greenness is not claimed for the double anywhere in [2].
- *The C-matrix of s_0, and a correction* (Theorem 8.1, Corollary 8.2). For every n with $3 \le n \le 16$ the C-matrix of s_0 is $-P_g$, where g is the deck involution exchanging $U_{i,j}$ and $\widetilde{U}_{i,j}$. Every entry of $-P_g$ is 0 or -1, so s_0 is a reddening sequence; but $-P_g \neq -I$, so s_0 is not a cluster DT-transformation in the sense of [2, Definition 7.1.3], while s_0 followed by g is one, and the two identities displayed in Remark 7.3.5 are exchanged (Remark 8.3). What the source draws from that remark — that $\mathcal{A}_n^{\mathrm{dbl}}$ admits the theta basis, for which a reddening sequence suffices — is unaffected, and so is its Theorem 7.3.1 for $\mathcal{A}_n$ itself, since g covers the identity of $\mathcal{A}_n$.
- *The round invariant of the double* (Theorem 9.1, Corollary 9.2). For $5 \le n \le 10$ and every r with $1 \le r \le m$, the red columns of the C-matrix after r copies of $s_1 \cdots s_m$ are exactly the vertices of the innermost r *levels* of $\mathcal{A}_n^{\mathrm{dbl}}$ — a level being both copies of one doubled main cycle, or, for even n, the merged innermost cycle — the green columns are exactly the rest, every entry lies in $\{-1, 0, 1\}$, and there are exactly $n(n-1)$ negative entries.
- *Supporting statements.* Propositions 2.4 and 7.1 record the lengths of the two words; Proposition 3.1 reproduces the values $n = 6, 8, 10$ of [2] together with $n = 4$, computed first and used as the control on the model; Propositions 2.3 and 7.1 check the two models against the quivers [2] draws; Proposition 7.2 records that g is a fixed-point-free automorphism of $\mathcal{A}_n^{\mathrm{dbl}}$; and Propositions 6.1 and 10.1 collect eleven negative controls.

What is not claimed. We do not prove the conjecture. Nothing below is a statement about all n except the two word-length counts of Propositions 2.4 and 7.1 and the maximality Lemma 2.1; Theorems 4.1, 4.2 and 5.1 are exhaustive verifications at finitely many values of n, and we say at each of them exactly what was searched. We do not claim that the round invariant of Theorem 5.1 holds for every even n, only that it holds at the seven values checked and that it is the pattern a proof would have to exploit. The same applies to the double: Theorems 8.1, 8.5, 8.6 and 9.1 are exhaustive verifications at the listed n and nothing more, and in particular we do not prove that the C-matrix of s_0 is $-P_g$ for every n — only that it is at $3 \le n \le 16$, and, by an argument we give but do not formalise (Remark 8.8), that it is not $-I$ for any $n \ge 3$ under any admissible reading of the cycle mutations. We take no position on the convention question raised in Remark 8.4, and we do not treat the coisotropic reduction listed beside the maximal green sequence question in the Conclusion of [2], which is not a finite computation as stated.

Changes from version 1. Version 1 of this paper treated $\mathcal{A}_n$ alone, at $n \le 20$, and listed the doubled quiver as untouched. Every statement of version 1 is kept here, with its number and its wording: Lemma 2.1, Remarks 2.2, 3.2, 5.3 and 6.2, Propositions 2.3, 2.4, 3.1 and 6.1, Theorems 4.1 and 5.1, and Corollary 5.2. New are Theorem 4.2 ($n = 22$ and $n = 24$, at the end of Section 4) and Sections 7–10 on the doubled quiver, including the correction to [2, Remark 7.3.5]. Sections 11 and 12 are brought up to date; nothing stated in version 1 is withdrawn.

Provenance. Everything quoted above is from [2]. The live listing page was consulted on 3 September 2026 and confirms that v3, of 8 June 2026, is the current version; the v3 source was then retrieved and read, and all quotations in this paper are from it. We are aware of no work that computes a maximal green sequence for this quiver beyond the three values in the Conclusion of [2]: two bibliographic databases report no works citing [2]; its own reference list, read in full, contains [1] and [11] but nothing that computes such a sequence for $\mathcal{A}_n$; a full-text search of a large snapshot of arXiv sources (some 7.6×10^5 documents, with [2] itself present as a positive control) returns no document co-occurring a green-sequence term with "symplectic groupoid" apart from [2] itself and one paper in which the two occurrences are unrelated. We note also that no general-purpose tool settles these cases out of the box: SageMath's cluster-algebra module exposes green and red vertices but no maximal-green-sequence predicate, and Lean's mathematical library contains no cluster algebra and no quiver mutation, so the development described in Section 12 builds the objects from scratch.

For the doubled quiver the same searches were run again, and again return nothing. The object is introduced in [2] itself; the two databases still report no work citing [2], so no one has repaired or extended its Remark 7.3.5 in print. A record-wise search of the same arXiv snapshot for documents carrying a green- or reddening-sequence term together with any of "punctured sphere", "doubled quiver", "quiver covering", "2 : 1 cover", "symplectic groupoid" or "Chekhov" returns six documents, and the only one carrying any of the last three keys is [2]. Existence theorems for quivers of triangulated surfaces do not reach the double. The signed adjacency matrix of an ideal triangulation of a bordered surface with marked points has every entry equal to 0, ±1 or ±2 [7, Definition 4.1], and the matrix of a tagged triangulation is by definition that of the underlying ideal one [7, Definition 9.6]; every tagged arc of a tagged triangulation can be flipped [7, Theorem 7.9] and a flip is a mutation of that matrix [7, Lemma 9.7], so the quivers of tagged triangulations of one surface are closed under mutation, and none of them has an exchange-matrix entry of absolute value 3. In an unformalised random walk in the mutation class of $\mathcal{A}_n^{\mathrm{dbl}}$ such an entry does appear, at $n = 4, 5, 6$ (after 14, 33 and 34 mutations of that walk). Only at $n = 3$, where 600 random mutations produce no such entry and [2] identifies the double with the quiver of a triangulation of the 4-punctured sphere, is existence of some maximal green sequence already known, by [10] (see Remark 8.7).

One attribution should be recorded. The Acknowledgments of [2] contain the sentence "I am grateful to John Machacek and Eric Bucher for their assistance in identifying the reddening sequence for even n." Neither name appears in the bibliography of [2], and their own output, enumerated on 3 September 2026, treats other families: their papers on reddening and maximal green sequences concern Banff quivers and the class $\mathcal{P}$, component-preserving mutations, red sizes of quivers, orientable surfaces, the n-torus, and mutation of infinite quivers, and none of them mentions the symplectic groupoid or this quiver. The credit is in any case for the reddening sequence — that is, for the word ω_0 — and not for its greenness, which is what is at issue here.

2. Preliminaries

Seeds, C-matrices, green vertices. Let $\varepsilon = (\varepsilon_{ij})$ be the exchange matrix of a quiver on the vertex set $I = \{1, \dots, N\}$: ε_{ij} is the number of arrows $i \to j$ minus the number of arrows $j \to i$, so ε is a skew-symmetric integer matrix. A *framed seed* is a pair (ε, C) of $N \times N$ integer matrices; the initial framed seed of a quiver is (ε, I). Mutation in direction $k \in I$ sends (ε, C) to (ε', C') with

$$\varepsilon'_{ij} = \begin{cases} -\varepsilon_{ij} & \text{if } i = k \text{ or } j = k, \\ \varepsilon_{ij} + \varepsilon_{ik} \max(0, \varepsilon_{kj}) + \max(0, -\varepsilon_{ik})\, \varepsilon_{kj} & \text{otherwise,} \end{cases} \tag{1}$$

$$C'_{ij} = \begin{cases} -C_{ij} & \text{if } j = k, \\ C_{ij} + C_{ik} \max(0, \varepsilon_{kj}) + \max(0, -C_{ik})\, \varepsilon_{kj} & \text{otherwise.} \end{cases} \tag{2}$$

Equation (1) is Fomin–Zelevinsky matrix mutation [5]; (2) is the C-matrix rule displayed in [2, §7.1], which we use in exactly that form — the source writes ε'_{kj} there for what is the exchange matrix of the seed being mutated, our ε_{kj}. The C-matrix records the tropical part of a seed with principal coefficients [6]; its columns are the *c-vectors*.

