METRIZABLE BETWEENNESS RELATIONS ON SIX POINTS: EXACT COUNTS, AND A FOUR-FACT OBSTRUCTION

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ABSTRACT. A metric d on a finite set induces the ternary relation $d(x,z)+d(z,y)=d(x,y)$, read as "z lies between x and y", and a ternary relation is *metrizable* when some metric induces it. A recent census of metrizable betweenness relations on n points prints $b_6 = 7\,238\,428$ and leaves the number $\bar{b}_6$ of isomorphism classes unknown. We recompute the census by exact integer face enumeration of the metric cone MET_6 — 296 extreme rays by double description over $\mathbb{Z}$, then a bitmask closure search over faces, with no linear programming at any point — and obtain

$$b_6 = 7\,221\,418, \qquad \bar{b}_6 = 11\,610, \qquad \bar{c}_6 = 10\,287.$$

The published b_6 is too large by 17 010; the corrected value now carried by the OEIS is right, and the two counts up to isomorphism, which are single-source there, are confirmed by an unrelated method. Behind the count is a structural fact. We isolate a six-point identity among eight triangle slacks which forces one "checkerboard" of four betweenness facts as soon as the complementary one holds, and use it to exhibit a ternary relation B^* on six points that satisfies the Mendris–Zlatoš axioms (B0)–(B4), has every one of its five-point restrictions metrizable, and is metrizable by no metric — with *four* betweenness facts, against the six of the only such structure we have found written down before. Exhaustive computation places 1 675 470 labelled relations, in 2 596 isomorphism classes, between the axioms and metrizability at six points, and shows that four facts is the exact minimum, attained by 270 relations in exactly two classes. That six is the least n at which the axioms fail to characterise metrizability is Mendris–Zlatoš's 1995 assertion, published without proof; we give a machine-checked proof of its existential half and of their own example. Every statement outside the census is formally verified in Lean 4.

1. Introduction

1.1. **Betweenness.** Let (X,d) be a finite metric space. For $x,z,y \in X$ write

$$x - z - y \quad :\Longleftrightarrow \quad d(x,z)+d(z,y)=d(x,y),$$

"z lies between x and y", and write B_d for the resulting ternary relation on X. A ternary relation B on a finite set X is *metrizable* if $B = B_d$ for some metric d on X. Which ternary relations are metrizable is an old question with two faces, one axiomatic and one enumerative, and this paper is about the place where the two meet.

On the enumerative side, let b_n be the number of metrizable betweenness relations on a fixed n-element set and $\bar{b}_n$ the number of them up to isomorphism, i.e. up to the action of the symmetric group. A subset $A \subseteq X$ is *convex* for B when $x,y \in A$ and $x - z - y$ force $z \in A$; write c_n and $\bar{c}_n$ for the corresponding counts of *convexities* induced by metrics. These four sequences are [7] A395237, A397184, A395250 and A395485. The Appendix of Vlasic's recent paper on metric segments and convexity [9] computes them for $n \le 6$ and prints, in its two tables,

(1)

n	1	2	3	4	5	6
b_n	1	1	4	74	8628	7 238 428
$\bar{b}_n$	1	1	2	9	122	≥ 6778
c_n	1	1	4	74	8508	6 460 153
$\bar{c}_n$	1	1	2	9	119	≥ 6778

with the remark that the lower bounds were obtained by Monte Carlo sampling and that "the exact value of $\bar{b}_6$ is currently unknown to us". The OEIS entries now disagree with (1) in three of the six-point cells: A395237 carries $b_6 = 7\,221\,418$, annotated "a(6) corrected by Tian Vlasic, Jul 04 2026" — four days after [9] was posted — and A397184 (keyword `hard`) and A395485 carry $\bar{b}_6 = 11\,610$ and $\bar{c}_6 = 10\,287$. The computer programs linked from A395237 and A397184 decide feasibility of a linear program in floating point, and the one attached to A395237 iterates only over $n < 6$, so it is not the program that produced either six-point value. The six-point entries in this picture therefore rest on unpublished floating-point computations, one of which has already been corrected once.

On the axiomatic side, Mendris and Zlatoš [6, p. 874] record five conditions — the ones [9] quotes in its Appendix, citing [6] — satisfied by every metric betweenness relation:

$$\begin{aligned} &\text{(B0)}\ \ x-y-x \Rightarrow x=y, \qquad \text{(B1)}\ \ x-x-y, \qquad \text{(B2)}\ \ x-y-z \Rightarrow z-y-x, \\ &\text{(B3)}\ \ x-y-z \wedge x-z-u \Rightarrow x-y-u, \qquad \text{(B4)}\ \ x-y-z \wedge x-z-u \Rightarrow y-z-u. \end{aligned} \tag{2}$$

A ternary structure satisfying (2) is a *betweenness space*. The conditions are older than [6], which records that A. Wald had characterised real metric betweenness in terms of (B0)–(B4) "in a slightly different formulation" together with topological and extension conditions [6, p. 874]; [6] is where they appear in the purely first-order form used here. Each axiom quantifies over at most four points, so a relation satisfies (2) exactly when all of its four-point restrictions do; (2) is a local condition, and metrizability, as we shall see, is not.

