# RESET THRESHOLDS OF MEDIA: A WITNESS AT EVERY NUMBER OF STATES, THE SIX-STATE EXAMPLE CORRECTED, AND THE EXACT MAXIMUM AT THREE TOKEN PAIRS

KOREA SUPERINTELLIGENCE LABS

Abstract. A *medium* in the sense of Eppstein and Falmagne is a deterministic finite automaton whose letters — its *tokens* — come in mutually reverse pairs and which satisfies three further axioms; equivalently it is the token system of a well-graded family of sets, and its transition graph is a partial cube. Eppstein and Falmagne proved that a medium on $n$ states has a reset word of length at most $n-1$. Volkov's list of results on the Černý conjecture records the class as one whose Černý function is known and equal to $n-1$, but flags the lower half of that claim: the bound $n-1$ "is claimed in [Eppstein&Falmagne:2008], although an example of a medium with reset threshold $n-1$ is is provided only for $n=6$."

We settle the existence question at every $n$, and we correct the record about the six-state example. For every $n \geq 1$ the *path medium* — the token automaton of the family of prefixes of an $(n-1)$-element set — is a medium on $n$ states with reset threshold exactly $n-1$, by an argument uniform in $n$. The six-state medium that Eppstein and Falmagne exhibit, the well-graded family of all one- and two-element subsets of a three-element set, has reset threshold 4, not 5: it is not a witness for the lower bound, and Eppstein and Falmagne never claim that it is. They produce it against a different hope, that a reset word of length $\tau/2$ — half the number of tokens — always exists.

It is that second quantity that we then measure exactly in the first case where the hope fails. Writing $E(d)$ for the largest reset threshold of a medium with $d$ token pairs ($\tau = 2d$ tokens), we prove $E(3) = 4$: every medium with three token pairs has a reset word of length 4, by exhaustion over all 256 subsets of the three-dimensional cube — hence over all media with three token pairs, modulo the equivalence of Eppstein and Falmagne, which we cite and do not reprove — and the hexagon attains 4. Computations outside the formal development, reported separately and labelled as such, place this in a wider picture: a census of all media on at most 14 states, which reproduces the eight published terms of OEIS `A343163` and adds five; the observation that within that range the media attaining $n-1$ are exactly the tree media; and $\mathrm{rt}(C_{2m}) = 2m-2$ for the even cycles up to $C_{20}$, where the excess $\mathrm{rt} - \tau/2 = m-2$ therefore reaches 8. Every statement below is machine-checked in Lean 4 except where the text says otherwise.

## 1. Introduction

**Reset thresholds.** A deterministic finite automaton $\mathcal{A} = \langle Q, \Sigma \rangle$ — a finite state set $Q$ with a right action $(q,a) \mapsto q \cdot a$ of a finite alphabet $\Sigma$, extended to words by $q \cdot (aw) = (q \cdot a) \cdot w$ — is *synchronizing* if some word $w \in \Sigma^*$ and some state $s \in Q$ satisfy $q \cdot w = s$ for all $q \in Q$. Such a $w$ is a *reset word*, and the minimum length of a reset word is the *reset threshold* $\mathrm{rt}(\mathcal{A})$. Černý's conjecture is that $\mathrm{rt}(\mathcal{A}) \leq (n-1)^2$ for every synchronizing automaton on $n$ states, and it is open; a large literature instead pins rt exactly on restricted classes. Volkov's list of results [8] is a running catalogue of that literature, one item per class, each item recording an upper bound, a lower bound and the method behind each. When the two bounds coincide — when the Černý function of the class is known — the item is marked with a dagger.

**Media.** Item C5$^\dagger$ of [8] is the class of *media*, introduced by Falmagne and studied algorithmically by Eppstein and Falmagne [3]. A medium is an automaton in which every letter $a$ has a unique *reverse* $b$ — meaning that $p \cdot a = r$ if and only if $r \cdot b = p$, for all distinct states $p, r$ — together with three further axioms recalled verbatim in §2: any two distinct states are joined by a word using no letter and its reverse, a word all of whose letters move the state they are applied to returns to its starting point exactly when every letter occurs as often as its reverse, and two such words that meet concatenate without a reverse pair. The letters of a medium are called *tokens* and come in $\tau/2$ mutually reverse pairs, where $\tau = |\Sigma|$. The motivating example, and by [3] essentially the only one, is the token system of a *well-graded family* of sets: a family $S$ of subsets of a finite ground set such that from any $P \in S$ and any other $R \in S$ one can move a single element into or out of $P$ and stay inside $S$, always in the direction of $R$. The transition graph of a medium is then a *partial cube*: an isometric subgraph of a hypercube.

Eppstein and Falmagne's first theorem [3, Theorem 1] is that a medium on $n$ states has a reset word of length at most $n-1$, found by a topological order of an auxiliary acyclic digraph on $Q$; this is the upper bound recorded at item C5$^\dagger$. Their introduction calls the bound tight, and [8] records the lower bound accordingly — but with an explicit caveat, which is the starting point of this paper:

> The lower bound $n-1$ is claimed in [Eppstein&Falmagne:2008], although an example of a medium with reset threshold $n-1$ is is provided only for $n = 6$.

(The doubled "is is" is the source's.) So the dagger on item C5 rests, as far as the printed record goes, on a single automaton at a single value of $n$. This raises the obvious question.

