METRIC BETWEENNESS ON FIVE POINTS: A COMPLETE CENSUS GRADED BY COLLINEAR TRIPLES

KOREA SUPERINTELLIGENCE LABS

ABSTRACT. A ternary relation B on a set X is a *metric betweenness relation* if there is a metric d on X with $B(x,z,y) \iff d(x,z)+d(z,y)=d(x,y)$. Vlasic (arXiv:2607.27222) enumerated these relations on finite sets by linear programming, obtaining $b_n = 1, 1, 4, 74, 8628, \dots$ for $n \le 6$, and quoted five axioms (B0)–(B4) of Mendris and Zlatoš that every metric betweenness relation satisfies. It is known that the axioms are not sufficient, that the smallest failures occur at six points (Mendris–Zlatoš), and that at five points they are exactly right (Chvátal, who counted 122 isomorphism types). We refine the five-point count into a complete census graded by the number k of *collinear* triples, that is, of triples one of whose points lies between the other two: for $k = 0, 1, \dots, 10$ the numbers of relations on a five-point set satisfying (B0)–(B4) are

$$1,\ 30,\ 285,\ 1100,\ 2010,\ 2142,\ 1640,\ 840,\ 390,\ 130,\ 60,$$

summing to 8628. The same eleven numbers count the metric betweenness relations on five points, because we also prove — by exhibiting, for each of the 8628 structures, an explicit integer metric on $\{0,1,2,3,4\}$ with all ten distances at most 7 — that at five points "satisfies (B0)–(B4)" and "is a metric betweenness relation" are the same condition. Every statement of this kind below is machine-checked in Lean 4, including the proof that the encoding used by the search is a bijection onto the class it is meant to count, and controls showing that neither (B3) nor (B4) is redundant. Separately, and outside the formal development, we record six-point data: $b_6 = 7221418$, which agrees with the corrected entry A395237 of the OEIS and not with the value 7238428 printed in the source; the number 8896888 of six-point relations satisfying (B0)–(B4), of which 1675470 are not metrizable; and the six-point distribution by k, whose zero at $k = 19$ is the case $n = 6$ of a theorem of Szabó.

1. Introduction

The objects. Let (X, d) be a metric space. Following Menger [11], the *metric segment* joining $x, y \in X$ is

$$[x,y] \ = \ \{\, z \in X : d(x,z) + d(z,y) = d(x,y)\,\},$$

and X carries the induced *betweenness relation* B_d, defined by $B_d(x,z,y) \iff z \in [x,y]$; the middle argument is the point that lies between. We quote the framing of the question from the source of this paper, [16]:

> Given a ternary relation B on a set X, it is natural to inquire whether B can be realized as the betweenness relation induced by some metric on X. Such ternary relations are referred to as *metrizable betweenness relations*.

We use *metrizable* and *metric betweenness relation* interchangeably. Write b_n for the number of metrizable betweenness relations on a labelled n-set and $\bar{b}_n$ for the number up to relabelling; c_n and $\bar{c}_n$ do the same for the induced convexities $\mathcal{C}_d = \{C \subseteq X : [x,y] \subseteq C \text{ for all } x,y \in C\}$. The source enumerated all four by "interpreting the problem as a linear feasibility problem and employing linear programming" [16], and registered them in the OEIS as A395237, A397184, A395250 and A395485 [13]. Their present values are

n	1	2	3	4	5	6
b_n	1	1	4	74	8628	7221418
$\bar{b}_n$	1	1	2	9	122	11610
c_n	1	1	4	74	8508	6460153
$\bar{c}_n$	1	1	2	9	119	10287

The same appendix of [16] quotes five axioms from Mendris and Zlatoš [12], verbatim:

> In a more general setting, one defines a *betweenness relation* on X to be a ternary relation B satisfying expected geometric axioms such as
> (B0) $B(x,y,x) \implies x = y$.
> (B1) $B(x,x,y)$
> (B2) $B(x,y,z) \implies B(z,y,x)$
> (B3) $B(x,y,z) \wedge B(x,z,u) \implies B(x,y,u)$
> (B4) $B(x,y,z) \wedge B(x,z,u) \implies B(y,z,u)$
> for all $x,y,z,u \in X$.

A ternary relation satisfying (B0)–(B4) will be called a *betweenness structure*. Every metric betweenness relation is one (Proposition 2.1 below); the converse is the question this paper is about.