1.2. **What is settled here.** *The census* (Section 3). We recompute all four sequences at $n = 6$ by an exact method that shares nothing with the one behind (1): the extreme rays of the metric cone MET_6 by a double description over the integers, then an enumeration of the faces meeting the positive orthant by bitmask closure. No linear program, floating point or otherwise, occurs anywhere. The outcome is

$$b_6 = 7\,221\,418, \qquad \bar{b}_6 = 11\,610, \qquad c_6 = 6\,460\,153, \qquad \bar{c}_6 = 10\,287.$$

So the published b_6 is too large by $17\,010$ and the corrected OEIS value is right; the two counts up to isomorphism that (1) reports only as lower bounds, and that the OEIS carries from a single source, are confirmed. A control that we regard as decisive accompanies this: $c_6 = 6\,460\,153$, the one six-point value on which (1) and the OEIS *agree*, is reproduced from the same face list, so the discrepancy is confined to b_6 and is not a systematic offset of our pipeline.

The exchange identity (Section 4). Eight triangle slacks arranged as a 2×4 "checkerboard" on two middles and four endpoints sum, along either diagonal class, to literally the same linear form in the distances. Since all eight are nonnegative on a metric, one checkerboard being tight forces the other. The configuration needs two middles and four endpoints — six points — and it is invisible to (2), which sees four points at a time.

A four-fact obstruction (Section 5). The relation B^* on $\{0, \ldots, 5\}$ whose only non-degenerate facts are

$$2-0-4, \qquad 3-0-5, \qquad 2-1-5, \qquad 3-1-4$$

is a betweenness space, all six of its five-point restrictions are metrizable, and it is metrizable by no metric. It has four betweenness facts, against the six of (A_2, T_2): the smallest member of the family (A_n, T_n) of [6, p. 878], every member of which with $n > 1$ is non-metrizable, and the only *six-point* non-metrizable betweenness space we have found written down anywhere. B^* is not isomorphic to it. The census shows that four is the exact minimum and that it is attained by 270 labelled relations in exactly two isomorphism classes, of which B^* represents one.

How much room there is between the two notions (Section 3). On six points, $8\,896\,888$ relations satisfy (2) and $7\,221\,418$ of them are metrizable, so $1\,675\,470$ labelled relations, falling into $2\,596$ isomorphism classes, separate the axioms from metrizability. Each of the $2\,596$ classes carries an exact rational certificate of non-metrizability with coefficients ± 1; $2\,547$ of them use eight triangle forms and are instances of the exchange identity.

1.3. **Priority.** The qualitative statement that (2) does not imply metrizability, and that six elements is the least size at which that happens, is not new. It is Mendris–Zlatoš [6, p. 880]:

> "Maybe it is worthwhile to mention here that, as one can easily check, all the betweenness spaces (i.e., all ternary structures satisfying the axioms (B0)–(B4)) with at most five elements are metrizable. The smallest nonmetrizable betweenness spaces have six elements — the structure (A_2, T_2) from part (v) of the proof of our theorem is an example."

and, on p. 874, "as is well known, the satisfaction of the axioms (B0)–(B4) is not sufficient for the metrizability of (A, T)". We state this plainly because the present development was written before [6] could be consulted, and because it bounds what is claimed below.

Three things are worth adding about that sentence. First, neither half of it is proved in [6]: the hedge is "as one can easily check", and no argument for either the five-point statement or the six-point one appears anywhere in the paper. Second, (A_2, T_2) is offered as *an* example, and minimality is asserted in the number of *elements*; no claim is made, and no question is raised, about how few betweenness facts a non-metrizable betweenness space can carry. Third, [6] contains no enumeration of any kind. Its concern is model-theoretic: its theorem [6, p. 877] is that the class $\mathcal{M}$ of all metrizable betweenness spaces — distances allowed to range over ordered Abelian groups — is elementary, has a set of universal Horn axioms, is recursively axiomatizable with decidable universal part, is not finitely axiomatizable, and has hereditarily undecidable finite part. A remark drawn from that proof is that "any axiomatization of the class $\mathcal{M}$ of all metrizable betweenness spaces has to use 'transitivities' with arbitrarily large number of variables" [6, p. 880]. It never counts betweenness spaces, metrizable or otherwise.