**Question 1.1.** For which $n$ does there exist a medium on $n$ states with reset threshold exactly $n-1$?

**What is settled here.** The answer is *every* $n$, and the witness is as simple as it could be.

**Theorem 1.2** (Theorem 3.2 and Corollary 3.3)**.** *For every $n \geq 1$ there is a medium on $n$ states whose reset threshold is exactly $n-1$: the token automaton of the family of prefixes $\emptyset \subset \{0\} \subset \{0,1\} \subset \cdots \subset \{0,\ldots,n-2\}$ of an $(n-1)$-element set, whose graph is the path on $n$ vertices.*

The proof is two lines and is uniform in $n$; we make no claim of depth for it. Its interest is that it closes the gap that [8] flags, at every $n$ rather than at one, and so justifies the dagger on item C5.

The second result concerns the example that [8] points at. The only six-state medium in [3], and the only medium there attached to reset words at all, appears in the closing remark of its section on reset sequences:

> We remark that better bounds on the reset sequence length, in terms of $\tau$ instead of $n$, may be possible: for instance, one can form a reset sequence with length $(\tau/2)^2$ simply by concatenating $\tau/2$ copies of a word $w$ containing all positive tokens for the content orientation of a state $Q$. However for some media $(\tau/2)^2$ may be larger than $n-1$. Since $\tau/2 \leq n-1$, it is natural to hope that a reset sequence of length $\tau/2$ always exists, but this is not true; for instance there is no such reset sequence for a medium with six states and six tokens, formed by the well-graded family of all 1- and 2-element subsets of a 3-element set.

That medium does not have reset threshold $n-1 = 5$.

**Theorem 1.3** (Theorem 4.2)**.** *The medium of* [3] *formed by the well-graded family of all one- and two-element subsets of a three-element set has six states, six tokens and reset threshold exactly* 4*. In particular it is not an example of a medium with reset threshold $n-1$.*

We stress what this does and does not say. Eppstein and Falmagne do assert that their bound is tight — their introduction reads "Our first result is a tight bound of $n-1$ on the length of the shortest reset sequence for a medium" — so [8] reports their *claim* accurately. What is not accurate is the attachment of that claim to this example: the example is produced against $\tau/2 = 3$, and it does refute $\mathrm{rt} \leq \tau/2$, since $4 > 3$. It falls short of $n-1$ by one. Theorem 1.2 supplies the missing witnesses.

The third result measures the quantity that Eppstein and Falmagne's remark is really about. Write $E(d)$ for the maximum of $\mathrm{rt}(\mathcal{A})$ over media $\mathcal{A}$ with $d$ token pairs, that is with $\tau = 2d$ tokens. Their hope was $E(d) = d$, and their example refutes it at $d = 3$. We determine $E(3)$ exactly.

**Theorem 1.4** (Theorem 5.3)**.** $E(3) = 4$. *That is: every medium with three token pairs has a reset word of length* 4*, and some medium with three token pairs — the hexagon — has no reset word of length* 3.

The upper half is an exhaustive statement about a class, not a computation on one automaton: it is verified over all 256 subsets of the three-dimensional cube, of which the well-graded ones using all three coordinates are kept. A caveat is stated precisely in §2 and repeated at the theorem: "every medium" here means "every token automaton of a well-graded family", which is all media by the equivalence of [3] — an equivalence we cite and do not reprove; the direction we need in the other sense, that every such token automaton really is a medium in the axiomatic sense of [8], is proved here in full and uniformly in the dimension (Proposition 2.3).

Finally, §7 reports computations that live outside the formal development and are labelled as such throughout: a census of all media on at most 14 states, extending the published terms of OEIS `A343163`; the observation that in that range the media with $\mathrm{rt} = n - 1$ are exactly the tree media; and the even cycles up to $C_{20}$, which give $\mathrm{rt} = \tau - 2$, so that the excess of rt over $\tau/2$ grows linearly across the range checked and reaches 8. These are computations in C, out of reach of a re-run inside the proof assistant at that size, and no theorem of this paper depends on them.

**Sources and conventions.** Quotations from [8] are from the LaTeX source of arXiv version 4 (13 January 2026), the latest at the time of writing; [8] is a living list, and a reader should check the current version. Quotations from [3], and the theorem and lemma numbers we give for it, are from the arXiv preprint `cs/0206033`, which Eppstein's own publication list gives as the preprint of the journal article; the journal version was not read.

## 2. Preliminaries

**Words, reset words, the threshold.** Throughout, $\mathcal{A} = \langle Q, \Sigma \rangle$ is a deterministic finite automaton with $Q$ finite and non-empty; $q \cdot w$ denotes the state reached from $q$ by reading $w \in \Sigma^*$ left to right, and $n = |Q|$. A word $w$ is a *reset word* if $p \cdot w = r \cdot w$ for all $p, r \in Q$, and $\mathrm{rt}(\mathcal{A}) = k$ means that some reset word has length $k$ and no shorter word is a reset word.