A vertex k is *green* at a seed when column k of C is nonnegative, and *red* when that column is nonpositive. By sign-coherence — [2, Theorem 7.1.6], quoting [4] — every column of a C-matrix is nonzero with all nonzero entries of one sign, so green and red is a dichotomy. We do not use that theorem anywhere below: "green" is spelled out directly as "the c-vector is nonnegative", which is a statement one can check without knowing that the alternative is nonpositivity.

Following [1, 8], a *green sequence* for a quiver is a word of mutations, each performed at a green vertex of the seed it acts on, starting from the initial framed seed; it is *maximal* when the seed it reaches has no green vertex. The source [2] does not restate this definition — it defines only reddening sequence and cluster DT-transformation, [2, Definition 7.1.3], quoted in Section 1 — and uses the phrase once, in its Conclusion; we take the definition from its own reference [1].

A word may also contain a relabelling $\pi_{p,q}$, which exchanges the labels p and q. It acts on ε by transposing rows p, q and columns p, q simultaneously, and on C by transposing columns p, q only: the rows of C are indexed by the initial seed and are not relabelled. Relabellings are not mutations and are not subject to the greenness condition.

Lemma 2.1 (maximality is automatic)**.** *At a seed whose C-matrix is $-I$, no vertex is green. Consequently, a word that starts at the initial framed seed of a quiver, mutates only at green vertices, and ends at a seed with $C = -I$, is a maximal green sequence.*

Proof. Column k of $-I$ is $-e_k$, whose k-th entry is $-1 < 0$, so column k is not nonnegative and k is not green. The consequence is the definition. □

This is why the conjecture of [2] is, for each fixed even n, a purely finite assertion: the terminal clause is already a theorem there, and only the step-by-step greenness is open.

The quiver $\mathcal{A}_n$. Let $n = 2m$ be even and $N = n(n-1)/2$. The vertices of $\mathcal{A}_n$ are $X_{l,j}$ for $1 \le l \le m$ and $j \in \mathbb{Z}/N_l$, where $N_l = n$ for $l < m$ and $N_m = m$; the index of $X_{l,j}$ is $n(l-1)+j$, and $\sum_l N_l = (m-1)n + m = N$. Reading [2, Proposition 3.3.4] off its two displayed subquivers, the arrows are

(3) $\quad X_{l,j} \to X_{l,j+1}$ $\qquad$ for $1 \le l \le m,\ j \in \mathbb{Z}/N_l$,

(4) $\quad X_{l,j} \to X_{l+1,j}, \quad X_{l+1,j} \to X_{l,j-1}$ $\qquad$ for $1 \le l \le m-2,\ j \in \mathbb{Z}/n$,

(5) $\quad X_{m-1,j} \to X_{m,j}, \quad X_{m,j} \to X_{m-1,j-1}, \quad X_{m,j} \to X_{m-1,m+j-1}$ $\qquad$ for the innermost pair,

indices of cycle m being read modulo m. The third line is the doubling forced by the second case of [2, Proposition 3.3.4], whose displayed formula $\tau^*_{m-1}(X_{m,j}) = X_{m,j} Y_{m-1,j-1} Y_{m-1,m+j-1}$ names two neighbours in cycle $m-1$ rather than one. The exchange matrix $\varepsilon(n)$ of $\mathcal{A}_n$ is obtained from (3)–(5) by counting arrows in each direction and subtracting; note that at $l = m$ the two arrows of (3) produced by j and $j+1$ can cancel, which is what happens when $N_m = 2$. Proposition 3.3.4 of [2] is stated for $n \ge 5$, so at $n = 4$ the recipe (3)–(5) extrapolates it; that this extrapolation is the source's own $\mathcal{A}_4$ is Proposition 2.3 below.

Remark 2.2 (an index dictionary in the source)*.* The explicit dictionary between Fock–Goncharov Z-labels and the X-labels, printed in [2, §3.2], disagrees by one in its middle entries with the cycle order displayed in the same subsection: it writes $X_{l,l+1} := Z_{(n-l,l,0)}$ where the cycle order gives $X_{l,l+2}$, and $X_{l,l} := Z_{(n-l,l-1,1)}$ where the cycle order gives $X_{l,l+1}$; its first three and its last entry agree — so it does not agree with itself either, since continuing its own middle indexing would end at $X_{l,n-1}$ rather than at the $X_{l,n}$ it prints. The cycle order is the reading that makes the count come out to $N_l = n$, and it is the one confirmed by the twenty X-labels printed inside the source's own figure for $\mathcal{A}_6$, every one of which the model used here reproduces. Nothing in this paper depends on the resolution: the model never uses Z-labels, and the dictionary was read only to confirm that the X-indexing is the source's own.

The word ω_0. For a chordless cycle with vertex set $J = \{j_1, \dots, j_L\}$ in cyclic order, [2, eq. (3.4)] defines the cycle mutation

$$\tau_J \ := \ \mu_{j_1} \circ \cdots \circ \mu_{j_{L-1}} \circ \pi_{j_{L-1}, j_L} \circ \mu_{j_{L-1}} \circ \cdots \circ \mu_{j_1}$$

(we write $L = |J|$ where [2] writes N, which is reserved here for the number of vertices of $\mathcal{A}_n$), where μ_k is mutation in direction k and π_{j_{L-1},j_L} switches the labels j_{L-1} and j_L. Composition is right to left, so as a sequence of operations τ_J is: mutate at $j_1, \dots, j_{L-1}$; switch the labels j_{L-1} and j_L; mutate at $j_{L-1}, \dots, j_1$. That is $2L-2$ mutations and one relabelling, and on the original labels the mutated vertices are $j_1, \dots, j_{L-1}, j_L, j_{L-2}, \dots, j_1$.

Write τ_l for τ_J with J the l-th main cycle, $w = \tau_1\tau_2\cdots\tau_m$ and $\omega_0 = w^m$, all applied left to right in the order just described.

Proposition 2.3 (the model reproduces the source's own $\mathcal{A}_4$)**.** *At $n = 4$ the recipe* (3)–(5) *produces exactly the twelve arrows of the $\mathcal{A}_4$-quiver drawn in* [2, §7.1]*, namely* $1 \to 2,\ 1 \to 5,\ 2 \to 3,\ 2 \to 6,\ 3 \to 4,\ 3 \to 5,\ 4 \to 1,\ 4 \to 6,\ 5 \to 2,\ 5 \to 4,\ 6 \to 1,\ 6 \to 3$*, with no arrow between* 5 *and* 6 *and no multiple arrows: every entry of $\varepsilon(4)$ is -1,* 0 *or* 1*. Moreover the cycle mutations produced by* [2, eq. (3.4)] *are exactly the words $\tau_1 = \mu_1\mu_2\mu_4\mu_3\mu_2\mu_1$ and $\tau_2 = \mu_6\mu_5$ of* [2, Example 7.1.7].

Proof sketch. Direct evaluation of (3)–(5) and of τ_J at $n = 4$, compared entry by entry with the source's figure and with its Example 7.1.7. Vertices $5, 6$ are the innermost main cycle, of length $N_2 = n/2 = 2$; the two arrows $X_{2,1} \to X_{2,2}$ and $X_{2,2} \to X_{2,1}$ of (3) cancel, which is why $\varepsilon(4)_{5,6} = 0$. That cancellation is a prediction of the recipe and the drawn quiver has it. On the words: the outer cycle gives, on the original labels, $1, 2, 3, 4, 2, 1$, which is $\mu_1\mu_2\mu_4\mu_3\mu_2\mu_1$ read right to left, and the inner cycle gives $5, 6$, which is $\mu_6\mu_5$. □

This proposition is a check of the model against published data, not a new claim; it was run before any $n \geq 12$ was attempted.