What this paper contributes on the axiomatic side is therefore: the quantitative layer (Section 3); the identity that explains the six-point obstruction and produces a strictly smaller witness than the published one (Sections 4–5); and formal proofs where [6] gives none, including for its own example (Section 6).

Two further papers, attached to A395237 and A397184 by the OEIS [7], have been read and do not bear on anything claimed here. Šimko [8] answers a question left open by [6]: $\mathcal{M}$ is *not* the least elementary class containing the real-metrizable betweenness spaces, the separating structure being infinite. His Theorem 2 turns on the assertion, made there without proof, that the two classes "contain the same finite structures" — the general form of the remark in Section 2 that rational distances lose nothing here. Hedlíková [5] studies ternary spaces under a related but not equivalent axiom set — [7] quotes it beside (B0)–(B4) as one of several inequivalent notions of betweenness — together with media and Chebyshev sets, and does not take up metrizability. Neither paper contains an enumeration, and neither exhibits a non-metrizable betweenness space.

1.4. **Verification.** Every statement in Sections 4–6 is formally verified in Lean 4; Section 8 says exactly what that means. The census of Section 3 is *not* part of the formal development: it is an exact integer computation carried out outside Lean, and every statement resting on it is labelled "Computation" rather than "Theorem" throughout. Section 7 records what a formally verified b_6 would cost, and the one lemma of it that is already in place.

2. Preliminaries

2.1. **Metrics, patterns, and the metric cone.** Throughout, $X = \{0, 1, 2, 3, 4, 5\}$ and a *metric* on X is a function $d\colon X \times X \to \mathbb{Q}$ that is symmetric, vanishes exactly on the diagonal, is strictly positive off it, and satisfies the triangle inequality. Rational rather than real values lose nothing here: every metrizability claim below is witnessed by an explicit *integer* vector, and every non-metrizability claim by a linear identity; both are valid over any linearly ordered field.

A metric on X is a vector $d \in \mathbb{Q}^{15}$, one coordinate per unordered pair, lying in the *metric cone*

$$\mathrm{MET}_6 \;=\; \left\{\, d \in \mathbb{R}^{15} \;:\; \sigma_d(u, m, v) \geq 0 \;\text{ for all triples}\,\right\}, \quad \sigma_d(u, m, v) = d(u, m) + d(m, v) - d(u, v),$$

the $3\binom{6}{3} = 60$ *triangle forms* $\sigma_d(u,m,v)$ being indexed by a three-element subset together with a choice of which of its points is the middle one. Nonnegativity of the coordinates is implied. By definition $u-m-v$ holds exactly when the form $\sigma_d(u,m,v)$ vanishes, so a betweenness relation is precisely the set of triangle forms tight at d, and

$$b_6 \;=\; \#\{\text{faces of } \mathrm{MET}_6 \text{ whose relative interior meets } \{d>0\}\}. \tag{3}$$

At most one point of a triple can be its middle: if $x-z-y$ and $x-y-z$ then adding the two equalities gives $d(y,z)=0$, hence $y=z$. Consequently a betweenness relation on X is determined by a *pattern*, a function from the 20 three-element subsets of X to $\{0,1,2,3\}$ recording which point of each triple is the middle one, 0 meaning none. The three degenerate shapes $x-x-y$, $x-y-y$ and $x-z-x \Leftrightarrow z=x$ are forced by the metric axioms and carry no information, so two metrics with the same pattern induce the same betweenness relation. This is proved in the formal development and is what makes the 4^{20} patterns the right finite object to search. We call the non-degenerate facts of a relation its *betweenness facts*, and count them as unordered data: $u-m-v$ and $v-m-u$ are one fact.

2.2. **Betweenness spaces.** We use the axioms exactly as (2) states them, with the middle argument in the middle position, which is also the convention of [6]. That every metric betweenness relation is a betweenness space is standard; we record it because the formal development proves it rather than citing it.

Proposition 2.1. *For every metric d on X, the relation B_d satisfies* (B0)–(B4)*.*

Proof. (B0), (B1), (B2) are the three degenerate shapes above together with the symmetry of d. For (B3) and (B4), suppose $d(x,y)+d(y,z)=d(x,z)$ and $d(x,z)+d(z,u)=d(x,u)$. Adding the triangle inequalities $d(x,u)\le d(x,y)+d(y,u)$ and $d(y,u)\le d(y,z)+d(z,u)$ to the two hypotheses gives $d(x,y)+d(y,u)\le d(x,u)$ and $d(y,z)+d(z,u)\le d(y,u)$, and the reverse inequalities are again triangle inequalities. □

Since each of (B0)–(B4) has at most four variables, a relation on X is a betweenness space if and only if all of its four-point restrictions are. Metrizability has no such description, and Sections 4–5 are about why.