What is known. Mendris and Zlatoš [12] record on p. 874 that "as is well known, the satisfaction of the axioms (B0) – (B4) is not sufficient for the metrizability of (A,T)", and locate the failure exactly (p. 880): "all the betweenness spaces (i.e., all ternary structures satisfying the axioms (B0) – (B4)) with at most five elements are metrizable. The smallest nonmetrizable betweenness spaces have six elements". Chvátal [1], working with the same class in a different notation — his (M1)–(M4) are (B0)–(B4) restricted to distinct points, and define the same class — states the five-point half quantitatively:

> No additional properties are required to characterize metric betweenness on five points: precisely 122 isomorphism types of ternary relations on a five-point set have properties (M1), (M2), (M3), (M4) and each of these 122 ternary relations is a metric betweenness.

He also exhibits two six-point relations B_1, B_2 on the complete quadrilateral that satisfy his axioms and are not metric betweennesses. Neither source publishes the enumeration behind the five-point statement. Szabó [15] studies the same class — his (P1)–(P4) are equivalent to (B0)–(B4) — graded by *co-size*, the number of triples with no middle, and proves structural results about the smallest attainable positive co-size.

The open questions the source itself records are about the values: it gives $\bar{b}_6 \geq 6778$ from Monte Carlo sampling and writes that "the exact value of $\bar{b}_6$ is currently unknown to us", and the same for $\bar{c}_6$. Both have since been determined and appear in the OEIS entries above.

Results. Our subject is the five-point case, made quantitative. Call a 3-element subset $\{x,y,z\}$ of X *collinear* for B when one of its three points lies between the other two, and let $k(B)$ be the number of collinear triples; for a five-point set k ranges over $0,\dots,10$.

(1) Exactly 8628 ternary relations on a five-point set satisfy (B0)–(B4) (Theorem 4.1). The number itself is A395237(5), whose count up to relabelling is A397184(5) = 122, Chvátal's count of isomorphism types; what is new is a machine-checked proof, including a proof that the encoding used by the exhaustive search is a bijection onto the class of relations it is meant to count, and controls showing that neither (B3) nor (B4) follows from the other four axioms (Proposition 4.3).
(2) Graded by k, the counts are $1, 30, 285, 1100, 2010, 2142, 1640, 840, 390, 130, 60$ (Theorem 4.2). This refinement is, to the best of our knowledge, new; see the discussion after the theorem.
(3) At five points, a ternary relation is a metric betweenness relation if and only if it satisfies (B0)–(B4) (Theorem 5.1). The substantive direction is proved by exhibiting one integer metric on $\{0,1,2,3,4\}$ for each of the 8628 structures, all ten distances lying in $\{1,\dots,7\}$,

and checking each. The mathematics is Mendris–Zlatoš's and Chvátal's; the proof and the explicit list of realizing metrics are new.

(4) Consequently the same eleven numbers count the betweenness relations of metrics on a five-point set graded by k (Corollary 5.2), and $b_5 = 8628$ becomes a count of metrics rather than of solutions of a feasibility problem.

Section 6 records the six-point obstruction in the same machine-checked form: Chvátal's B_1 satisfies (B0)–(B4), is realized by no metric, and each of its six five-point restrictions is realized by an explicit integer metric. Section 7 explains the polyhedral reformulation on which the six-point computations run, and reports those computations; nothing in that part of Section 7 is machine-checked, and it is labelled accordingly.

What rests on exhaustive computation. Theorems 4.1, 4.2 and 5.1 and Corollary 5.2 are proved by exhaustive search, inside the proof assistant and with no appeal to an external program: a single pass over the $4^{10} = 1\,048\,576$ codes described in Section 3 settles the first two, and a second pass over 8628 integer distance tables settles the third. What makes these searches into proofs about ternary relations rather than about bit patterns is the encoding lemma (Lemma 3.1) and the equivalence of the boolean test with the axioms (Lemma 3.2), both proved without computation. The six-point numbers of Section 7 rest on external programs and are reported as computational evidence only.

2. Definitions and elementary facts

Throughout, X is a set and $d : X \times X \to \mathbb{R}$ is a *metric*: $d(x,x) = 0$, $d(x,y) > 0$ for $x \neq y$, $d(x,y) = d(y,x)$, and $d(x,z) \leq d(x,y) + d(y,z)$. We write

$$B_d(x,z,y) \;:\Longleftrightarrow\; d(x,z) + d(z,y) = d(x,y), \qquad \delta_d(x,z,y) \;=\; d(x,z) + d(z,y) - d(x,y)$$

for the betweenness relation of d and for its *defect*, so that $\delta_d \geq 0$ and $B_d(x,z,y)$ says $\delta_d(x,z,y) = 0$. A ternary relation B on X is *metrizable* if $B = B_d$ for some metric d on X.

The degenerate triples are forced for every metric: $B_d(x,x,y)$ and $B_d(x,y,y)$ always hold, and $B_d(x,z,x)$ holds only for $z = x$. So all the information in B_d sits on the $\binom{|X|}{3}$ triples of distinct points, and by Proposition 2.1(2) each such triple carries at most one middle: a triple is in one of four states.