Proposition 2.4 (the length of the word)**.** *For $n \in \{4, 6, 8, 10, 12, 14, 16\}$ the quiver $\mathcal{A}_n$ has $n(n-1)/2$ vertices and the word ω_0 consists of exactly $n^2(n-2)/2$ mutations, the pairs (vertices, mutations) being* $(6, 16)$, $(15, 72)$, $(28, 192)$, $(45, 400)$, $(66, 720)$, $(91, 1176)$, $(120, 1792)$. *More generally, for every even $n = 2m \geq 4$, counting gives $n(n-1)/2$ vertices and $n^2(n-2)/2$ mutations together with m^2 relabellings; at $n = 18$ and $n = 20$ that is* $(153, 2592)$ *and* $(190, 3600)$.

Proof. Counting. τ_l contributes $2N_l - 2$ mutations and one relabelling, the l-th main cycle having length $N_l = n$ for $l < m$ and $N_m = m$; hence w has $(m-1)(2n-2) + (2m-2)$ mutations and m relabellings, and $\omega_0 = w^m$ has $m\big((m-1)(2n-2) + 2m - 2\big) = 4m^2(m-1) = n^2(n-2)/2$ mutations and m^2 relabellings. The vertex count is $\sum_l N_l = (m-1)n + m = n(n-1)/2$. The tabulated values were in addition evaluated directly from the definition of ω_0, and that evaluation is what is machine-checked; the general count is the display above, which is not. □

3. The published values, reproduced

Proposition 3.1. *ω_0 is a maximal green sequence for $\mathcal{A}_n$ at $n = 4, 6, 8, 10$: each of its* 16*,* 72*,* 192*,* 400 *mutations is at a green vertex, the exchange matrix returns unchanged, and the final C-matrix is exactly $-I$.*

Proof sketch. For each n, run the word ω_0 from the initial framed seed $(\varepsilon(n), I)$ using (1) and (2), testing before each of the 16, 72, 192, 400 mutations that the column of C indexed by the mutated vertex is nonnegative, and comparing the final $2N \times N$ pair against $(\varepsilon(n), -I)$. This is one exhaustive pass per n; no search over words is involved, since the word is explicit. Maximality is Lemma 2.1. □

The values $n = 6, 8, 10$ are exactly those reported in the Conclusion of [2]. The case $n = 4$ is its Example 7.1.7, which exhibits the same word as a reddening sequence and says nothing about its greenness; we include it because it is the smallest case and the one whose quiver the source draws. Proposition 3.1 claims no priority; it is the control that fixes the model, the indexing, the reading of (1)–(2) and the reading of the word before anything new is computed. All four were reproduced in two independent unformalised implementations before the formal development was written.

Remark 3.2 (the reading of the word). [2, Theorem 3.3.1] states that τ_J does not depend on the order of $j_1, \dots, j_L$ *as a cluster transformation*; the intermediate seeds do depend on it, so a green-sequence statement could in principle depend on the reading. The source's own two displays of the word differ in the order they write — [2, Theorem 7.3.1] writes $(\tau_1\tau_2\cdots\tau_m)^m$ and [2, Example 7.1.7] writes $\tau_2\tau_1\tau_2\tau_1$ at $n = 4$, which as a right-to-left composition applies τ_1 first. Outside the formal development we evaluated four readings at $n = 4, 6, 8$ — cycle order forwards or reversed, τ_1 first or τ_m first — and all four are green at every step and end at $-I$. The reading committed to below is: cycle order forwards, τ_1 first.

4. New cases for $\mathcal{A}_n$: $n = 12, \dots, 24$

Theorem 4.1. *Let $n \in \{12, 14, 16, 18, 20\}$. Then $\omega_0 = (\tau_1 \cdots \tau_{n/2})^{n/2}$ is a maximal green sequence for the Chekhov–Shapiro quiver $\mathcal{A}_n$. Explicitly, starting from the initial framed seed $(\varepsilon(n), I)$:*

1. *every one of the* 720*,* 1176*,* 1792*,* 2592*,* 3600 *mutations of ω_0 is performed at a green vertex of the seed it acts on;*
2. *the exchange matrix reached at the end of ω_0 is $\varepsilon(n)$ again;*
3. *the C-matrix reached at the end of ω_0 is exactly $-I$, so by Lemma 2.1 no green vertex remains.*

Here $\mathcal{A}_n$ has 66*,* 91*,* 120*,* 153*,* 190 *vertices respectively.*

Proof sketch, and what was computed. Clause (1) is an exhaustive computation, and clauses (2) and (3) are the terminal comparison of one $2N \times N$ integer array against $(\varepsilon(n), -I)$. For each of the five values of n the computation is a single pass down the explicit word ω_0: maintain the pair (ε, C) as one $2N \times N$ integer array initialised at $(\varepsilon(n), I)$; for each of the $n^2(n-2)/2$ mutation steps, first test that all N entries of the column of C indexed by the mutated vertex are ≥ 0, then apply (1) and (2) to every one of the $2N^2$ entries; for each of the m^2 relabelling steps, permute as described in Section 2; and at the end compare the array with $(\varepsilon(n), -I)$. The number of entry updates is $n^2(n-2)/2 \cdot 2 \cdot (n(n-1)/2)^2$, that is 6.3×10^6 at $n = 12$ and 2.6×10^8 at $n = 20$. Nothing is searched: the word is given by [2, Theorem 7.3.1] and only its greenness is in question. Clauses (1)–(3) are verified simultaneously in that one pass, and the conclusion "maximal green sequence" follows from (1), (3) and Lemma 2.1. The arithmetic is exact integer arithmetic throughout; no floating point and no modular reduction enters. The same computation at $n = 6, 8, 10$ agrees with the values reported in [2] (Proposition 3.1), which is the control on the whole pipeline. □

The cost of that computation grows like n^7, and at $n = 22$ it passes what an interpreted evaluation will do in reasonable time. Compiling the evaluator to a shared library — the lever named, and not used, in version 1 of this paper — removes the obstacle, and the next two values follow.

Theorem 4.2. *Let $n \in \{22, 24\}$. Then $\omega_0 = (\tau_1 \cdots \tau_{n/2})^{n/2}$ is a maximal green sequence for $\mathcal{A}_n$. Explicitly, starting from the initial framed seed $(\varepsilon(n), I)$: every one of the* 4840*, respectively* 6336*, mutations of ω_0 is performed at a green vertex; the exchange matrix reached at the end is $\varepsilon(n)$ again; and the C-matrix reached at the end is exactly $-I$, so by Lemma 2.1 no green vertex remains. Here $\mathcal{A}_n$ has* 231*, respectively* 276*, vertices.*

Proof sketch, and what was computed. Word for word the computation of Theorem 4.1, at two further values of n: one pass down ω_0, testing the N entries of one column of C before each mutation and applying (1)–(2) to all $2N^2$ entries after it, and one terminal comparison against $(\varepsilon(n), -I)$. The number of entry updates is 5.2×10^8 at $n = 22$ and 9.7×10^8 at $n = 24$. The evaluator is the same one, compiled rather than interpreted; the quiver and the word it is run on are the objects of Section 2, and that they are is itself part of the machine-checked statement (Section 12). □

Theorems 4.1 and 4.2 add seven confirming cases to the three on which the conjecture of [2] was based, and move the smallest open even value from 12 to 26. They do not make the conjecture a theorem, and the evidence they add is of the same kind as the evidence already there. What follows is of a different kind.

5. How ω_0 reddens the quiver

The word ω_0 is w^m with $w = \tau_1 \cdots \tau_m$, and [2] says nothing about the $m-1$ intermediate seeds. They are extremely regular. Write C_r for the C-matrix reached after r copies of w, so $C_0 = I$ and, by [2, Theorem 7.3.1], $C_m = -I$. Call a vertex *inner of depth* r when it lies in one of the innermost r main cycles, that is in cycle $m, m-1, \ldots, m-r+1$.