3. The census

This section reports an exact computation carried out outside the formal development. Its statements are labelled "Computation" for that reason: they are not theorems of the Lean development, and nothing in the later sections depends on them.

3.1. **Method.** By (3) the problem is a face enumeration. For a pointed cone a face is generated by the extreme rays it contains, so a form vanishes on a face exactly when it vanishes on every extreme ray of that face. Writing $T(r)\subseteq\{0,\ldots,59\}$ for the set of forms tight at a ray r and $S(r)$ for the support of r (the set of coordinates where r is nonzero), the face generated by a tight set T has tight set and support

$$\mathrm{cl}(T)=\bigcap\{T(r): T\subseteq T(r)\},\qquad \mathrm{supp}(T)=\bigcup\{S(r): T\subseteq T(r)\},$$

and T is a betweenness relation exactly when $\mathrm{cl}(T)=T$ and $\mathrm{supp}(T)$ is all 15 coordinates. A breadth-first search from $T=\emptyset$, adding one form at a time and discarding any T whose support is not full, enumerates them; support only shrinks as T grows, so the pruning is exact. All of this is integer bitmask arithmetic. The extreme rays are produced beforehand by a double description procedure over exact integers, started from the nonnegative orthant and adding the triangle inequalities one at a time with the combinatorial adjacency test. The whole computation is exact; no linear program is solved at any point.

3.2. **Controls.** The following were checked, and their expected values fixed, before the six-point answers were read. The extreme-ray counts of $\mathrm{MET}_3, \dots, \mathrm{MET}_6$ come out as $3, 7, 25, 296$, matching the published values [2, Table 1] for the metric cone [3]. The values $b_3, b_4, b_5 = 4, 74, 8628$, $\bar{b}_3, \bar{b}_4, \bar{b}_5 = 2, 9, 122$, $c_3, c_4, c_5 = 4, 74, 8508$ and $\bar{c}_3, \bar{c}_4, \bar{c}_5 = 2, 9, 119$ — every cell on which (1) and the OEIS agree — are reproduced. So is $c_6 = 6\,460\,153$, the one six-point cell on which they agree, computed from the same face list by pushing each relation to its convexity. The enumerated set of relations is stable under the action of the symmetric group with no escapes; the 11 610 orbit sizes $720/|\mathrm{Aut}|$ sum to exactly 7 221 418 and the 10 287 convexity orbit sizes to exactly 6 460 153; at most one point of each triple is a middle in every enumerated relation; and each of the 7 221 418 relations was verified to be induced by an explicitly exhibited integer metric, the largest distance occurring being 386.

Computation 3.1. On six points, $b_6 = 7\,221\,418$, $\bar{b}_6 = 11\,610$, $c_6 = 6\,460\,153$ and $\bar{c}_6 = 10\,287$.

Thus the value $b_6 = 7\,238\,428$ printed in (1) is too large by 17 010, and the value 7 221 418 to which A395237 was corrected is correct; the values $\bar{b}_6 = 11\,610$ and $\bar{c}_6 = 10\,287$, which (1) reports only as ≥ 6778 and which the OEIS carries from a single source, are confirmed by an independent exact method. The discrepancy $17\,010 = 2 \cdot 3^5 \cdot 5 \cdot 7$ is not a multiple of 720, so it is not accounted for by whole missing orbits. Why the published value is too large we do not know, and what follows is a conjecture and not a result: the programs that produced 7 238 428 are not public and we have not seen them; the two programs that are public — those linked from A395237 and A397184 — decide feasibility numerically, each strict inequality being imposed as ≥ 1, which is legitimate since the cone is homogeneous but makes a near-degenerate face the hard case; and an overcount is the failure mode we would expect there. We claim the value $b_6 = 7\,221\,418$, not a diagnosis of the other one.

Computation 3.2. On six points, exactly 8 896 888 ternary relations satisfy (B0)–(B4), of which 7 221 418 are metrizable. Hence exactly 1 675 470 labelled relations, falling into 2 596 isomorphism classes, are betweenness spaces that are not metrizable. The least number of betweenness facts in such a relation is four, attained by exactly 270 labelled relations forming exactly two isomorphism classes, of orbit sizes 90 and 180.

The count 8 896 888 was obtained twice: by a depth-first enumeration of the patterns satisfying (B0)–(B4), run under two different orderings of the 20 triples and visiting 35 157 115 and 60 490 879 nodes with the same answer; and by testing (B0)–(B4) directly on all 8 896 888 resulting patterns. Each of the 2 596 classes carries a certificate of non-metrizability found by an exact rational simplex — a linear identity between two sums of triangle forms, in the sense of Section 4 — with all coefficients ± 1 and none of them using a coordinate-positivity row. Of these, 2 547 use eight forms and 49 use twelve. The eight-form certificate is the exchange identity of Theorem 4.1 below, so the identity is not one obstruction among many: it accounts for all but 49 of the 2 596 classes.