**Volkov's axioms.** We use the definition of a medium exactly as item C5 of [8] states it. Two letters $a, b \in \Sigma$ are *reverses* of each other if, for any distinct states $p, r \in Q$, one has $p \cdot a = r$ if and only if $r \cdot b = p$. A word is *consistent* if it does not contain two letters that are reverses of each other. A word $w$ is *vacuous* if each letter $a$ occurs in $w$ as often as the reverse of $a$ does. A word $a_1 \cdots a_\ell$ is *stepwise effective for a state* $q$ if $q \cdot a_1 \neq q$ and $q \cdot a_1 \cdots a_i \neq q \cdot a_1 \cdots a_{i-1}$ for $i = 2, \ldots, \ell$. A *medium* is an automaton satisfying

(i) every letter has a unique reverse;
(ii) for any distinct states $p, r$ there is a consistent word $w$ with $p \cdot w = r$;
(iii) if $w$ is stepwise effective for $q$, then $q \cdot w = q$ if and only if $w$ is vacuous;
(iv) if $p \cdot u = r \cdot v$ where $u, v$ are consistent, $u$ stepwise effective for $p$ and $v$ stepwise effective for $r$, then $uv$ is consistent.

The letters of a medium are its *tokens*; by (i) they fall into $\tau/2$ reverse pairs, where $\tau = |\Sigma|$.

**Well-graded families and their token automata.** Fix $d \geq 0$ and identify the subsets of $\{0, \ldots, d-1\}$ with the vertices $\{0,1\}^d$ of the $d$-cube. For $v \in \{0,1\}^d$, a coordinate $k$ and a bit $b$, write $v[k := b]$ for $v$ with its $k$-th coordinate replaced by $b$.

**Definition 2.1.** A *well-graded family* in the $d$-cube is a non-empty finite set $S \subseteq \{0,1\}^d$ such that

(a) for all $p, r \in S$ with $p \neq r$ there is a coordinate $k$ with $p_k \neq r_k$ and $p[k := r_k] \in S$, and
(b) every coordinate $k$ is used: some $p \in S$ has $p[k := \neg p_k] \in S$.

Condition (a) says exactly that the subgraph of the $d$-cube induced by $S$ is isometric, i.e. that $S$ is a partial cube; condition (b) says that the ambient dimension is not padded.

**Definition 2.2.** The *token automaton* $\mathcal{M}(S)$ of a well-graded family $S$ in the $d$-cube has state set $S$ and alphabet $\{0,\dots,d-1\}\times\{0,1\}$ of size $\tau = 2d$; the token $(k,b)$ acts by

$$v\cdot(k,b) \;=\; \begin{cases} v[k:=b], & \text{if } v[k:=b]\in S,\\ v, & \text{otherwise.}\end{cases}$$

We write $k^+$ for $(k,1)$ and $k^-$ for $(k,0)$; these are Falmagne's positive and negative tokens $i_k$ and $d_k$ for the ground-set element $k$.

**Proposition 2.3.** *For every $d$ and every well-graded family $S$ in the $d$-cube, the token automaton $\mathcal{M}(S)$ satisfies axioms (i)–(iv), i.e. it is a medium. The reverse of $(k,b)$ is $(k,\neg b)$.*

*Proof sketch.* The proof is elementary, uniform in $d$, and formalised in full; we indicate the four steps. (i) $(k,\neg b)$ is a reverse of $(k,b)$ directly from the definition of the action; uniqueness needs an edge in direction $k$, which is condition (b) of Definition 2.1: if $(k',b')$ is also a reverse of $(k,b)$ it must send the head of that edge back to its tail, and only $(k,\neg b)$ does. (ii) Induct on the Hamming distance from $p$ to $r$: well-gradedness supplies a coordinate to flip towards $r$, and the resulting word uses each coordinate at most once, hence contains no reverse pair. (iii) The invariant is a signed count: if $w$ is stepwise effective for $q$ then, for each coordinate $k$, the number of occurrences of $k^+$ minus the number of occurrences of $k^-$ equals $(q\cdot w)_k - q_k$ read as integers, because a token that moves the state must set its coordinate to the value the state did not have. Both directions of (iii) follow. (iv) A consistent stepwise effective word has pairwise distinct coordinates, and each of its tokens $(k,b)$ has $b = (p\cdot u)_k$; so if $p\cdot u = r\cdot v = z$, then every token of $u$ and of $v$ carries the bit $z_k$ at its coordinate $k$, and $uv$ cannot contain both $k^+$ and $k^-$. □

The converse — that every medium is isomorphic to the token automaton of a well-graded family — is the equivalence stated in [3], where the proof is omitted. We cite it and do not reprove it. Consequently the exhaustive statement of §5 is proved for token automata of well-graded families, and is a statement about all media modulo that cited equivalence. Apart from the appeal to [3, Theorem 1] for the upper bound $n-1$ in Corollary 3.3 and Remark 3.5, this is the only place in the paper where a published result is used as a hypothesis rather than merely as context.