Theorem 5.1 (the round invariant)**.** *Let* $n \in \{4, 6, 8, 10, 12, 14, 16\}$*,* $m = n/2$*, and* $1 \leq r \leq m$*. Then*

(1) *a column of* C_r *is nonpositive if and only if its vertex is inner of depth* r*;*
(2) *a column of* C_r *is nonnegative if and only if its vertex is not inner of depth* r*;*
(3) *every entry of* C_r *lies in* $\{-1, 0, 1\}$*;*
(4) C_r *has exactly* $N = n(n-1)/2$ *negative entries.*

In particular one round of w *reddens the innermost still-green main cycle and changes the colour of no other vertex.*

Proof sketch, and what was computed. For each of the seven values of n there is one pass down ω_0, pausing at each of the m round boundaries to evaluate (1)–(4) on the current C-matrix: for each of the N columns, the two Boolean equalities of (1) and (2) against membership in the innermost r cycles, and the bound of (3); then the total count of negative entries for (4). The cost of the pause is $O(N^2)$ per round, so the whole check costs about what the plain greenness sweep of Theorem 4.1 costs. In the formal development clauses (1)–(3) are extracted as statements about the entries of C_r at $n = 6$, $n = 12$ and $n = 16$; at the remaining four values, and for clause (4) at all seven, the content is that of the same verified Boolean predicate, which an unconditional bridge lemma identifies entry by entry with C_r. The largest case checked is $n = 16$, where C_r is 120×120 and there are eight rounds. □

Corollary 5.2. *At* $n = 10$ *the numbers of red columns of* $C_0, C_1, \ldots, C_5$ *are* 0, 5, 15, 25, 35, 45*: the innermost cycle has* $N_5 = m = 5$ *vertices and cycles* $1, \ldots, 4$ *have* $n = 10$ *each.*

Proof. $C_0 = I$ has no nonpositive column. For $r \geq 1$, Theorem 5.1(1) makes the red columns of C_r the vertices inner of depth r, so their number is $N_m + N_{m-1} + \cdots + N_{m-r+1}$; that sum is evaluated here at $n = 10$. □

Remark 5.3 (why this is the shape of a proof, and why it is not one). Theorem 5.1 is a verified pattern, not an induction. What it says is that the colouring of the seed at every round boundary is determined by r alone: cycles $m, \ldots, m-r+1$ red, cycles $m-r, \ldots, 1$ green, all entries in $\{-1, 0, 1\}$. If that were established for all even n — by induction on r — then the conjecture of [2] would reduce to a statement about the *inside* of a single cycle mutation: given that cycles $m, \ldots, m-r+1$ are red and the rest are green, are all $2N_l - 2$ mutations of τ_l green? That is a question about one chordless cycle and its two neighbouring cycles, i.e. about bounded data, and its answer would not depend on n. The source already has machinery of the right shape. [2, Lemma 7.3.2] states $w_0^*(\mathcal{C}_i) = \mathcal{C}_i^{-1}$ for the Casimirs $\mathcal{C}_i = \prod_j X_{i,j}$ attached to the main cycles, and its proof passes through the one-round relations, displaying $w^*(\mathcal{C}_i) = \mathcal{C}_{i-1}$ for $2 \leq i \leq m-1$ together with two different formulas at $i = 1$ and $i = m$: an index shift of one per round, the same one-cycle-per-round bookkeeping that Theorem 5.1 exhibits at the tropical level.

Theorem 5.1 is not, however, a formal consequence of that lemma. A Casimir is a product over a whole main cycle, so tropicalising these relations constrains only the *sum* of the c-vectors of the vertices of one cycle, and sign-coherence does not turn a nonnegative sum over a cycle into nonnegative summands. Nor does the aggregate see clauses (3) and (4). Those two clauses are in fact where the c-vectors are least like monomials: at $n = 6$ and $r = 1$, Theorem 5.1 gives three red columns — the innermost cycle has $N_m = m = 3$ vertices — and exactly $N = 15$ negative entries, which must therefore all lie in those three columns of height 15; at $n = 8$ and $r = 1$ it is 28 negative entries in four columns. We attempt no such induction here.

6. Controls

A computation that only ever returns "yes" proves nothing about the predicate it evaluates. The following five statements, all at $n = 6$, are the falsifications. For a word u write $R(u) =$ (all mutations green, $C = -I$ at the end, quiver returned) for the three Booleans that Theorem 4.1 asserts.

Proposition 6.1. *At $n = 6$:*

(1) (the order, not the multiset). *Transposing the first two mutations of ω_0 — a word with the same multiset of steps — gives $R = (false, false, false)$: some mutation is at a vertex that is not green, the final C-matrix is not $-I$, and the quiver does not return.*
(2) (too short). *Deleting the last mutation of ω_0 gives $R = (true, false, false)$: every mutation is still green, but the final C-matrix is not $-I$ and the quiver does not return.*
(3) (too long). *Running ω_0 twice gives $R = (false, false, true)$: the quiver returns, as it must, but some mutation is at a red vertex and the final C-matrix is not $-I$.*
(4) (not vacuous). *After the first mutation of ω_0, vertex 1 is not green: its c-vector has a negative entry.*
(5) (not degenerate). *$\varepsilon(6)_{1,2} = 1$ and $\varepsilon(6)_{2,1} = -1$; and $\varepsilon(4)_{5,6} = 0$, the two vertices of the innermost main cycle of $\mathcal{A}_4$ being non-adjacent.*

Proof sketch. Each item is one run of the same engine on the stated word, or a single matrix entry. Item (1) matters because [2, Theorem 3.3.1] makes τ_J order-independent as a cluster transformation, so the greenness statement is precisely what the order carries; item (3) is the maximality of ω_0 witnessed as a failure, since after one copy every c-vector is $-e_i$ and the first mutation of the second copy is therefore at a red vertex; item (4) shows that the greenness test of Theorem 4.1 is not satisfied by every vertex of every seed, so clause (1) of that theorem is not vacuously true. □

Item (2) shows that the terminal clause of Theorem 4.1 is not an artefact of the greenness sweep — greenness survives the deletion and the C-matrix condition does not. It does not by itself show that no *proper prefix* of ω_0 already reaches $-I$; that stronger statement was checked at $n = 6$ over all prefixes, outside the formal development, and the first prefix reaching $-I$ is the whole word.

Remark 6.2 (computations outside the formal development). The same three checks — all mutations green, $C = -I$, quiver returned — were run in an independent implementation in C, outside the formal development and with no machine-checked guarantee, for every even n from 4 to 30. All of them pass; the largest, $n = 30$, has 435 vertices and 12600 mutations and takes 13 seconds on one core. At the even $n \leq 12$ the c-vectors were in addition inspected after every step of the word, and none is sign-mixed — the empirical shadow of the sign-coherence theorem quoted as [2, Theorem 7.1.6]. These runs are recorded because they say where the mathematics stops as opposed to where the machine-checked development stops; they are not part of any statement above.

7. The doubled $\mathcal{A}_n$-quiver

The quiver. The doubled $\mathcal{A}_n$-quiver $\mathcal{A}_n^{\mathrm{dbl}}$ is [2, Definition 3.4.1]: glue a flipped second copy of $\mathcal{A}_n$ to $\mathcal{A}_n$ along the amalgamated variables and then unfold each vertex $X_{i,j}$ into $U_{i,j}$ and $\widetilde{U}_{i,j}$. The source defines it by a picture, drawn there three times, and records that it is a $2:1$ quiver covering of $\mathcal{A}_n$. It has $n(n-1)$ vertices, and unlike $\mathcal{A}_n$ it is used for every n, odd as well as even. Its main cycles are fixed by [2, Proposition 3.4.3], which we quote:

> Consider the doubled $\mathcal{A}_n$-quiver. On each cycle $U_{i,1} \to U_{i,2} \to \cdots \to U_{i,n} \to U_{i,1}$ and $\widetilde{U}_{i,1} \to \widetilde{U}_{i,2} \to \cdots \to \widetilde{U}_{i,n} \to \widetilde{U}_{i,1}$ for $i \neq n/2$, there are natural cycle mutations

f_i and $\widetilde{f}_i$ associated to the cycles (3.4). ... When n is even, the cycle $\widetilde{U}_{n/2,1} \to \widetilde{U}_{n/2,2} \to \cdots \to \widetilde{U}_{n/2,n/2} \to U_{n/2,1} \to U_{n/2,2} \to \cdots \to U_{n/2,n/2} \to \widetilde{U}_{n/2,1}$ induces a cycle mutation $f_{n/2} = \widetilde{f_{n/2}}$.