The two minimal classes are, in the notation $m \mid u, v$ for $u - m - v$,

$$\{\, 0 \mid 2,4;\;\; 0 \mid 3,5;\;\; 1 \mid 2,5;\;\; 1 \mid 3,4 \,\}, \qquad \{\, 0 \mid 2,4;\;\; 3 \mid 0,5;\;\; 2 \mid 1,5;\;\; 1 \mid 3,4 \,\},$$

of orbit sizes 90 and 180. The first is the relation B^* of Section 5: two middles, two endpoint pairs, one checkerboard present and the complementary one absent. The second obeys a different four-plus-four identity, with four distinct middles arranged in the cycle $0 \to 2 \to 1 \to 3 \to 0$ alternating between the two outer points 4, 5; it is verified in the census but is not part of the formal development.

We have not found a count of non-metrizable betweenness spaces anywhere in the literature, nor any enumeration in [6]. The nearest published quantities we are aware of are asymptotic, and count a coarser object: Cizma and Linial [1, Thm. 5.5] prove that MET_n has at most $2^{2.38n^2(1+o(1))}$ faces, improving a bound of $2^{2.72n^2}$ that they report as mentioned without proof in Graham, Yao and Yao [4]. That count is over *all* faces of MET_n, including those whose relative interior contains only vectors with a vanishing coordinate — pseudometrics, whose induced relations violate (B0). It is therefore an asymptotic upper bound for b_n rather than a count of it, and it gives no value at $n = 6$. Section 5.1 of [1] is, incidentally, the only place we found where the correspondence between faces of a metric cone and betweenness relations is stated at all.

The whole computation runs in under five CPU-minutes with a peak resident set of about 300 megabytes: 0.04 seconds for the extreme rays, 114 seconds for the six-point face enumeration, 11 seconds for the orbits and convexities, 22 seconds for the certificate analysis and 123 seconds for the axiom comparison.

It was carried out by thirteen short purpose-written programs in C, covering double description, face enumeration, orbits and convexities, witness verification, the defect analysis, locality under two triple orderings, an exact rational simplex, the symmetric-group closure of the certificates, the axiom comparison, and the emission of the data reused in Lean. None of them is distributed with the present version of this work, which contains the Lean development and this paper and nothing else; they will be archived, with their inputs and outputs, as a reproducible artefact accompanying the submission. Until then the census rests on the controls above, and not on code the reader can run.

4. The exchange identity

Theorem 4.1 (Exchange identity and exchange law)**.** *Let $d\colon X \times X \to \mathbb{Q}$ be arbitrary and let $a, b, c, e, g, h \in X$. Then*

$$\sigma_d(c,a,e)+\sigma_d(g,a,h)+\sigma_d(c,b,h)+\sigma_d(g,b,e) \;=\; \sigma_d(c,a,h)+\sigma_d(g,a,e)+\sigma_d(c,b,e)+\sigma_d(g,b,h), \tag{4}$$

identically in d. Consequently, if d is a metric and

$$c - a - e, \qquad g - a - h, \qquad c - b - h, \qquad g - b - e$$

all hold, then so do

$$c - a - h, \qquad g - a - e, \qquad c - b - e, \qquad g - b - h.$$

Proof. Expanding σ, both sides of (4) carry the eight terms $d(c,a)$, $d(a,e)$, $d(g,a)$, $d(a,h)$, $d(c,b)$, $d(b,h)$, $d(g,b)$, $d(b,e)$ with coefficient $+1$ — each of the two middles a, b is paired with each of the four endpoints c, e, g, h exactly once, on both sides — and subtract the four cross terms $d(c,e)$, $d(g,h)$, $d(c,h)$, $d(g,e)$, again on both sides. The two sides are therefore the same linear form, and no hypothesis on d is used; in particular the six points need not be distinct. For the second statement, the four hypotheses say that the four forms on the left of (4) vanish, so the four forms on the right sum to zero; each of them is nonnegative because d is a metric, so each vanishes. $\square$

The configuration is a 2×4 array: two middles a, b and four endpoints partitioned into the pairs $\{c, g\}$ and $\{e, h\}$. The eight slacks split into two "checkerboards" of four, and (4) says the two checkerboards have equal total slack, so one being tight forces the other. Two middles and four endpoints is six points, which is why the phenomenon cannot appear earlier; and the statement relates eight triples at once, so it is invisible to (B0)–(B4), each of which sees four points.

Three controls fix what Theorem 4.1 does and does not say. Each is witnessed by an explicit integer metric on X, and each is verified by evaluating the relevant betweenness facts on that metric.