**$\Theta$-classes and the token count.** Writing $\rho$ for graph distance, two edges $e=\{x,y\}$ and $f=\{u,v\}$ stand in the Djoković–Winkler relation $\Theta$ [2, 9] when $\rho(x,u)+\rho(y,v)\neq\rho(x,v)+\rho(y,u)$; on a partial cube $\Theta$ is an equivalence relation, and its classes are the $\Theta$-*classes*. For the graph of $\mathcal{M}(S)$ with $S$ as in Definition 2.1 they are exactly the $d$ coordinate directions. Indeed the embedding $S\subseteq\{0,1\}^d$ is isometric, so $\rho$ is Hamming distance and the difference $\rho(x,u)+\rho(y,v)-\rho(x,v)-\rho(y,u)$ is summed coordinate by coordinate: a coordinate on which both $e$ and $f$ are constant contributes 0; a coordinate flipped by exactly one of them contributes 0 as well; and if $e$ and $f$ flip the same coordinate, that coordinate contributes $\pm 2$. So $e\,\Theta\,f$ exactly when $e$ and $f$ flip the same coordinate, and condition (b) of Definition 2.1 makes all $d$ resulting classes non-empty. Thus "$d$ $\Theta$-classes" and "$\tau = 2d$ tokens" say the same thing, and we use the two interchangeably; the formal development speaks of the dimension $d$ throughout, so nothing below depends on this paragraph beyond the choice of words.

**Definition 2.4.** $E(d)$ is the maximum of $\mathrm{rt}(\mathcal{A})$ over all media $\mathcal{A}$ with $d$ $\Theta$-classes, i.e. with $\tau = 2d$ tokens.

**A lower-bound lemma.** One elementary lemma carries every lower bound in the paper that is not an exhaustive search.

**Lemma 2.5.** *Let $\mathcal{A} = \langle Q,\Sigma\rangle$ be an automaton in which every letter fixes all states but at most one. Then every reset word of $\mathcal{A}$ has length at least $|Q|-1$.*

*Proof.* A map $f\colon Q\to Q$ that fixes every point except possibly one satisfies $|T|\le|f(T)|+1$ for every $T\subseteq Q$, since $T\setminus\{x\}$ injects into $f(T)$ when $x$ is the moved point. Composing, a word of length $\ell$ maps $Q$ onto an image of size at least $|Q|-\ell$. A reset word has image of size 1, so $\ell\ge|Q|-1$. □

## 3. A medium with reset threshold $n-1$, at every $n$

**Definition 3.1.** For $m \geq 0$ let $P_m \subseteq \{0,1\}^m$ be the family of *prefixes* $\emptyset, \{0\}, \{0,1\}, \ldots, \{0,\ldots,m-1\}$, that is the $m+1$ vertices $p^{(i)}$, $0 \leq i \leq m$, with $p^{(i)}_k = 1$ iff $k < i$. It is a well-graded family in the $m$-cube using every coordinate, and the graph of $\mathcal{M}(P_m)$ is the path on $m+1$ vertices. We call $\mathcal{M}(P_m)$ the *path medium* on $n = m+1$ states.

**Theorem 3.2.** *For every $m \geq 0$ the path medium $\mathcal{M}(P_m)$ is a medium on $m+1$ states, and*

$$\mathrm{rt}\big(\mathcal{M}(P_m)\big) \;=\; m.$$

*Proof.* That $\mathcal{M}(P_m)$ is a medium is Proposition 2.3; that it has $m+1$ states is the injectivity of $i \mapsto p^{(i)}$.

*Upper bound.* The word $0^+\,1^+\cdots(m-1)^+$ has length $m$ and resets. Indeed one checks by induction on $t$ that after its first $t$ tokens the state that started at $p^{(i)}$ sits at $p^{(\max(i,t))}$: the token $j^+$ moves $p^{(i)}$ only when $i = j$, in which case it moves it to $p^{(i+1)}$, because $p^{(i)}[j := 1] = p^{(i+1)}$ lies in $P_m$ while for $i < j$ the vertex $p^{(i)}[j := 1]$ is not a prefix and the token acts trivially. After all $m$ tokens every state sits at $p^{(m)}$.

*Lower bound.* Each token of $\mathcal{M}(P_m)$ fixes all states but at most one: $k^+$ moves only $p^{(k)}$, and $k^-$ moves only $p^{(k+1)}$, since a prefix strictly shorter than $k+1$ cannot gain the element $k$ and stay a prefix, and one strictly longer than $k+1$ cannot lose it. Lemma 2.5 then gives $\ell \geq (m+1) - 1 = m$ for every reset word. □

**Corollary 3.3.** *For every $n \geq 1$ there is a medium on $n$ states with reset threshold exactly $n-1$; one may take the token automaton of the prefix family in the $(n-1)$-cube. In particular, taken together with the upper bound of* [3, Theorem 1]*, the Černý function of the class of media equals $n-1$ at every $n \geq 1$, and the lower bound recorded at item C5 of* [8] *is witnessed at every $n$ — including at $n = 6$, where the example* [8] *points at is not in fact a witness (Theorem 1.3).*

*Proof.* Apply Theorem 3.2 with $m = n-1$; the upper bound $n-1$ is [3, Theorem 1], and the witness shows it is attained. □

Both halves of Theorem 3.2 are uniform in $m$: no case is checked by evaluation, at any $m$. We record the two negative controls that guard the statement against a misplaced quantifier, at the value of $n$ that the source discusses.

**Proposition 3.4.** *The path medium on six states has reset threshold neither* 6 *nor* 4*.*

*Proof.* Not 6, because the explicit word of length 5 from the upper-bound half resets it; not 4, because Lemma 2.5 forbids a reset word of length 4 on six states when every letter moves at most one state. □

*Remark* 3.5. The same two lines apply to any medium whose graph is a tree: in a tree every $\Theta$-class is a single edge, so every token moves exactly one state, Lemma 2.5 gives $\mathrm{rt} \geq n-1$, and [3, Theorem 1] gives $\mathrm{rt} \leq n-1$. Only the path case is part of the formal development; the general tree statement is not, and §7 reports computational evidence that the trees are the *only* media attaining $n-1$.