Proposition 2.1 (Mendris–Zlatoš; the axioms as quoted in [16])**.** *Let d be a metric on a set X. Then*

(1) *B_d satisfies (B0), (B1), (B2), (B3) and (B4);*

(2) (uniqueness of the middle) *if $B_d(x,z,y)$ and $B_d(z,x,y)$ then $x = z$.*

Proof sketch. Each item is a two-line consequence of the triangle inequality: adding the defining equalities of the two hypotheses of (B3) or (B4) and cancelling gives the conclusion after one application of $\delta_d \geq 0$ in each direction. (B3) is Menger's. Formally, each of the seven statements is discharged by linear arithmetic over the ordered field of reals from the four metric axioms; no finiteness is used, and the carrier is an arbitrary type. □

The following lemma is elementary but is the engine of every construction and every computation in this paper. It is the shadow, for genuine metrics, of a correspondence published for pseudometrics by Cizma and Linial [3], who remark that much of what they say there "is not specific to the metric cone and holds just as well for any polyhedral cone".

Lemma 2.2. *Let d_1, d_2 be metrics on X and $c > 0$. Then*

$$B_{d_1+d_2} = B_{d_1} \cap B_{d_2}, \qquad B_{c\cdot d_1} = B_{d_1}.$$

Consequently the metrizable relations on X are closed under intersection of any nonempty finite family, and are invariant under positive scaling.

Proof. $\delta_{d_1+d_2} = \delta_{d_1} + \delta_{d_2}$, and both summands are nonnegative, so the sum vanishes exactly when both do; scaling multiplies the defect by $c > 0$. The closure statements follow because a sum of metrics is a metric. □

Lemma 2.2 is what makes an intersection-based search for realizing metrics correct: to realize a prescribed relation it suffices to intersect the betweenness relations of metrics that all contain it, until the intersection is exactly it. This is how the 8628 metrics of Theorem 5.1 were produced.

Lemma 2.3 (witnesses and heredity)**.** (1) *If $p : X \to \mathbb{R}$ is injective then $d(x, y) = |p(x) - p(y)|$ is a metric, and $p(x) \le p(z) \le p(y)$ implies $B_d(x, z, y)$.*

(2) *The discrete metric realizes the trivial relation: $B_{\mathrm{disc}}(x, z, y)$ holds exactly when $z = x$ or $z = y$.*
(3) *Metrizability is hereditary: if B is metrizable on $\{0, \dots, n-1\}$ and e is an injection of $\{0, \dots, m-1\}$ into $\{0, \dots, n-1\}$, then $(x, z, y) \mapsto B(ex, ez, ey)$ is metrizable.*

Together with Lemma 2.2 these say that the metrizable relations on a finite set are closed under intersection, with the trivial relation as their least element and the linear orders among their maximal elements. That the system is not closed under the dual operation, and that the statements above are not vacuous, is the content of the controls:

Proposition 2.4 (controls)**.** (1) *If $x \ne y$ in X, the all-true relation on X is not metrizable.*

(2) *If $X \ne \emptyset$, the all-false relation on X is not metrizable.*
(3) *There are metrizable relations B_1, B_2 on a three-point set whose union is not metrizable.*

Proof. (1) violates (B0) and (2) violates (B1). For (3), take the three points on a line in the two orders that put the second and then the first point in the middle; both relations are metrizable by Lemma 2.3(1), and their union has two middles on one triple, contradicting Proposition 2.1(2). □

Item (3) is what stops Lemma 2.2 from reading as a triviality about lattices of relations: the metrizable relations are a meet-subsemilattice and not a sublattice.

3. Codes for five-point structures

Fix $X = \{0, 1, 2, 3, 4\}$ and let $T_0 < T_1 < \cdots < T_9$ be its ten 3-subsets in increasing lexicographic order. By Proposition 2.1(2) — proved for betweenness structures from (B0), (B2) and (B4) alone — each triple of a betweenness structure is in one of four states: no middle, or one of its three points as the middle. A *code* is therefore an element w of $\{0, 1, 2, 3\}^{10}$, written as an integer $w < 4^{10} = 1\,048\,576$ with two-bit fields, whose t-th field is 0 if T_t has no middle and $s \in \{1, 2, 3\}$ if the s-th point of T_t (in increasing order) is the middle.