Proposition 4.2. *The hypotheses of Theorem 4.1 are simultaneously satisfiable: the metric with $d(0,1) = 4$, $d(i,j) = 3$ for every other pair meeting $\{0,1\}$, $d(2,4) = d(2,5) = d(3,4) = d(3,5) = 6$ and $d(2,3) = d(4,5) = 3$ satisfies $2 - 0 - 4$, $3 - 0 - 5$, $2 - 1 - 5$ and $3 - 1 - 4$.*

Without Proposition 4.2 the exchange law could be vacuously true. By the law, the metric of Proposition 4.2 also satisfies the four conclusions — so the configuration of eight facts *is* metrizable, and it is exactly the omission of the second checkerboard that is not.

Proposition 4.3. *All four hypotheses are needed. There is a metric on X satisfying $3 - 0 - 5$, $2 - 1 - 5$ and $3 - 1 - 4$ for which all four of $2 - 0 - 5$, $3 - 0 - 4$, $2 - 1 - 4$ and $3 - 1 - 5$ fail.*

Proposition 4.4. *Two betweenness facts do not force two more: there is a metric on X satisfying $2-1-5$ and $3-1-4$ for which both $2-0-5$ and $3-0-4$ fail.*

5. A non-metrizable betweenness space with four facts

Definition 5.1. B^* is the ternary relation on $X = \{0, \dots, 5\}$ holding exactly at the degenerate triples $x-x-y$ and $x-y-y$ together with

$$2-0-4, \qquad 3-0-5, \qquad 2-1-5, \qquad 3-1-4$$

and their reversals.

So B^* is the exchange configuration of Theorem 4.1 with $a = 0$, $b = 1$, $\{c, g\} = \{2, 3\}$ and $\{e, h\} = \{4, 5\}$, one checkerboard present and the other absent.

Proposition 5.2. *B^* is a betweenness space: it satisfies* (B0)–(B4)*.*

Proof. Each axiom has at most four variables ranging over a six-element set, so this is a finite check. It was carried out exhaustively over all $6^4 = 1296$ instantiations of the four-variable axioms (B3) and (B4), and over the smaller instantiation sets of (B0)–(B2), by kernel evaluation. $\square$

Proposition 5.3. *For every $w \in X$ there is a metric d_w on X such that B^* and B_{d_w} agree on every triple avoiding w. In particular every five-point restriction of B^* is metrizable.*

Proof. Six explicit integer vectors in $\mathbb{Z}^{15}$, one per w, are exhibited; the entries lie between 27 and 72. For each, positivity of the 15 entries and nonnegativity of the 60 triangle forms are integer inequalities, checked by evaluation, so each vector is a metric; and the agreement of B^* with B_{d_w} on triples avoiding w is a finite check over the 216 ordered triples of the remaining five points. The vectors themselves were extracted from the face enumeration of Section 3, but nothing about their provenance enters the proof: once written down they are verified from scratch. $\square$

Theorem 5.4. *B^* is the betweenness relation of no metric on X.*

Proof. Suppose d is a metric with $B_d = B^*$. Then $2-0-4$, $3-0-5$, $2-1-5$ and $3-1-4$ hold for d; these are the hypotheses of the exchange law of Theorem 4.1 with $a = 0$, $b = 1$, $c = 2$, $e = 4$, $g = 3$, $h = 5$. So $2-0-5$ holds for d, hence for B^*; but $2-0-5$ is not among the facts of Definition 5.1. $\square$

Proposition 5.5. *None of the six metrics d_w of Proposition 5.3 induces B^* globally.*

Proof. Immediate from Theorem 5.4, each d_w being a metric on all of X. $\square$

Proposition 5.5 is the control that keeps Proposition 5.3 from being read as more than it says: the six metrics realise B^* on five points each and none realises it on six, which by Theorem 5.4 is automatic but worth recording, since a stronger reading of Proposition 5.3 would contradict Theorem 5.4.

Combining Propositions 5.2 and 5.3 with Theorem 5.4 gives the two statements of Section 1.3 in the form in which they are verified. Both are due to [6], and we state them as such.

Theorem 5.6 ([6, p. 880])**.** *There is a ternary relation on a six-point set every five-point restriction of which is the betweenness relation of a metric, and which is itself the betweenness relation of no metric. Metrizability is therefore not a five-point condition.*

Theorem 5.7 ([6, p. 874, p. 880])**.** *The relation of Theorem 5.6 may moreover be taken to satisfy* (B0)–(B4)*. Hence the betweenness axioms do not imply metrizability.*

Proof. Both are witnessed by B^*: Proposition 5.3 and Theorem 5.4 give the first, and Proposition 5.2 adds the second. $\square$

That six is the *least* n at which such a relation exists is asserted in [6, p. 880] and is not part of the formal development; in the present work it rests on the exhaustive computation of Section 3, which finds $4, 74, 8628$ betweenness spaces on $3, 4, 5$ points and finds all of them metrizable.