## 4. The six-state example of Eppstein and Falmagne

**Definition 4.1.** Let $H \subseteq \{0,1\}^3$ be the family of all one- and two-element subsets of $\{0,1,2\}$, that is $H = \{\,\{0\},\{1\},\{2\},\{0,1\},\{0,2\},\{1,2\}\,\}$. It is a well-graded family in the 3-cube using every coordinate; $\mathcal{M}(H)$ has six states and $\tau = 6$ tokens, and its graph is the hexagon.

**Theorem 4.2.** *$\mathcal{M}(H)$ is a medium with six states and six tokens, and* $\mathrm{rt}(\mathcal{M}(H)) = 4$*.*

*Proof.* Medium: Proposition 2.3. Six states: the three singletons and the three two-element subsets.

*Upper bound.* The word $0^+ \, 1^- \, 0^+ \, 2^+$ of length 4 is a reset word; this is checked by evaluating the action on each of the six states, a computation the kernel performs by unfolding the definitions.

*Lower bound.* No word of length 3 resets $\mathcal{M}(H)$. This is a finite search: the alphabet has six tokens, so there are $6^3 = 216$ words of length 3, and each is applied to all six states and rejected. The search is exhaustive and was run by compiled evaluation inside the proof assistant (see §8); it was independently reproduced by a breadth-first search over image sets outside the formal development. Words of length at most 2 are excluded too: appending letters to a reset word gives a reset word, so a short reset word would pad out to one of length 3. □

**Proposition 4.3.** $\mathrm{rt}(\mathcal{M}(H)) \neq 5$. *Hence the well-graded family of all one- and two-element subsets of a three-element set is not an example of a medium with reset threshold* $n-1$.

*Proof.* A reset word of length 4 exists, so the minimum is at most $4 < 5$. □

*Remark* 4.4. Three separate assertions should be kept apart. (1) Eppstein and Falmagne assert that the bound $n-1$ is tight; that assertion is correct, by Corollary 3.3. (2) [8] reports (1) accurately, and adds that an example is provided only for $n = 6$. (3) The $n = 6$ example that [3] does provide is not an example of tightness: its reset threshold is $4 = n - 2$. It is provided against $\tau/2 = 3$, and it is a correct counterexample to that: $4 > 3$. What is missing from the printed record — and what §3 supplies — is an example attaining $n - 1$.

## 5. Three token pairs: $E(3) = 4$

The remark of [3] quoted in §1 asks, in effect, for the maximum of rt in terms of $\tau$ rather than $n$. Their hope $\mathrm{rt} \leq \tau/2$ is $E(d) \leq d$ in the notation of Definition 2.4, and their six-state example refutes it at $d = 3$. The exact value at $d = 3$ is 4, and both halves are proved here.

**Definition 5.1.** For $m \geq 2$ let $C^{(m)} \subseteq \{0,1\}^m$ consist of the vertices whose support is a down-set or an up-set of the linear order $0 < 1 < \cdots < m-1$; that is, the $2m$ subsets $\emptyset, \{0\}, \{0,1\}, \ldots, \{0, \ldots, m-1\}$ and $\{m-1\}, \{m-2, m-1\}, \ldots$. It is a well-graded family in the $m$-cube using every coordinate, and the graph of $\mathcal{M}(C^{(m)})$ is the even cycle $C_{2m}$. For $m = 3$ the graph is the hexagon again, in a different embedding from that of Definition 4.1.

**Proposition 5.2.** $\mathrm{rt}\big(\mathcal{M}(C^{(2)})\big) = 2$ *and* $\mathrm{rt}\big(\mathcal{M}(C^{(3)})\big) = 4$*; both are media, on* 4 *and* 6 *states respectively. In the second case* $\mathrm{rt} = \tau - 2 = 4$ *exceeds the value* $\tau/2 = 3$ *hoped for in* [3].

*Proof.* Medium in both cases: Proposition 2.3. For $m = 2$ the word $0^+ \, 1^+$ resets, and none of the 4 words of length 1 does — a kernel computation. For $m = 3$ the word $0^+ \, 1^+ \, 0^+ \, 2^+$ resets (every state is sent to $\{0,1,2\}$), and none of the $6^3 = 216$ words of length 3 does; the latter is an exhaustive search run by compiled evaluation, as in Theorem 4.2. □

**Theorem 5.3.** *Every medium with three* $\Theta$*-classes — equivalently with* $\tau = 6$ *tokens — has a reset word of length* 4*; and some such medium, namely* $\mathcal{M}(C^{(3)})$*, has none of length* 3*. Hence*

$$E(3) \;=\; 4 \;=\; 2 \cdot 3 - 2 \;>\; 3 \;=\; \tau/2.$$

*Here "every medium with three* $\Theta$*-classes" means every token automaton of a well-graded family in the* 3*-cube using all three coordinates; by the equivalence of* [3]*, which we cite and do not reprove, these are all media with three* $\Theta$*-classes up to isomorphism.*

*Proof.* The lower half is Proposition 5.2 (or Theorem 4.2, which gives the same value on an isomorphic medium).