Lemma 3.1 (the encoding is a bijection)**.** *Let* $\mathrm{rel}(w)$ *be the ternary relation on X whose degenerate triples are as forced and whose triple T_t is decoded from the t-th field of w as above. Then*

(1) $\mathrm{rel}(w)$ *satisfies (B0), (B1) and (B2) for every $w < 4^{10}$;*
(2) *every relation C satisfying (B0)–(B4) equals* $\mathrm{rel}(w)$ *for some $w < 4^{10}$;*
(3) rel *is injective on* $\{w : w < 4^{10}\}$.

Proof sketch. (1) is immediate from the decoding table. For (2), read the slots off C: uniqueness of the middle makes the assignment well defined, and three two-line exclusion lemmas from (B0), (B2), (B4) show that all three "middle" statements about a triple are recovered from its slot. (3) is packing and unpacking of two-bit fields. Every step is a proof, not a computation: the finite content is confined to the decoding table, which is checked exhaustively on the 125 argument triples of X. □

The point of Lemma 3.1 is that the counts below are counts of *ternary relations* — of elements of the 2^{125}-element set of all ternary relations on a five-point set — and not counts of the codes that a program happens to enumerate.

Lemma 3.2 (the search tests the axioms)**.** *Let* $\mathrm{axOK}(w)$ *be the boolean program that checks (B3) and (B4) on all* 5^4 *quadruples of* $\mathrm{rel}(w)$ *by four nested loops. Then* $\mathrm{axOK}(w) = \mathrm{true}$ *if and only if* $\mathrm{rel}(w)$ *satisfies (B0)–(B4).*

Proof sketch. Four induction lemmas, one per loop, convert the bounded recursion into the corresponding bounded quantifier; (B0), (B1), (B2) come from Lemma 3.1(1). The proof uses no evaluation of the program. □

4. The census at five points

Theorem 4.1. *Exactly* 8628 *ternary relations on a five-point set satisfy the axioms (B0)–(B4).*

Theorem 4.2 (the graded census)**.** *For* $k = 0, 1, \dots, 10$*, the number of ternary relations on a five-point set that satisfy (B0)–(B4) and have exactly* k *collinear triples is*

k	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
count	*1*	*30*	*285*	*1100*	*2010*	*2142*	*1640*	*840*	*390*	*130*	*60*

The eleven numbers sum to 8628*.*

Proof of Theorems 4.1 and 4.2. By Lemmas 3.1 and 3.2 the set of relations in question is the image under the injection rel of the set of codes $w < 4^{10}$ with $\mathrm{axOK}(w) = \mathrm{true}$, and the number of collinear triples is computed from the relation itself. A single exhaustive pass over all 1 048 576 codes evaluates axOK and the collinearity count and returns the twelve numbers 8628 and $1, 30, 285, 1100, 2010, 2142, 1640, 840, 390, 130, 60$; transporting along the injection turns them into cardinalities of sets of relations. The search is exhaustive: nothing is pruned and no symmetry is quotiented out. □

The same twelve numbers are reproduced outside the proof assistant by two further decision procedures, sharing no code with the development above: one tests each code's five four-point restrictions against a precomputed table of the 74 valid four-point codes, the other checks the (B3)/(B4) quantifier directly on all 5^4 quadruples. The total 8628 agrees with A395237(5) [13], whose count up to relabelling, A397184(5) = 122, is Chvátal's [1]. The grading appears to be new. We searched for it in the OEIS, where the row, each of its suffixes, and its reversal return no matching sequence, and where the revision histories of the four sequences above carry no graded data; and in a full-text corpus of arXiv mathematics, where the distinctive fragments `2010, 2142` and `1640, 840, 390` do not occur at all, and no paper containing any of the phrases “betweenness structure”, “metrizable betweenness”, “metric betweenness”, “pseudometric betweenness” or “collinear triple(s)” contains the number 2142 as a standalone token. Szabó [15] defines exactly this grading — “We will denote the set of betweenness structures of order n and co-size m by $B(n, m)$” — but every occurrence of the notation in his paper is an emptiness or membership statement rather than a cardinality, the string $|B(n, m)|$ never appears, and the paper contains no tables at all. Chvátal states 122 flat, with no breakdown; the nine four-point isomorphism types he lists explicitly do yield, by inspection, the unlabelled four-point row $1, 1, 3, 2, 2$, but he does not state it as a distribution. The published program behind A395237 accumulates precisely this distribution into a counter and then returns only its total.

The extreme entries are the expected ones. The unique structure with $k = 0$ is the trivial relation, realized by the discrete metric (Lemma 2.3(2)); the $60 = 5!/2$ structures with $k = 10$ are the betweenness relations of the linear orders on five points, a distinct relation for each order up to reversal. (At four points the analogous top entry is 15, not $4!/2 = 12$; see Remark 7.4.)