What the formal development adds to Theorems 5.6 and 5.7 is a proof, since [6] gives none, and a smaller witness. The example named in [6] carries six betweenness facts; B^* carries four, and by Computation 3.2 four is the exact minimum. That the two examples are genuinely different is Proposition 6.6 below.

6. The published example, machine-checked

The witness [6] names is (A_2, T_2), the case $n = 2$ of the family $A_n = \{a_0, b_0, \ldots, a_n, b_n\}$ of [6, p. 878], where T_n consists of the degenerate triples together with (a_i, b_i, b_{i+1}) and (b_i, a_i, a_{i+1}) and their reversals, indices modulo $n+1$. Relabelling $a_i \mapsto i$ and $b_i \mapsto 3+i$, its six non-degenerate facts are

$$0-3-4, \qquad 1-0-3, \qquad 1-4-5, \qquad 2-1-4, \qquad 2-5-3, \qquad 0-2-5, \tag{5}$$

a six-cycle in which every point is a middle exactly once. Write B^{MZ} for this relation. All of the following are stated in, or immediately implied by, [6]; none is proved there. The paper says of the axioms only "It can easily be verified that any of the ternary structures (A_n, T_n) satisfies the axioms (B0)–(B4)" [6, p. 879].

Proposition 6.1. *B^{MZ} is a betweenness space.*

Proposition 6.2. *For every $w \in X$ there is a metric on X whose betweenness relation agrees with B^{MZ} on every triple avoiding w. In particular every five-point restriction of B^{MZ} is metrizable, as* [6, p. 880] *asserts of every five-element betweenness space.*

Both are finite checks of the same shape as Propositions 5.2 and 5.3; the six witness vectors have entries between 18 and 47. The non-metrizability rests on a twelve-term identity in place of the eight-term (4).

Lemma 6.3. *For every metric d on X,*

$$\begin{aligned}\sigma_d(0,1,4) + \sigma_d(2,0,3) + \sigma_d(0,3,5) + \sigma_d(1,2,5) + \sigma_d(1,4,3) + \sigma_d(2,5,4)\\ = \sigma_d(1,0,3) + \sigma_d(0,2,5) + \sigma_d(0,3,4) + \sigma_d(2,1,4) + \sigma_d(1,4,5) + \sigma_d(2,5,3).\end{aligned}$$

The six forms on the right are exactly the six facts (5) *of B^{MZ}.*

Proof. Both sides expand to the same linear form in the fifteen distances once the symmetry $d(u, v) = d(v, u)$ is used. The identity was found by an exact rational simplex run on the certificate of Section 3 and then verified directly. ☐

Theorem 6.4 ([6, p. 879])**.** *B^{MZ} is the betweenness relation of no metric on X.*

Proof. A metric inducing B^{MZ} makes the six forms on the right of Lemma 6.3 vanish, so the six nonnegative forms on the left sum to zero and each vanishes. In particular $0-1-4$ holds, which is not a fact of B^{MZ}. ☐

The route of [6] is different and reaches the same place. Their p. 878 lemma is a four-point *inequality* — "given any metric d on the four-element set $\{a_0, a_1, b_0, b_1\}$, the conditions $T_d(a_0, b_0, b_1)$ and $T_d(b_0, a_0, a_1)$ imply $d(a_0, b_0) \le d(a_1, b_1)$" — which they chain around the cycle to force $d(a_0, b_0) = \cdots = d(a_n, b_n)$, and from that they read off the forced triples $T_n(a_i, a_{i+1}, b_{i+1})$ and $T_n(b_i, b_{i+1}, a_{i+1})$. Those six triples are precisely the six forms on the left of Lemma 6.3. The argument as given on p. 879 stops there, without saying that the forced triples are not in T_n; but the observation is not absent from [6]. Two pages later, among the closing remarks, the Horn formula α_n "forbidding (A_n, T_n)" is written down [6, p. 881] with premises the facts of T_n and conclusion $T(x_0, x_1, y_1)$ — for $n = 2$ exactly the triple $0-1-4$ that does the work above. What is added here is a proof.

Theorem 6.5. *B^{MZ} is a betweenness space, every five-point restriction of it is metrizable, and it is metrizable by no metric.*

Proof. Propositions 6.1 and 6.2 and Theorem 6.4. □

Theorem 6.5 is the assertion of [6, p. 880], verified for the example that paper names. Finally, the two witnesses are distinguished.

Proposition 6.6. *No bijection of X carries B^{MZ} to B^*.*

Proof. Every point of X is the middle of exactly one fact of B^{MZ}, whereas only 0 and 1 are ever middles of B^*. A bijection carrying one relation to the other would send the four points $2, 3, 4, 5$, each a middle of B^{MZ}, to four distinct middles of B^*, of which there are two. □

An independent check in the census agrees: the orbit of B^{MZ} under the symmetric group has 120 members and B^* is not among them, its own orbit having 90.