The upper half is an exhaustive search over the class, and we describe exactly what was run. The 3-cube has $2^3 = 8$ vertices, hence $2^8 = 256$ subsets $S \subseteq \{0,1\}^3$. For each of the 256 the two conditions of Definition 2.1 are tested — (a) well-gradedness, a check over the ordered pairs of distinct elements of $S$, and (b) that all three coordinates are used — and the subsets failing either test are discarded. For each surviving $S$ the $6^4 = 1296$ words of length 4 in the six tokens of $\mathcal{M}(S)$ are searched until one is found that maps all of $S$ to a single vertex. Every one of the surviving subsets yields such a word, which is the claim. The whole of this — the 256 filters and the 1296-word searches for the survivors — is one exhaustive computation, run by compiled evaluation inside the proof assistant, and it is what discharges the theorem; the passage from "all subsets of the cube that pass the two tests" to "all media with three Θ-classes" is Proposition 2.3 in one direction and the cited equivalence of [3] in the other. □

*Remark* 5.4. $E(1) = 1$ and $E(2) = 2$ by the same style of exhaustion in dimensions 1 and 2; the formula $E(d) = 2d - 2$ therefore begins at $d = 2$. Only the case $d = 3$ is stated as a theorem here.

## 6. A control: Černý's automaton is not a medium

The bound of item C5 is far below the general Černý bound, so it is worth confirming that the hypothesis in it is doing real work.

**Proposition 6.1.** *Let $C_3$ be Černý's three-state automaton on $\{0, 1, 2\}$ with letters $a$ and $b$, where $0 \cdot a = 1$ and $q \cdot a = q$ for $q \neq 0$, and $q \cdot b = q + 1 \bmod 3$. Then $\mathrm{rt}(C_3) = 4 = (3-1)^2$, and $C_3$ is not a medium: no letter of it has a reverse, so axiom (i) already fails.*

*Proof.* The word *abba* resets $C_3$, and none of the $2^3 = 8$ words of length 3 does; both are kernel computations over a three-state automaton. For the second assertion, the reverse condition is a statement about the 6 ordered pairs of distinct states and the 4 ordered pairs of letters, and it fails for every pair; what the formal development records is the consequence, that axiom (i) fails. □

So $\mathrm{rt}(C_3)$ is twice the value $n - 1 = 2$ that item C5 gives for media on three states, and the gap is not an accident of size: the medium axioms are what make the linear bound possible. Černý's series and the value $(n-1)^2$ are classical [1] and are quoted in [8]; only the two statements above are ours.

## 7. Computations outside the formal development

This section reports numbers obtained by ordinary programs, not inside the proof assistant. Nothing in §§3–6 depends on any of them. They are included because they place the theorems above in their range and because several of them appear to be new. The programs are: a generator of connected bipartite graphs (`nauty-geng` [6]), a filter in C that computes the Djoković–Winkler classes of a graph, builds the induced hypercube labelling and keeps the graph when that labelling is isometric, and a breadth-first search over image subsets that computes rt. A second, independent implementation in Python was used as a cross-check.

**A census of media.** Since media are, up to isomorphism, the token automata of well-graded families, and the graphs of the latter are exactly the partial cubes, counting media on $n$ states is counting partial cubes on $n$ vertices up to isomorphism.

*Remark* 7.1 (computation, not part of the formal development). The number of media on $n$ states, for $n = 1, \ldots, 14$, is

$$1,\ 1,\ 1,\ 3,\ 4,\ 12,\ 25,\ 79,\ 212,\ 731,\ 2427,\ 9098,\ 34614,\ 140326.$$

The first eight terms are exactly the published terms of OEIS `A343163`, a sequence entered after a communication from D. E. Knuth and carrying the keyword `more` — as of 3 September 2026 it still lists exactly those eight; the ninth, 212, is the first entry of row 9 of the associated triangle `A343168`, whose published data run to that row. The remaining five terms appear to be new. The generation used the density bound of [3, Lemma 4]: a medium has at most $n \log_2 n$ effective transitions, that is at most $\frac{1}{2} n \log_2 n$ edges, a bound attained by the hypercubes, so the cap discards nothing; $n = 14$ meant filtering 21,806,966 generated graphs. Two checks were run besides the agreement with `A343163`: the two implementations agree on every $n \leq 9$, and the number of media in the census whose graph is a tree is $3, 6, 11, 23, 47, 106, 235, 551, 1301, 3159$ for $n = 5, \dots, 14$, which is A000055, the number of unlabelled trees.