So for odd $n = 2m+1$ there are $2m$ main cycles, all of length n; for even $n = 2m$ there are $2(m-1)$ main cycles of length n together with one merged cycle of length n carrying both $U_{m,\cdot}$ and $\widetilde{U}_{m,\cdot}$. Either way the vertex count is $n(n-1)$.

Write o for the number of *doubled* main cycles, $o = m-1$ for even n and $o = m$ for odd n, and put $A^1_{l,j} = U_{l,j+1}$, $A^0_{l,j} = \widetilde{U}_{l,j+1}$ for $1 \le l \le o$ and $j \in \mathbb{Z}/n$; for even n write V_j, $j \in \mathbb{Z}/n$, for the merged innermost cycle, so that $V_j = U_{m,j+1}$ for $j < m$ and $V_j = \widetilde{U}_{m,j+1-m}$ for $j \ge m$. The index of $A^c_{l,j}$ is $2n(l-1) + cn + j + 1$ and that of V_j is $2no + j + 1$, so vertex 1 is $\widetilde{U}_{1,1}$ and vertex $n+1$ is $U_{1,1}$. In these coordinates the arrows of $\mathcal{A}^{\mathrm{dbl}}_n$ are

$$A^c_{l,j} \to A^c_{l,j+1}, \qquad V_j \to V_{j+1} \ \ (n \text{ even}) \qquad \text{for } 1 \le l \le o,\ c \in \{0,1\},\ j \in \mathbb{Z}/n, \tag{6}$$

$$A^c_{l,j} \to A^c_{l+1,j}, \qquad A^c_{l+1,j} \to A^c_{l,j-1} \qquad \text{for } 1 \le l \le o-1, \tag{7}$$

$$A^c_{o,j} \to V_{j+cm}, \qquad V_j \to A^c_{o,\,j-cm-1} \qquad \text{for even } n, \tag{8}$$

$$A^c_{o,j} \to A^{1-c}_{o,\,j+m} \qquad \text{for odd } n. \tag{9}$$

Lines (6) and (7) are the two copies of the corresponding lines (3)–(4) for $\mathcal{A}_n$, copy by copy; the two copies meet only at the innermost cycle, through (8) or (9). We write $\varepsilon^{\mathrm{dbl}}(n)$ for the resulting exchange matrix on $n(n-1)$ vertices.

The transcription (6)–(9) is the one place where a misreading would invalidate everything below, and it is the one place where [2] gives pictures rather than formulas. Proposition 7.1 is the check against those pictures.

The word s_0. Each f_i, $\widetilde{f}_i$ and (for even n) f_m is the cycle mutation of [2, eq. (3.4)] over the cycle named, spelled out exactly as in Section 2: for a cycle of length L it is $2L-2$ mutations and one relabelling. Following [2, Definition 3.4.4],

$$s_i := f_i \widetilde{f}_i \ \ (1 \le i < m), \qquad s_m := \begin{cases} f_m & n \text{ even}, \\ f_m \widetilde{f_m} f_m & n \text{ odd}, \end{cases} \qquad s_0 := (s_1 s_2 \cdots s_m)^m,$$

the last being the word of [2, Remark 7.3.5], all applied left to right as in Section 2: s_1 before s_2, and inside s_i the cycle mutation f_i over the U-copy before $\widetilde{f}_i$ over the $\widetilde{U}$-copy.

Proposition 7.1 (the model reproduces the source's three drawings)**.** *In the source's own vertex names, the arrows produced by* (6)–(9) *are exactly the arrows of the three doubled quivers drawn in* [2]*: all* 12 *arrows of the doubled* $\mathcal{A}_3$*-quiver, all* 50 *of the doubled* $\mathcal{A}_5$*-quiver and all* 78 *of the doubled* $\mathcal{A}_6$*-quiver — every arrow drawn, and no arrow that is not drawn. Moreover, for* $3 \le n \le 18$ *the quiver* $\mathcal{A}^{\mathrm{dbl}}_n$ *has* $n(n-1)$ *vertices and the word* s_0 *consists of exactly* $n(n-1)^2$ *mutations; for every* $n \ge 3$ *that count is what* (6)–(9) *and the definition of* s_i *give.*

Proof sketch. For the drawings: the three pictures were transcribed cell by cell into labelled pairs $U_{i,j}, \widetilde{U}_{i,j}$, and the list of arrows produced by (6)–(9), rewritten in the same names, was compared with them as an equality of lists — so neither a missing nor an extra arrow can pass. The transcription is the one human step, and for the figure that carries the doubled $\mathcal{A}_5$- and $\mathcal{A}_6$-quivers it is a reduction rather than a reading off. That figure draws the two quivers side by side as strips with their boundary vertices repeated: 24 nonempty cells in the $\mathcal{A}_5$ strip carrying 20 distinct labels, 35 in the $\mathcal{A}_6$ strip carrying 30, and 138 drawn lines between them. Three of the 138 are marked headless in the source and are drawing connectors rather than arrows — two end at the figure's empty spacer cells and the third joins the two strips — and the remaining 135, de-duplicated by the pair of labels at their ends, are exactly 50 arrows in the first strip and 78 in the second. The two label counts, 20 and 30, are $n(n-1)$ at $n = 5$ and at $n = 6$, which is the check that no vertex was lost in the reduction. For the count: a cycle mutation over a cycle of length n is $2n-2$ mutations, so s_i is $4n-4$ mutations for $i<m$, and s_m is $2n-2$ for even n and $6n-6$ for odd n; one round $w = s_1 \cdots s_m$ is therefore $(m-1)(4n-4)+(2n-2) = 2(n-1)^2$ mutations for even $n=2m$ and $(m-1)(4n-4)+(6n-6) = 2n(n-1)$ for odd $n=2m+1$, and m rounds give $n(n-1)^2$ in both cases. The tabulated values at $3 \le n \le 18$ were in addition evaluated directly from the definition of s_0, and that evaluation is what is machine-checked; the general count is the display just given, which is not. $\square$

The deck involution. [2] defines g to be the involution that exchanges the variables $\widetilde{U}_{i,j}$ and $U_{i,j}$. On vertices it is

$$g\big(A^c_{l,j}\big) = A^{1-c}_{l,j}, \qquad g(V_j) = V_{j+m} \quad (n \text{ even}),$$

the second line being the same exchange, since V_j is a U for $j<m$ and a $\widetilde{U}$ for $j \ge m$. Let P_g be its permutation matrix, $(P_g)_{ij} = 1$ if $g(i) = j$ and 0 otherwise. As a step in a word, g is the product of the $n(n-1)/2$ label switches $\pi_{v,g(v)}$ over the orbits of g; being a relabelling, it is not subject to the greenness condition and acts on a C-matrix by permuting columns.

Proposition 7.2 (g is an automorphism). *For $3 \le n \le 12$: g is an involution of the vertex set of $\mathcal{A}_n^{\mathrm{dbl}}$ without fixed points, and it is an automorphism of $\mathcal{A}_n^{\mathrm{dbl}}$, that is $\varepsilon^{\mathrm{dbl}}(n)_{g(i)g(j)} = \varepsilon^{\mathrm{dbl}}(n)_{ij}$ for all vertices i,j.*

Proof sketch. Exhaustive evaluation at each of the ten values of n: $g(g(v)) = v$, $g(v) \ne v$ and $g(v)$ a vertex, for each of the $n(n-1)$ vertices v; and the displayed equality for each of the $n^2(n-1)^2$ pairs. $\square$

That g is an automorphism is what makes "a Donaldson–Thomas transformation up to g" a statement about the quiver rather than an accident of labelling.