The census predicate is not over-determined:

Proposition 4.3 (independence of (B3) and (B4))**.** *On a five-point set there is a relation satisfying (B0), (B1), (B2) and (B3) but not (B4), and a relation satisfying (B0), (B1), (B2) and (B4) but not (B3).*

Proof. For the first, take the relation whose only collinear triples are $\{0,1,2\}$ and $\{0,1,3\}$ with middle 1, and $\{0,2,3\}$ with middle 2: it satisfies (B3), while $B(0,1,2)$ and $B(0,2,3)$ hold but $B(1,2,3)$ fails, contradicting (B4). For the second, take the relation whose only collinear triples are $\{0,1,2\}$ with middle 1 and $\{0,1,3\}, \{0,2,3\}$ with middle 0: it satisfies (B4), while $B(2,1,0)$ and $B(2,0,3)$ hold but $B(2,1,3)$ fails, contradicting (B3). (B0), (B1), (B2) hold for both by Lemma 3.1(1). Both verifications are finite checks over 5^4 quadruples. □

Dropping an axiom therefore changes the answer, and by a lot: an external run of the cross-check program of Section 4 with (B4) disabled counts 30353 relations instead of 8628, and with (B3) disabled 20178. Those two totals are computations outside the formal development and are quoted here only to show that 8628 answers a question with content.

5. Metrizability at five points

Theorem 5.1. *A ternary relation on a five-point set is the betweenness relation of a metric if and only if it satisfies (B0)–(B4). In particular the number of metric betweenness relations on a five-point set is* $b_5 = 8628$.

Proof sketch. The forward implication is Proposition 2.1. For the converse, a list of 8628 integer distance tables on $\{0,1,2,3,4\}$ is supplied, one for each code accepted by axOK and in the same order, each given by its ten pairwise distances, all of which lie in $\{1,\dots,7\}$ and hence are single decimal digits. For every row of the list it is checked that the ten entries are positive, that all 5^3 instances of the triangle inequality hold, and that the induced betweenness relation is *exactly* the relation coded by the corresponding code — both inclusions, on all 5^3 argument triples. Casting the integer table to a real-valued metric, and Lemma 3.1, turn this into the statement about relations. The check is exhaustive over the 8628 rows. □

The construction of the list uses Lemma 2.2 and nothing else. Fix a target structure C. Among integer metrics on five points with distances in $\{1,\dots,5\}$, retain those whose betweenness relation contains C; greedily add to a running sum the retained metric that shrinks the intersection of the betweennesses fastest, and stop when the intersection equals C. By Lemma 2.2 the sum of the chosen metrics realizes exactly the intersection of their betweenness relations, so the output is a metric with betweenness relation exactly C. In the event, every one of the 8628 structures was reached with at most two summands, and the largest distance occurring is 7. This is a search, not a proof: its output is a certificate, and the certificate is what is verified.

Corollary 5.2 (the graded census, for metrics)**.** *For* $k = 0,1,\dots,10$*, the number of ternary relations on a five-point set that are the betweenness relation of a metric and have exactly* k *collinear triples is*

$$1,\ 30,\ 285,\ 1100,\ 2010,\ 2142,\ 1640,\ 840,\ 390,\ 130,\ 60.$$

Proof. Theorem 4.2 and Theorem 5.1. □

Two consistency checks were built into the search. The code all of whose triples have their median point as middle satisfies the axioms and has $k = 10$; the metric the greedy construction returned for it is $d(i,j) = |i-j|$ on $\{0,1,2,3,4\}$, which is the expected answer at the top of the range. And the code declaring the smallest point of every triple to be its middle does *not* satisfy the axioms, so the sweep is not counting every code.

6. Six points: the first obstruction

At six points the two classes part company. We record the separating example in the same verified form; the mathematics is Chvátal's.

Theorem 6.1 (Chvátal's B_1)**.** *Let W be the ternary relation on $\{0,1,2,3,4,5\}$ whose only collinear triples are*

$$B(2,0,5), \quad B(3,0,4), \quad B(2,1,4), \quad B(3,1,5).$$

Then W satisfies (B0)–(B4), W is not the betweenness relation of any metric, and each of the six five-point restrictions of W is *the betweenness relation of a metric.*

Proof sketch. That W satisfies the axioms is a finite check over the 6^4 quadruples. Suppose d realizes W. The four displayed triples give

$$d(2,0)+d(0,5)=d(2,5), \qquad d(3,0)+d(0,4)=d(3,4),$$
$$d(2,1)+d(1,4)=d(2,4), \qquad d(3,1)+d(1,5)=d(3,5),$$

while the failures of $B(2,0,4)$, $B(3,0,5)$, $B(2,1,5)$, $B(3,1,4)$ make the four corresponding triangle inequalities strict. Adding the first two strict inequalities and substituting the first two equalities gives $d(2,4)+d(3,5)<d(2,5)+d(3,4)$; adding the last two and substituting the last two equalities gives the reverse strict inequality. For the last clause, six explicit integer metrics on five points, with entries in $\{2,3,4,6\}$, are exhibited and checked, one per omitted point. $\square$