**Which media attain $n - 1$.**

*Remark* 7.2 (computation, not part of the formal development). Over the census of Remark 7.1, the maximum of rt over media on $n$ states is $n-1$ for every $n \le 14$, and it is attained *exactly* by the tree media; the maximum over the non-tree media is exactly $n-2$ for $4 \le n \le 14$. The direction "tree $\Rightarrow \mathrm{rt} = n-1$" is Remark 3.5, of which the path case is Theorem 3.2; the converse, "non-tree $\Rightarrow \mathrm{rt} \le n-2$", is at present a conjecture supported by the census and by nothing else. If it holds in general, item C5 sharpens to: the media attaining the bound $n-1$ are precisely the tree media.

| $n$ | media | max rt | max rt over non-trees | tree media |
|---:|---:|---:|---:|---:|
| 1 | 1 | 0 | — | 1 |
| 2 | 1 | 1 | — | 1 |
| 3 | 1 | 2 | — | 1 |
| 4 | 3 | 3 | 2 | 2 |
| 5 | 4 | 4 | 3 | 3 |
| 6 | 12 | 5 | 4 | 6 |
| 7 | 25 | 6 | 5 | 11 |
| 8 | 79 | 7 | 6 | 23 |
| 9 | 212 | 8 | 7 | 47 |
| 10 | 731 | 9 | 8 | 106 |
| 11 | 2427 | 10 | 9 | 235 |
| 12 | 9098 | 11 | 10 | 551 |
| 13 | 34614 | 12 | 11 | 1301 |
| 14 | 140326 | 13 | 12 | 3159 |

**How far rt can exceed $\tau/2$.**

*Remark* 7.3 (computation, not part of the formal development). For $2 \le m \le 10$ the even-cycle medium $\mathcal{M}(C^{(m)})$ of Definition 5.1 has $2m$ states, $\tau = 2m$ tokens and

$$\mathrm{rt}\big(\mathcal{M}(C^{(m)})\big) \;=\; 2m-2 \;=\; \tau - 2,$$

computed by breadth-first search over image subsets (so up to $C_{20}$, on 20 states). The reset word is uniform in $m$ — $0^+ \, 1^+ \cdots (m-2)^+ \, (m-3)^+ \cdots 0^+ \, (m-1)^+$, of length $2m-2$ — and only the matching lower bound is obtained by search. So the excess $\mathrm{rt} - \tau/2 = m-2$ grows linearly across the range checked and reaches 8 at $C_{20}$: the hexagon of [3] is the first member of a family whose shortfall against the hoped-for $\tau/2$ does not stay bounded by 1. Whether it is bounded by any constant at all is exactly the question whether $\mathrm{rt}(C_{2m}) = 2m-2$ holds for every $m$, which we leave as a conjecture: both halves of that equality are checked here only for $m \le 10$, and nothing else in the paper uses it. Lemma 2.5 does not reach the lower half: a $\Theta$-class of a cycle consists of two antipodal edges, so each token of $\mathcal{M}(C^{(m)})$ moves two states rather than one, and the counting argument that gives $n-1$ for the trees gives only $\lceil (2m-1)/2 \rceil = m$ here. The cases $m = 2, 3$ are Proposition 5.2 and are part of the formal development; $m \ge 4$ is not.

*Remark* 7.4 (computation, not part of the formal development). A second, independent census — all subsets of the $d$-cube containing a fixed vertex, filtered for well-gradedness and coordinate use — gives

$$E(1) = 1, \qquad E(2) = 2, \qquad E(3) = 4, \qquad E(4) = 6,$$

so $E(d) = 2d-2$ for $2 \le d \le 4$, attained in each case by the even cycle $C_{2d}$; the numbers of families enumerated are $1, 4, 51, 3189$ respectively. The value $E(3) = 4$ is Theorem 5.3 and is part of the formal development; $E(4) = 6$ is not. Moreover no medium anywhere in the census of Remark 7.1 has $\mathrm{rt} > 2d-2$, which makes $E(d) = 2d-2$ for $d \ge 2$ the natural conjecture. This census and the census of Remark 7.1 agree on every pair $(n, d)$ they share.

**What the finite checks cost.**

*Remark* 7.5 (measurement, not part of the formal development). The finite searches above sit close to a hard boundary, and it is worth recording where it runs. Inside the proof assistant, exhausting all 216 words of length 3 over the six tokens of a medium with three Θ-classes takes about 13 seconds of compiled evaluation. The corresponding check one step up — all $8^5 = 32768$ words of length 5 over the eight tokens of $C_8$, which is what a formal proof of $\mathrm{rt}(C_8) = 6$ would need — did not finish: one eighth of it, 4096 words with the first token fixed, ran for more than ten minutes without returning, so the whole lower bound is at least two processor-hours and was not attempted. The apparently cheaper route is worse: a breadth-first search over image sets carried as a set of sets of cube vertices touches 137 sets rather than 32768 words at $C_8$, yet on the hexagon it returns the 27 sets at depth 3 instantly and does not return at depth 4 within four minutes, because every insertion compares cube vertices as functions and the deduplication is quadratic in the frontier. The affordable redesign, not carried out here, is a bitmask encoding of cube vertices with a proved bridge to the functional one. All figures were measured on a single machine, with Lean's own code running through its IR interpreter rather than a compiled shared library; they are orders of magnitude, not benchmarks.

## 8. Formal verification

Every lemma, proposition, theorem and corollary of §§2–6 is machine-checked in Lean 4 [4] against Mathlib [5], and every definition there is transcribed; the exceptions, labelled as such where they occur, are the remarks of §7, the general tree statement of Remark 3.5, the values $E(1)$ and $E(2)$ of Remark 5.4, the identification of the Θ-classes with the coordinate directions in §2 — which is terminological, the formal statements naming the dimension $d$ throughout — the general-$m$ assertions of Definition 5.1, formalised only at $m = 2, 3$, the descriptions of transition graphs as paths, cycles and hexagons, which are glosses — the formal statements name the token action and not a graph — and the bibliographic and historical statements of §1. Volkov's four axioms are transcribed from the quotation in §2 rather than restated, and an automaton is a bare transition function $Q \times \Sigma \to Q$, so that $\mathrm{rt}(\mathcal{A}) = k$ unfolds to "a reset word of length $k$ exists and every reset word has length at least $k$". There is no incomplete proof and no added axiom.