8. The word s_0 on the doubled quiver

Theorem 8.1 (the C-matrix of s_0). *Let $3 \le n \le 16$, and run $s_0 = (s_1 \cdots s_m)^m$ from the initial framed seed $(\varepsilon^{\mathrm{dbl}}(n), I)$ of $\mathcal{A}_n^{\mathrm{dbl}}$. Then*

1. *the exchange matrix reached at the end is $\varepsilon^{\mathrm{dbl}}(n)$ again;*
2. *the C-matrix reached at the end is exactly $-P_g$;*
3. *it is* not *$-I$: its $(1,1)$ entry is 0;*
4. *the word $s_0 g$ — s_0 followed by the relabelling g — ends at the seed $(\varepsilon^{\mathrm{dbl}}(n), -I)$, and every one of the mutations in it is at a green vertex.*

Proof sketch, and what was computed. One pass down s_0 and then down the $n(n-1)/2$ label switches of g, from the initial framed seed, exactly as in the proof of Theorem 4.1: the pair (ε, C) is carried as one $2n(n-1) \times n(n-1)$ integer array; for each of the $n(n-1)^2$ mutations the column of C indexed by the mutated vertex is tested for nonnegativity and then (1)–(2) are applied to every entry; the array is compared with $(\varepsilon^{\mathrm{dbl}}(n), -P_g)$ at the end of s_0, its C-block entry $(1,1)$ is read off, and the array is compared with $(\varepsilon^{\mathrm{dbl}}(n), -I)$ after the label switches. Clauses (1)–(4) are verified simultaneously in that one pass. The number of entry updates is $2n^3(n-1)^4$, that is 4.2×10^8 at $n = 16$; the arithmetic is exact integer arithmetic throughout. Nothing is searched: the word is the one [2] writes down. $\square$

Every entry of P_g is 0 or 1, so clause (2) says in particular that every entry of the final C-matrix is nonpositive. Reading clauses (1)–(3) against [2, Definition 7.1.3] — "*a sequence of cluster mutations is called a reddening sequence if all entries of the associated C-matrix are nonpositive. Furthermore, if the C-matrix is exactly $-I$ and the underlying quiver is preserved, the sequence is called a cluster Donaldson–Thomas (DT) transformation*" — gives:

Corollary 8.2. *For* $3 \le n \le 16$*, the word* s_0 *of* [2, Remark 7.3.5] *is a reddening sequence for* $\mathcal{A}_n^{\mathrm{dbl}}$ *but is not a cluster DT-transformation in the sense of* [2, Definition 7.1.3]*, while* s_0 *followed by the deck involution* g *is one.*

Remark 8.3 (the two displayed identities). [2, Remark 7.3.5] supports its assertion with the two identities $s_0^*(\mathcal{C}_i) = \mathcal{C}_i^{-1}$ and $s_0^*(\widetilde{\mathcal{C}}_i) = (\widetilde{\mathcal{C}}_i)^{-1}$, where $\mathcal{C}_i = \prod_j U_{i,j}$ and $\widetilde{\mathcal{C}}_i = \prod_j \widetilde{U}_{i,j}$ are the Casimirs of the two copies of the i-th main cycle. Theorem 8.1 exchanges them. The source reads the C-matrix of a cluster transformation τ off the tropicalisation by $[\tau^*(X_i')] = \prod_j X_j^{c_{ji}^{\tau}}$ [2, Theorem 7.1.2], so $C^{s_0} = -P_g$ with $(P_g)_{ij} = 1$ exactly when $g(i) = j$ says $[s_0^*(X_i')] = X_{g(i)}^{-1}$ at every vertex i, g being an involution. Tropicalisation is a homomorphism of semifields [2, Definition 7.1.1], so multiplying that over one main cycle gives $[s_0^*(\mathcal{C}_i)] = \widetilde{\mathcal{C}}_i^{-1}$ and $[s_0^*(\widetilde{\mathcal{C}}_i)] = \mathcal{C}_i^{-1}$, the two displayed identities with the tildes crossed over. Reading Theorem 8.1 in the source's variables is that translation, and it is not part of what is machine-checked; what is verified is the C-matrix.

Remark 8.4 (how wide the correction is). Part of the literature calls a reddening sequence a Donaldson–Thomas transformation when its C-matrix is $-P$ for a permutation matrix P inducing an automorphism of the quiver — equivalently, when its tropical action is $x \mapsto -x$ followed by an automorphism. Under that convention, and given Proposition 7.2, s_0 *is* a Donaldson–Thomas transformation and only the two displayed identities of Remark 8.3 are wrong. Corollary 8.2 states the failure under the convention [2] itself writes down, which asks for the C-matrix to be exactly $-I$. In either reading the identities have to be exchanged, and in either reading the conclusion the source draws — that $\mathcal{A}_n^{\mathrm{dbl}}$ admits the theta basis, which needs only a reddening sequence — is untouched, as is [2, Theorem 7.3.1] for $\mathcal{A}_n$, since g covers the identity of $\mathcal{A}_n$.

We turn to what [2] does not claim.

Theorem 8.5 (a maximal green sequence on the double, odd n)**.** *Let* $n \in \{3, 5, 7, 9, 11, 13, 15\}$ *and* $m = (n-1)/2$*, so that* $s_m = f_m \widetilde{f_m} f_m$*. Then* $s_0 = (s_1 \cdots s_m)^m$ *is a maximal green sequence for* $\mathcal{A}_n^{\mathrm{dbl}}$*: every one of its* 12*,* 80*,* 252*,* 576*,* 1100*,* 1872*,* 2940 *mutations is performed at a green vertex, the exchange matrix returns unchanged, and no vertex of the seed reached at the end is green, so the sequence cannot be extended. Here* $\mathcal{A}_n^{\mathrm{dbl}}$ *has* 6*,* 20*,* 42*,* 72*,* 110*,* 156*,* 210 *vertices.*

Theorem 8.6 (a maximal green sequence on the double, even n)**.** *Let* $n \in \{4, 6, 8, 10, 12, 14, 16\}$ *and* $m = n/2$*, so that* $s_m = f_m$ *is the cycle mutation over the merged innermost cycle of* [2, Proposition 3.4.3]*. Then* $s_0 = (s_1 \cdots s_m)^m$ *is a maximal green sequence for* $\mathcal{A}_n^{\mathrm{dbl}}$*: every one of its* 36*,* 150*,* 392*,* 810*,* 1452*,* 2366*,* 3600 *mutations is performed at a green vertex, the exchange matrix returns unchanged, and no vertex of the seed reached at the end is green. Here* $\mathcal{A}_n^{\mathrm{dbl}}$ *has* 12*,* 30*,* 56*,* 90*,* 132*,* 182*,* 240 *vertices.*

Proof sketch, and what was computed. Both theorems are read off the same single pass as Theorem 8.1, one pass per n: the greenness of the mutated vertex is tested before each of the $n(n-1)^2$ mutations, and the exchange matrix is compared at the end. Maximality is *not* automatic from Lemma 2.1 here, because the C-matrix reached is $-P_g$ and not $-I$; instead all $n(n-1)$ columns of that C-matrix are tested, and none is nonnegative. The largest case is $n = 16$: 3600 mutations on 240 vertices. □

For odd n the contrast with the standard quiver is sharp: by [2, Theorem 7.3.7] the quiver $\mathcal{A}_n$ admits no reddening sequence at all when n is odd, hence no maximal green sequence either, while by Theorem 8.5 its double admits one — and the word that works is the one [2] writes down for the double.

Remark 8.7 (the case $n = 3$). At $n = 3$ the existence of *some* maximal green sequence for $\mathcal{A}_n^{\mathrm{dbl}}$ is already known: [2] identifies the doubled $\mathcal{A}_3$-quiver with the quiver of a triangulation of the 4-punctured sphere, hence of finite mutation type, and [10] proves that a mutation finite cluster quiver has a maximal green sequence unless it arises from a once-punctured closed marked surface or is one of the two quivers in the mutation class of X_7. Consistently with that, $n = 3$ is the only value at which our random walks in the mutation class of $\mathcal{A}_n^{\mathrm{dbl}}$ produced no entry of absolute value 3 (Section 1, Provenance). What Theorem 8.5 adds at $n = 3$ is that this particular word is one, and Theorem 8.1 that its C-matrix is $-P_g$.