7. Towards a verified count

The census of Section 3 is exact but not formally verified, and it is the one part of this paper that is not. We record what closing that gap costs, since the gap is not one of principle.

Two one-way tools already carry the whole verification burden. A vector of 15 positive integers whose 60 triangle forms are nonnegative is a metric, hence realises the pattern it computes; and a linear identity $\sum_{f \in P} \sigma_f = \sum_{f \in M} \sigma_f$ between two lists of triangle forms, compared as coefficient vectors in $\mathbb{Z}^{15}$, refutes every pattern declaring all of M tight and some member of P strict — the certificate used in Theorem 4.1 and Lemma 6.3, in its general form. Both are proved in the formal development. What is missing is the enumeration that runs them across the 4^{20} patterns, and its combinatorial core is the following.

Proposition 7.1. *Let $k \geq 0$, let $\mathrm{leaf}\colon \mathbb{N} \to \mathbb{N}$, and let ok be a Boolean test taking a depth and a partial code. The pruned base-4 depth-first search over k digits, which fixes the digits of a code in order, applies ok at depth t to the truncation of the code modulo 4^{t+1}, and abandons a branch as soon as the test fails, returns exactly*

$$\sum_{c < 4^k} \big[\, \textit{every check passes at } c \,\big] \cdot \mathrm{leaf}(c).$$

Proposition 7.1 is proved by induction on the number of free digits and knows nothing about metrics: its content is that applying the test to the truncation is what makes a prune transfer to every completion. With it in place, a verified b_6 is data plumbing — a table of the 74 metrizable four-point patterns to prune with, the 296 extreme rays to build witnesses from, and a table of 21 060 refuting identities (the symmetric-group closure of one certificate per class) — plus a bridge from patterns to codes. The pruned search visits 35 157 115 nodes in the better of the two triple orderings, and at a leaf either 7.7 extreme rays on average are live and their sum is the witness metric, or a refuting identity is found after 422 tests on average under a two-lowest-form index; those two averages are counts taken from the census and do not depend on the machine. The price does. One block of 384 932 nodes, run through the `sweep` of the formal development, took 11.5 seconds on the machine used here — 30 μs per node, extrapolating to about 35 CPU-minutes for the whole search. That figure and the 30 μs behind it are single-machine timings of one implementation, reported as they were recorded, and are quoted for the order of magnitude only; they are not reproducible constants. A single monolithic evaluation exceeds the ten-minute limit, so the search would be split into the 74 natural blocks, the largest of 3 845 123 nodes. This work has been priced and deliberately not done; the corresponding claims are the Computations of Section 3 and are labelled as such throughout.

8. Verification

Every numbered Theorem, Proposition and Lemma in this paper has been formally verified in Lean 4 (toolchain `v4.32.0`) against *Mathlib*; repository reference to be added at submission. The development contains no `sorry`, and — unusually for a paper with this much finite search in it — no appeal to compiled evaluation either: every one of these statements depends on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`, so all of the finite checks are carried out by the kernel itself. Those checks are: the $6^4 = 1296$ instantiations of the four-variable axioms for each of Propositions 5.2 and 6.1; for each of Propositions 5.3 and 6.2, six integer vectors, each with 15 positivity checks, 60 triangle-form checks and an agreement check over the 216 ordered triples of the retained five points; and one integer vector with the same checks for each of Propositions 4.2, 4.3 and 4.4. Theorem 4.1 and Lemma 6.3 use no finite search at all: the first is a ring identity, the second a linear consequence of the symmetry of d, and the deductions from them are ordinary inequality reasoning. Proposition 7.1 is an induction with no data in it.

The census of Section 3 is outside all of this. It was carried out by purpose-written programs in C using exact integer and exact rational arithmetic throughout; its statements are labelled Computation, carry no formal verification, and are not used in the proof of any Theorem, Proposition or Lemma above. The controls listed in Section 3 — in particular the reproduction of every published value on which (1) and the OEIS agree, including the six-point value c_6 — are what we offer in their place. Where the census and the formal development overlap they agree: the witness metrics of Propositions 5.3 and 6.2 came out of the face enumeration and are re-verified from scratch in Lean, and the non-isomorphism of Proposition 6.6 is confirmed by the orbit computation.

References

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[9] T. Vlasic, *Segments and convexity in metric spaces*, arXiv:2607.27222v1 (2026), Appendix. Still at v1 on 2026-08-29; the table quoted as (1) is Tables 1 and 2 of that version, and should be re-read if a later version appears, since the OEIS correction postdates v1 by four days.