Lemma 2.5, Proposition 2.3, Theorem 3.2, Corollary 3.3, Proposition 3.4, Proposition 4.3, Proposition 6.1 and the case $m = 2$ of Proposition 5.2 depend only on the standard axioms of the system, with no compiled evaluation — which is the formal counterpart of Theorem 3.2 and Proposition 2.3 holding at every $n$ and every $d$ rather than at checked values. Exactly three statements use compiled evaluation, one axiom each, and each is a finite exhaustion described in the proof where it occurs: the 216-word search behind the lower bound of Theorem 4.2; the 216-word search behind the case $m = 3$ of Proposition 5.2; and the 256-subset, 1296-word exhaustion behind the upper half of Theorem 5.3. No other statement in the paper rests on compiled evaluation, and the statements that make those searches mean their theorems — that the enumeration of words of a given length is complete, and that a reset word for the token action on labels is a reset word for the automaton — use none of it. The whole development compiles in about three minutes of processor time. The repository is private; repository reference to be added at submission.

## References


[1] J. Černý, *Poznámka k homogénnym experimentom s konečnými automatmi*, Matematicko-fyzikálny časopis Slovenskej Akadémie Vied **14** (1964), 208–216; English translation: *A note on homogeneous experiments with finite automata*, Journal of Automata, Languages and Combinatorics **24** (2019), no. 2–4, 123–132, DOI `10.25596/jalc-2019-123`. Not read directly; cited here only for the automata $C_n$ and the value $(n-1)^2$, both of which are quoted in [8]. The page range of the 1964 note is the one given both by [8] and by the bibliography of [3]; part of the literature gives 208–215 instead. The details of the 2019 translation are from the publisher's page for that issue.

[2] D. Ž. Djoković, *Distance-preserving subgraphs of hypercubes*, Journal of Combinatorial Theory, Series B **14** (1973), no. 3, 263–267; DOI `10.1016/0095-8956(73)90010-5` (details from Crossref). Cited only for the origin of the relation Θ; not read.

[3] D. Eppstein and J.-C. Falmagne, *Algorithms for media*, Discrete Applied Mathematics **156** (2008), no. 8, 1308–1320; DOI `10.1016/j.dam.2007.05.035` (volume, issue, pages, year and doi confirmed against Crossref and against the first author's own publication list, which gives `arXiv:cs/0206033` as the preprint of this article). All quotations and all numbered references in this paper are to that preprint, which we compiled from its own LaTeX source so as to resolve its numbering exactly; the journal version was not read. In the preprint the reset-sequence theorem is Theorem 1, the equivalence between oriented media and well-graded families is Lemma 3, and the density bound is Lemma 4; [8] cites the journal version for the same reset-sequence result as its "Theorem 1".

[4] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, in: Automated Deduction — CADE 28, Lecture Notes in Computer Science **12699**, Springer, 2021, pp. 625–635; DOI `10.1007/978-3-030-79876-5_37`.

[5] The mathlib Community, *The Lean mathematical library*, in: Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020), ACM, 2020, pp. 367–381; DOI `10.1145/3372885.3373824`.

[6] B. D. McKay and A. Piperno, *Practical graph isomorphism, II*, Journal of Symbolic Computation **60** (2014), 94–112; DOI `10.1016/j.jsc.2013.09.003` (details from Crossref). The generator `geng` of the `nauty` distribution, version 2.7r3 — the program `nauty-geng` of the Debian package `nauty 2.7r3+ds-1` — was used for the census of §7.

[7] OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, `https://oeis.org`. Sequences `A343163` (column 1 of `A343168`: graphs on $n$ nodes isometrically embeddable in a hypercube, that is partial cubes), `A343168` (the triangle of isometric embeddings into Hamming graphs), `A343162` (its row sums) and `A000055` (unlabelled trees), all consulted on 3 September 2026. The first three were entered by N. J. A. Sloane in April 2021 from an email of D. E. Knuth, and `A343162` refers for the topic to D. E. Knuth, *The Art of Computer Programming*, Volume 4B, §7.2.2.3, where an exercise on it was announced; the entry, written while that volume was in preparation, gives no exercise number.

[8] M. V. Volkov, *List of Results on the Černý Conjecture and Reset Thresholds for Synchronizing Automata*, arXiv: 2508.15655, cs.FL, 29 pages; no journal reference. All quotations are from the LaTeX source of version 4 (13 January 2026); the item quoted is C5$^\dagger$ (Media), together with the global conventions on synchronization and the convention governing the dagger. Re-checked on 3 September 2026: the abstract page still lists v4 as the latest of four versions, the source file is byte-identical to the one quoted, and item C5 reads as above. This is a living list and a reader should check the current version.

[9] P. M. Winkler, *Isometric embedding in products of complete graphs*, Discrete Applied Mathematics **7** (1984), no. 2, 221–225; DOI `10.1016/0166-218X(84)90069-6` (details from Crossref). Cited only for the origin of the relation $\Theta$; not read.