Remark 8.8 (the reading of the cycle mutations). As in Remark 3.2, τ_J — here f_i and $\widetilde{f_i}$ — is order-independent as a cluster transformation [2, Theorem 3.3.1] but not as a word, and [2, Proposition 3.4.3] does not prescribe the order in which the vertices of a cycle are listed. Could another admissible reading give $-I$? Outside the formal development we searched exhaustively over an independent cyclic offset for each cycle, both traversal directions, and both round orders $s_1 \cdots s_m$ and $s_m \cdots s_1$: 36 variants at $n = 3$, 256 at $n = 4$, 2500 at $n = 5$ and 31104 at $n = 6$, and none of them gives $C = -I$. There is also an argument, which we give but do not formalise, covering every $n \geq 3$ and every such reading. By [2, Theorem 3.3.1] a cycle mutation does not depend on the order of its index set as a cluster transformation, and by [2, Theorem 7.1.2] the C-matrix is the tropicalisation of that transformation read in the final labelling; so two readings of s_0 give C-matrices differing by a column permutation π which is a product of transpositions (j_{L-1}, j_L), each supported inside a single cycle-mutation index set J. For $n \geq 3$ the double has at least one doubled main cycle, and for such a cycle J is one copy, which g maps to the other copy, disjoint from J; so π preserves J setwise while g does not, $g \circ \pi \neq \mathrm{id}$, and $C = -P_{g\circ\pi} \neq -I$ for every reading. We record also that $(s_1 \cdots s_m)^{2m}$ has $C = +I$ at $n = 3, 4, 5, 6$, so on the tropical part s_0 acts as an involution.

9. How s_0 reddens the double

The double has the same round structure as $\mathcal{A}_n$, one level at a time. Group the $n(n-1)$ vertices into m *levels*: level l, for $1 \leq l \leq o$, is the pair of copies $A^0_{l,\cdot}, A^1_{l,\cdot}$ of the l-th doubled main cycle, of size $2n$; and for even n level m is the merged innermost cycle $V_\cdot$, of size n. So there are m levels in both parities, of total size $2nm = n(n-1)$ for odd n and $2n(m-1) + n = n(n-1)$ for even n. Call a vertex *inner of depth* r when it lies in one of the levels $m, m-1, \ldots, m-r+1$.

Theorem 9.1 (the round invariant of s_0)**.** *Let $n \in \{5, 6, 7, 8, 9, 10\}$, $m = \lfloor n/2 \rfloor$ and $1 \leq r \leq m$, and write C_r for the C-matrix reached after r copies of $w = s_1 \cdots s_m$ on $\mathcal{A}_n^{\mathrm{dbl}}$. Then*

1. *a column of C_r is nonpositive if and only if its vertex is inner of depth r;*
2. *a column of C_r is nonnegative if and only if its vertex is not inner of depth r;*
3. *every entry of C_r lies in $\{-1, 0, 1\}$;*
4. *C_r has exactly $n(n-1)$ negative entries.*

In particular one round of w reddens the innermost still-green level and changes the colour of no other vertex.

Proof sketch, and what was computed. As for Theorem 5.1: for each of the six values of n there is one pass down s_0, pausing at each of the m round boundaries to evaluate (1)–(4) on the current C-matrix — the two Boolean equalities and the entry bound for each of the $n(n-1)$ columns, then the total count of negative entries. In the formal development clauses (1)–(3) are extracted as statements about the entries of C_r at $n = 7$, $n = 8$ and $n = 10$; at the remaining three values, and for clause (4) at all six, the content is that of the same verified Boolean predicate, which an unconditional bridge lemma identifies entry by entry with C_r. The largest case checked is $n = 10$, where C_r is 90×90 and there are five rounds. □

Corollary 9.2. *The innermost r levels of $\mathcal{A}_n^{\mathrm{dbl}}$ hold $2nr$ vertices when n is odd and $n(2r-1)$ when n is even, so by Theorem 9.1(1) the numbers of red columns of $C_0, C_1, \ldots, C_m$ are $0, 14, 28, 42$ at $n = 7$ and $0, 8, 24, 40, 56$ at $n = 8$.*

Proof. $C_0 = I$ has no nonpositive column. For $r \geq 1$, Theorem 9.1(1) makes the red columns of C_r the vertices inner of depth r, and these are r levels of $2n$ vertices each when n is odd, and the merged level of n vertices together with $r-1$ levels of $2n$ vertices when n is even; the sums are evaluated here at $n = 7$ and $n = 8$. □

Unlike the standard quiver, where the innermost main cycle has half the length of the others, every level of the double has the same size $2n$ except the merged one, which is the innermost level when n is even and does not exist when n is odd. That is the one place where the two round invariants differ, and it is why the red-column counts of Corollary 9.2 run $2n, 4n, \dots, n(n-1)$ for odd n but $n, 3n, \dots, n(n-1)$ for even n, the innermost level contributing n and not $2n$.

10. Controls for the doubled word

Proposition 10.1. *With $R(u)$ as in Section 6, at $n = 6$ and at $n = 7$ — one value of each parity:*

(1) (the parities are not interchangeable). *Building s_m by the other parity's recipe — the triple $f_m\widetilde{f_m}f_m$ at even n, where $f_m = \widetilde{f_m}$ is the merged cycle, and the bare f_m at odd n — gives $R = (\mathit{false}, \mathit{false}, \mathit{true})$: the quiver still returns, but some mutation is at a red vertex and the final C-matrix is not $-I$.*
(2) (too short). *Deleting the last step of s_0g — that step being one of the label switches of g, since g comes last — gives $R = (\mathit{true}, \mathit{false}, \mathit{false})$.*
(3) (too long). *Running s_0g twice gives $R = (\mathit{false}, \mathit{false}, \mathit{true})$.*
(4) (not vacuous). *Mutating the initial framed seed $(\varepsilon^{\mathrm{dbl}}(n), I)$ at vertex 1 leaves vertex 1 not green: its c-vector is then $-e_1$.*
(5) (not degenerate). *At $n = 6$: $\varepsilon^{\mathrm{dbl}}(6)_{1,2} = 1$, $\varepsilon^{\mathrm{dbl}}(6)_{2,1} = -1$ and $\varepsilon^{\mathrm{dbl}}(6)_{1,7} = 0$, the vertices 1 and 7 being the two copies $\widetilde{U}_{1,1}$ and $U_{1,1}$ of one vertex of $\mathcal{A}_6$, so the covering really is $2:1$; and every entry of $\varepsilon^{\mathrm{dbl}}(6)$ is -1, 0 or 1.*
(6) (the assertion of Remark 7.3.5, as a control). *The raw word s_0 gives $R = (\mathit{true}, \mathit{false}, \mathit{true})$.*

Proof sketch. Each item is one run of the same engine on the stated word, or a single matrix entry. Item (1) is the sharp one: it shows that the split of s_m by the parity of n in [2] is load-bearing, and that the recipe for one parity does not work for the other. Item (2) shows that the terminal clause is not an artefact of the greenness sweep, greenness surviving the deletion and the C-matrix condition not; item (3) is the maximality of s_0g witnessed as a failure; item (4) shows that the greenness test of Theorems 8.5 and 8.6 is not satisfied by every vertex of every seed, so clause (1) of those theorems is not vacuously true; item (6) exhibits Theorem 8.1(3) in the report form, and is the literal reading of [2, Remark 7.3.5]. □

As in Section 6, item (2) does not by itself show that no *proper prefix* of s_0g already reaches $(\varepsilon^{\mathrm{dbl}}(n), -I)$; that stronger statement was checked over all prefixes at $n = 6$ and at $n = 7$, outside the formal development, and the first prefix reaching it is the whole word.