Under the relabelling $a_1=0$, $a_2=1$, $b_1=2$, $b_2=3$, $c_1=5$, $c_2=4$, the four triples of W are exactly the hypothesis of Chvátal's condition (M5), and W is his relation B_1, of which he writes "B_1 is not a metric betweenness, since it lacks property (M5)"; the argument above is his certificate. That six is the least size at which the axioms fail is Mendris and Zlatoš's statement quoted in Section 1; at five points it is Theorem 5.1 above. Mendris and Zlatoš give a different six-point example, with six collinear triples rather than four.

Proposition 6.2. *Deleting any one of the four collinear triples of W yields a metrizable relation on six points, and each of the four differs from W.*

Proof. Four explicit integer metrics on six points, with entries in $\{12,13,16,17,21,24,25\}$, one per deleted triple, each checked to be a metric and to induce exactly the stated relation. $\square$

Proposition 6.2 is not a surprise — realizing a prescribed three-triple relation on six points is a small feasibility problem, and the census reported in Remark 7.3 shows that *every* six-point betweenness structure with at most three collinear triples is metrizable — but it makes the minimality of W precise in the second direction: the obstruction needs all six points and all four triples.

7. The metric cone, and the six-point numbers

Apart from Proposition 7.1, nothing in this section is machine-checked.

The reformulation. For distinct x,z,y, the statement $B_d(x,z,y)$ says that one of the $3\binom{n}{3}$ triangle inequalities defining the *metric cone*

$$\mathrm{MET}_n \;=\; \{\, \rho \in \mathbb{R}^{\binom{n}{2}} : \rho_{ik}+\rho_{kj}-\rho_{ij} \ge 0 \text{ for all triples} \,\}$$

is tight at $\rho=d$. Two metrics induce the same betweenness relation exactly when they make the same inequalities tight, that is, when they lie in the relative interior of the same face of MET_n; the faces arising from genuine metrics are those whose relative interior misses every coordinate hyperplane. Hence b_n is the number of faces of MET_n of full support. (For background on this cone see [5, 6]; nothing is lost by working over $\mathbb{R}$, since Šimko [14] records that the betweenness spaces metrizable by a metric with values in an arbitrary ordered field and those metrizable by a real-valued metric "contain the same finite structures".) The pseudometric form of the correspondence is proved by Cizma and Linial [3]; the passage between the two levels is the following splitting.

Proposition 7.1. *Let ρ be a pseudometric on X and let B_ρ be its betweenness relation. Then*

(1) $\rho(x,y)=0$ *if and only if $B_\rho(x,y,x)$, so the zero-set of ρ is determined by B_ρ;*
(2) *that zero-set is an equivalence relation on X;*

(3) *B_ρ is invariant under replacing any argument by a point at distance zero from it; and*
(4) *the restriction of ρ to any set of representatives of the zero-classes is a metric, whose betweenness relation pulls back to B_ρ along the map sending each point to its representative.*

Proof. Four applications of the triangle inequality, on an arbitrary carrier. $\square$

Counting the faces of MET_n by the partition supplied by Proposition 7.1 gives $\sum_k S(n,k)\,b_k = f(\mathrm{MET}_n)$, where S is a Stirling number of the second kind and $f(\mathrm{MET}_n)$ is the total number of faces. That identity, and the face counts entering it, are computations; they are recorded in Remark 7.5.

Computations outside the formal development. *Nothing in the remainder of this section is machine-checked.* The numbers below come from two exact integer programs: a double-description computation of the extreme rays of MET_n followed by a breadth-first enumeration of the intersection-closure of their sets of tight inequalities — correct by Lemma 2.2, since the betweenness relation of a sum of metrics is the intersection of their betweenness relations — and, independently, a depth-first search over the states of the $\binom{n}{3}$ triples that checks (B3) and (B4) on each quadruple as soon as its four triples are assigned. The two share no code and agree wherever they overlap. We report them as computational evidence.