Remark 10.2 (computations outside the formal development). The same checks — all mutations green, the quiver returned, $C = -P_g$, $C = -I$ after the relabelling g — were run in independent implementations in C and in Python, outside the formal development and with no machine-checked guarantee, for every n from 3 to 20, in both parities; all of them pass, and the whole sweep takes 12 seconds on one core. No c-vector is sign-mixed at any step of any of these runs. Independently of all of this, the quiver $\mathcal{A}_n$ itself was rebuilt from scratch from the Fock–Goncharov triangle quiver — vertices $Z_{(i,j,k)}$ with $i + j + k = n$ minus the three corners, arrows $Z_v \to Z_w$ for $w - v \in \{(0, 1, -1), (1, -1, 0), (-1, 0, 1)\}$, counted with weight $\frac{1}{2}$ when both ends lie on one side of the triangle and 1 otherwise, then forget, amalgamate, unfreeze and relabel by the cycle order — and reproduces the exchange matrix used here exactly at $n = 4, 6, 8, 10, 12$. That reconstruction also covers odd n, which $\mathcal{A}_n$ is not used for above: for $n = 2m + 1$ the innermost main cycle carries the chords $X_{m,j} \to X_{m,j+m}$ in addition to the cycle itself, and it is those chords that fix the arrows (9) joining the two copies of the innermost cycle of the double. These runs are recorded because they say where the mathematics stops as opposed to where the machine-checked development stops; they are not part of any statement above.

11. What remains open

The conjecture itself. A proof for all even n would settle the question rather than extend its evidence, and Remark 5.3 says where we would start: prove the round invariant of Theorem 5.1 by induction on r, then analyse a single τ_l under the colouring that invariant supplies. The second half is a bounded question, about one chordless cycle and its neighbours. Theorem 9.1 says that the double has the same structure, one level per round, so a proof of either would be a template for the other.

Why the deck involution appears. Theorem 8.1 says that s_0 acts on the tropical part of the seed of $\mathcal{A}_n^{\mathrm{dbl}}$ as $x \mapsto -x$ followed by g, at $3 \le n \le 16$. A proof of that identity for all n is the natural next statement, and it would give [2, Theorem 7.3.1] downstairs as a corollary, g covering the identity of $\mathcal{A}_n$; it would also say why the two Casimir identities of [2, Remark 7.3.5] come out exchanged. We do not have it.

Larger n, machine-checked. The cost of the verification behind Theorem 4.1 grows like n^7 — word length $n^2(n-2)/2$ times $2(n(n-1)/2)^2$ entry updates per mutation — and the measured constant over $n = 12, \ldots, 20$ is 3.5 to 4.8 microseconds per entry update under the interpreter, which puts $n = 22$ and $n = 24$ at about 35 and 66 minutes in a single file. That is why version 1 of this paper stopped at $n = 20$. Compiling the evaluator to a shared library instead — possible because the computational layer imports nothing — was measured here at 25 to 44 times faster, and $n = 22$ and $n = 24$ together then take 206 seconds (Theorem 4.2). The same lever puts the standard word at $n = 26, 28, 30$ at about 5.5, 9 and 15 minutes and the doubled word at $n = 17, \ldots, 20$ at about 16 minutes in total; none of these was run, because the statement at one more n decides nothing that the values already verified do not, and all of them are verified outside the formal development already (Remarks 6.2 and 10.2). The two round invariants cost about what the greenness sweeps cost, which is why they stop at $n = 16$ and $n = 10$.

Odd n, for $\mathcal{A}_n$ itself. Nothing to compute: [2, Theorem 7.3.7] rules out even a reddening sequence. A machine-checked refutation would have to formalise that argument, which passes through the Markov quiver as a full subquiver and is not a finite computation. On the double, by contrast, odd n is where Theorem 8.5 lives, and the natural question is what the $2:1$ covering does to that obstruction.

The sibling item. The Conclusion of [2] lists, beside the maximal green sequence question, the completion of the *coisotropic* reduction for general n — its Conjecture 6.2.4 — which it records as known for $n = 5, 6, 8, 10$. (Version 1 of [2] calls this the Hamiltonian reduction; v3, the version cited throughout, does not use that phrase.) That is not a finite decidable computation as stated and we do not treat it.

12. Formal verification

Every lemma, proposition, theorem and corollary above is formalised and machine-checked in Lean 4 [9]; the exceptions, labelled as such in the text, are Remarks 2.2, 3.2, 5.3, 6.2, 8.3, 8.4, 8.7, 8.8 and 10.2, the two sentences after Propositions 6.1 and 10.1 about proper prefixes, the general-n counts of Propositions 2.4 and 7.1, whose proofs are the elementary displays given there and whose tabulated values are what the development checks, and two derivations made in the text from machine-checked statements: Corollary 8.2, which reads clauses (1)–(4) of Theorem 8.1 against [2, Definition 7.1.3] using only that the entries of P_g are 0 and 1 by construction, and Corollary 9.2, which counts the levels named in Theorem 9.1(1). (The corresponding counts for $\mathcal{A}_n$, Corollary 5.2, are machine-checked.) The development is sixteen files and uses only the matrix definitions of Lean's mathematical library, which contains no cluster algebra and no quiver mutation, so seeds, the mutation rules (1)–(2), the cycle mutation, ω_0, s_0, greenness and the notion of maximal green sequence are all defined from scratch. Statements are about $N \times N$ matrices over $\mathbb{Z}$ in the sense of that library; because iterating (1)–(2) on a function-valued matrix builds an unusable closure, the evaluation runs on a materialised array of $2N$ rows, and four unconditional bridge lemmas plus an induction over the word identify the two layers, so that no statement is about arrays. Three of the files evaluate their Booleans through a shared library compiled from an import-free copy of that array engine rather than through the interpreter, which is what makes Theorems 4.2, 8.1, 8.5 and 8.6 affordable; the duplication is not taken on trust, and the identity of the duplicated quiver and word with the originals is itself machine-checked at $n = 22$ and $n = 24$. There is no incomplete proof. The axiom set of every headline declaration was printed. The reasoning layer — the bridge lemmas for both engines, the maximality Lemma 2.1, and the extraction of Theorems 5.1, 8.1, 8.5, 8.6 and 9.1 from their verified predicates — depends on propositional extensionality, quotient soundness and choice alone. Each finite verification additionally depends on compiled evaluation, through exactly one such axiom per named computation: one for each of the five values of Theorem 4.1, one for each of the four of Proposition 3.1, seven for the seven values of Theorem 5.1, one for Corollary 5.2, one for each of the separately evaluated statements in Propositions 2.3, 2.4 and 6.1, and, for the doubled material, one for each of the fourteen values $3 \leq n \leq 16$ of Theorem 8.1 and Theorems 8.5–8.6, one for each of the three drawings and one for the table of sizes in Proposition 7.1, one for Proposition 7.2, one for each of the six values of Theorem 9.1, fourteen for Proposition 10.1 — two for each of its items (1)–(4) and (6), one at $n = 6$ and one at $n = 7$, and four for the four separately evaluated statements of its item (5), which is at $n = 6$ alone — and three for each of the two values of Theorem 4.2 — the run itself and the two comparisons that identify the duplicated definitions with the originals. Every such axiom stands between a single explicit Boolean — the outcome of one pass down one explicit word — and the corresponding statement about matrices; everything that turns the Boolean into that statement is proved. As a further guard against a mistranscribed quiver, the mutation rule was implemented in the source's max form (1) in the formal development and in the equivalent form $\varepsilon_{ij} + \frac{1}{2}\big(|\varepsilon_{ik}|\varepsilon_{kj} + \varepsilon_{ik}|\varepsilon_{kj}|\big)$ in the independent unformalised implementations, and all of them agree at every n computed — on the standard quiver, where the values of [2] are the control, as well as on the double. The files of version 1 check in about 36 CPU-minutes, dominated by $n = 20$, and the eight files added here in about eight more, of which $n = 22, 24$ together and the double at $n = 13, \ldots, 16$ account for six. The repository is private; repository reference to be added at submission.

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