Remark 7.2 (the value of b_6). The face enumeration gives $b_6 = 7221418$, in agreement with the present value of A395237 and not with the value 7238428 printed in Table 1 of [16]. The two are consistent in the following sense: the OEIS entry records that $a(6)$ was supplied on 24 May 2026 and corrected on 4 July 2026 by the author of [16], four days after the e-print was posted; as of 3 September 2026 the e-print has only the version of 30 June 2026 and so carries the earlier number. The computation reported here was gated on reproducing $b_3 = 4$, $b_4 = 74$, $b_5 = 8628$, $\bar{b}_3, \bar{b}_4, \bar{b}_5 = 2, 9, 122$, $c_3, c_4, c_5 = 4, 74, 8508$, $\bar{c}_3, \bar{c}_4, \bar{c}_5 = 2, 9, 119$ and the extreme-ray counts $3, 7, 25, 296$ of $\mathrm{MET}_3, \ldots, \mathrm{MET}_6$ [4], and was cross-checked by the Stirling identity above and by summing the 11610 relabelling classes. Similarly, where [16] reports $\bar{b}_6 \geq 6778$ and $\bar{c}_6 \geq 6778$ from Monte Carlo sampling and calls the exact values unknown, the enumeration gives $\bar{b}_6 = 11610$ and $\bar{c}_6 = 10287$, which are the values now in A397184 and A395485; and $c_6 = 6460153$, the value in A395250 since May 2026, is confirmed. The 296 extreme rays of MET_6 fall into 8 orbits under relabelling, of sizes $6, 10, 10, 15, 15, 60, 90, 90$, agreeing with [2] against the value 7 reported in [4].

Remark 7.3 (the axiom class at six points). The depth-first search counts $4, 74, 8628, 8896888$ relations satisfying (B0)–(B4) on $3, 4, 5, 6$ points. Comparing with b_n: the axioms characterize metrizability exactly for $n \leq 5$ — which is Theorem 5.1 at $n = 5$ — and at $n = 6$ there are $8896888 - 7221418 = 1675470$ betweenness structures that no metric realizes. Graded by the number of collinear triples, those 1675470 begin at $k = 4$: there are none with $k \leq 3$, exactly 270 with $k = 4$, and none with $k \geq 18$. The 270 fall into two classes under relabelling, of sizes 90 and 180, and these are exactly Chvátal's B_1 and B_2, the hypotheses of his conditions (M5) and (M6). So his two obstructions are also all the obstructions with the fewest collinear triples; Mendris and Zlatoš's six-point example has six and is a third class.

Remark 7.4 (the distribution by collinear triples). The metrizable relations on n points, graded by the number k of collinear triples, are distributed as $1, 12, 30, 16, 15$ for $n = 4$ ($k = 0, \ldots, 4$), as the row of Corollary 5.2 for $n = 5$, and for $n = 6$ ($k = 0, \ldots, 20$) as

$$1,\ 60,\ 1350,\ 15000,\ 92325,\ 339984,\ 802020,\ 1293660,\ 1499130,\ 1294220,\ 880968,$$
$$498420,\ 265215,\ 127440,\ 67590,\ 26640,\ 10755,\ 5040,\ 1240,\ 0,\ 360.$$

The top entry is $n!/2$ at $n = 5$ and $n = 6$, where a structure with every triple collinear is a linear order; at $n = 4$ it is 15, the twelve orders together with three relations realized by the four vertices of an ℓ^1 unit square, in which all four triples are collinear but the middles form a 4-cycle. The zero at $k = 19$ says that a six-point metric never has exactly 19 of its 20 triples collinear. That is not special to metrics and it is not new. Szabó [15] calls the number of non-collinear triples the co-size, writes $B(n,m)$ for the set of betweenness structures of order n and co-size m, and sets $\tau(n,0) = \min\{m > 0 : B(n,m) \neq \emptyset\}$, the least positive co-size attained at order n. Theorem 1(1) of his paper (numbering of the arXiv version, v3) states $\tau(n,0) = \max\{1, n-4\}$ for all $n \geq 3$, which at $n = 6$ reads $\tau(6,0) = 2$ — no six-point betweenness structure has exactly one non-collinear triple. The absence of non-metrizable structures with $k \geq 18$ noted in Remark 7.3 follows from the same theorem: co-size 1 is empty by part (1), so the six-point structures of co-size 2 are exactly the *quasilinear* ones, the nonlinear structures of least co-size; part (2) lists the quasilinear structures up to isomorphism, and each entry of that list is the betweenness structure of a weighted graph and hence metrizable. The paper [15] is cited by [16]. The counts themselves we have not found published: the OEIS returns no match for the four-point or six-point rows, and the program behind A395237 computes this distribution internally and discards it.

Remark 7.5 (faces of the metric cone). The total face counts are $f(\mathrm{MET}_n) = 8, 106, 9484, 7356040$ for $n = 3, 4, 5, 6$, and the identity $\sum_k S(n,k)\, b_k = f(\mathrm{MET}_n)$ holds at each of those four values. By dimension, the f-vectors are $(1,3,3,1)$ for MET_3; $(1,7,21,34,30,12,1)$ for MET_4; $(1,25,195,785,1880,2742,2380,1160,285,30,1)$ for MET_5; and

$$(1,\ 296,\ 6465,\ 55730,\ 260430,\ 751173,\ 1422055,\ 1821675,\ 1595850,$$
$$950970,\ 377979,\ 96825,\ 15180,\ 1350,\ 60,\ 1)$$

for MET_6, the entry in dimension 1 being the number of extreme rays and the entry in codimension 1 the number of facets in each case. Ranks were computed twice, modulo two different primes, and agree at all 7 356 040 faces of MET_6. Cizma and Linial [3] bound the number of faces asymptotically and name the f-vector of the metric cone as an open quest; the ancestor of the correspondence, Graham, Yao and Yao [8], uses a related indexing only to bound a face count from above. We caution a reader comparing tables that the metric *polytope* has a different face lattice, and that published partial f-vectors for it are not the numbers above.

Remark 7.6 (cost). The whole six-point computation — extreme rays, the 7 356 040 faces, the relabelling orbits, the induced convexities and the two-prime rank pass — takes 127 CPU-seconds and 415 MB in exact integer arithmetic; the separate depth-first census of the axiom class at $n = 6$ takes about six CPU-minutes. Seven points are out of reach by this route: MET_7 has 55226 extreme rays [7], and the face count has grown by factors of 13, 89 and 776 over $n = 3, \ldots, 6$.

8. What remains open

Verifying the six-point count. The five-point census of Sections 4 and 5 is a proof; the six-point numbers of Section 7 are not. The lower bound $b_6 \geq 7221418$ is certificate-shaped, since by Lemma 2.2 a chain of faces can be generated from the extreme rays with two small numbers per face. The upper bound is the obstacle: $b_6 \leq 7221418$ needs the *completeness* of the list of 296 extreme rays of MET_6, which no certificate about individual faces carries. A direct exhaustive sweep at six points in the style of Section 4 would visit $4^{20} \approx 1.1 \times 10^{12}$ codes at roughly 230 elementary tests each; at the rate our compiled cross-check program achieves that is about 225 CPU-hours, and inside the proof assistant's evaluator, at the interpretation penalty of 413 measured on exactly this code, about 93 000 CPU-hours. We have not attempted it.

A local criterion is not available. One might hope to certify the six-point count by a condition on five-point restrictions. There is none: a computation reported here finds that exactly 8896888 six-point relations have all six of their five-point restrictions metrizable, i.e. precisely the betweenness structures, so no test reading only five-point restrictions separates the 7221418 metrizable ones. The reason is structural rather than accidental — at five points metrizable and axiom-satisfying coincide (Theorem 5.1), while the axioms themselves only constrain four points at a time.

Refinements at five points. Two are within reach of the machinery above and are not done here: the same census for the induced convexities, whose total at five points is $c_5 = 8508$ against

$b_5 = 8628$, and the refinement of the 8628 structures into Chvátal's 122 isomorphism types, which needs a canonical form under the 120 relabellings.

Isomorphism at six points. $\bar{b}_6 = 11610$ is now in the OEIS and is confirmed by the computation of Remark 7.2, but a verified proof would need the same canonical form over 720 relabellings.

9. Formal verification

Every numbered statement of Sections 2–6, together with Proposition 7.1, is formalized and machine-checked in Lean 4 [9] on top of its mathematical library [10]; the exceptions, marked as such where they appear, are the numerical statements of Section 7 after Proposition 7.1, the two totals 30353 and 20178 quoted after Proposition 4.3, the sentence about the two external decision procedures after Theorem 4.2, and the prices in Section 8. The development is four files — the definitions and the elementary theory of Section 2 over an arbitrary carrier type, the pseudometric splitting, the six-point obstruction, and the five-point census — and contains no incomplete proof. The axiom set of each headline statement was printed. Sections 2 and 6 and Proposition 7.1 use no compiled evaluation at all: every finite check there is performed by the kernel. The census rests on exactly two compiled-evaluation axioms, one for each exhaustive pass — the sweep over the 4^{10} codes behind Theorems 4.1 and 4.2, and the verification of the 8628 integer metrics behind Theorem 5.1 — while everything between and around them is kernel-checked: the encoding bijection (Lemma 3.1), the equivalence of the boolean test with the five axioms (Lemma 3.2), the cast of the integer distance tables to real metrics, and the implication from metrizable to axiom-satisfying. The two controls of Proposition 4.3 depend on no axioms whatsoever. Checking the census file, which is the expensive one because it carries both exhaustive passes, takes about 480 CPU-seconds with a peak of 7.1 GB; the other three files are cheap. The repository is private; repository reference to be added at submission.

References

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