THE SUMSET SIZES OF FOUR-ELEMENT SETS OF INTEGERS: $\mathcal{R}_{\mathbb{Z}}(h,4)$ FOR $h \le 6$ AND NATHANSON'S OPEN PROBLEMS

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Abstract. For a finite set A of integers let hA be the set of sums of h not necessarily distinct elements of A, and let $\mathcal{R}_{\mathbb{Z}}(h,k) = \{|hA| : A \subseteq \mathbb{Z},\ |A| = k\}$. Nathanson determined $\mathcal{R}_{\mathbb{Z}}(h,3)$ for every h and wrote that for $k = 4$ "the problem is still open: Compute $\mathcal{R}_{\mathbb{Z}}(h,4) \subseteq [3h+1, \binom{h+3}{3}]$". The case $h = 3$, $\mathcal{R}_{\mathbb{Z}}(3,4) = [10,20] \setminus \{11\}$, is in the literature (Nathanson; Rajagopal). We determine the next three values exactly: $\mathcal{R}_{\mathbb{Z}}(4,4) = [13,35] \setminus \{14,15,18,20,22,28\}$, $\mathcal{R}_{\mathbb{Z}}(5,4) = [16,56] \setminus \{17,18,19,22,23,25,28,40,49\}$ and $\mathcal{R}_{\mathbb{Z}}(6,4) = [19,84] \setminus \{20,21,22,23,26,27,28,30,32,34,36,44,50,55,77\}$. Of the thirty gaps, fifteen are Rajagopal's published set $\Delta_{h,4}$, one (32 at $h = 6$) follows from a diameter bound he sketches, three are the value $5h$ that he conjectured from experiment to be a gap for every $h \ge 4$, and eleven (22, 28; 28, 40, 49; 34, 36, 44, 50, 55, 77) appear to be new; two further patterns, $6h - 2$ and $\binom{h+3}{3} - 7$, are gaps at $h = 4, 5, 6$ and not at $h = 3$. We also answer Problems 1 and 2 of Nathanson's paper on tetrahedral differences for $h \le 8$ and $h \le 7$ respectively, and show that the family $\{0,1,a,b\}$ of his Problem 2 realises every size in $\mathcal{R}_{\mathbb{Z}}(h,4)$ except the B_h value for $h = 3,4,5$ but not for $h = 6$, where it misses $39 = |6\{0,2,5,7\}|$. The method replaces the diameter box of size $\approx 2.8 \times 10^{15}$ that the literature proposes for $h = 5$ by a certificate indexed by the finite difference set $T_h - T_h$ of the exponent tetrahedron: the size of hA depends on $A = \{0,a,b,c\}$ only through the set of exponent differences orthogonal to (a,b,c), and that set is either trivial, or determined by a cross product of two differences, or by a single difference. All statements are verified in Lean 4 against Mathlib.

1. Introduction

Throughout, A is a finite set of integers, $h \ge 1$, and

$$hA = \{a_1 + \cdots + a_h : a_i \in A\}$$

is its h-fold sumset (the summands need not be distinct). If $|A| = k$ then

$$h(k-1)+1 \ \le\ |hA| \ \le\ \binom{h+k-1}{k-1}, \tag{1}$$

with equality on the left exactly when A is an arithmetic progression [7] and on the right exactly when A is a B_h-set, i.e. when every element of hA has a unique representation up to the order of the summands. Sets near either end of (1) are classical objects — Freiman's theory of small doubling [9, 8] on the left, Sidon and B_h-sets on the right — but the *range* of $|hA|$ over all k-element sets was studied systematically only recently. Following Nathanson [2, 1] write

$$\mathcal{R}_{\mathbb{Z}}(h,k) = \{\, |hA| : A \subseteq \mathbb{Z},\ |A| = k \,\},$$

so that (1) reads $\mathcal{R}_{\mathbb{Z}}(h,k) \subseteq [h(k-1)+1, \binom{h+k-1}{k-1}]$, where $[u,v]$ denotes an interval of integers. Since $|hA|$ is invariant under affine maps $A \mapsto \lambda A + \mu$ ($\lambda \ne 0$), one has $\mathcal{R}_{\mathbb{Z}}(h,1) = \{1\}$, $\mathcal{R}_{\mathbb{Z}}(1,k) = \{k\}$, $\mathcal{R}_{\mathbb{Z}}(h,2) = \{h+1\}$; Erdős and Szemerédi [6] stated, and Nathanson [2] proved, that $\mathcal{R}_{\mathbb{Z}}(2,k) = [2k-1, \binom{k+1}{2}]$; and Nathanson [2, Theorem 9] proved

$$\mathcal{R}_{\mathbb{Z}}(h,3) = \left\{ \binom{h+2}{2} - \binom{t}{2} : t \in [1,h] \right\}$$

for every h, so that $\mathcal{R}_{\mathbb{Z}}(3,3) = \{7,9,10\}$ and, in general, the sizes of hA for $|A| = 3$ are differences of triangular numbers. The founding source of the present paper, Nathanson's *Additive sumset sizes with tetrahedral differences* [1], proves that for $|A| = 4$ the h numbers $\binom{h+3}{3} - \binom{i_0+2}{3}$, $i_0 \in [0,h-1]$, are always sumset sizes — they are the "popular" sizes seen in experiments [12, 15] — and states, immediately after the $k = 3$ formula (Section 1 of the compiled e-print of version 2):

"For $k = 4$, the problem is still open: Compute $\mathcal{R}_{\mathbb{Z}}(h, 4) \subseteq \left[3h + 1, \binom{h+3}{3}\right]$."

The case $h = 3$ of this problem is in fact settled in the literature: Nathanson [2, Theorem 8, eq. (21)] proves $\mathcal{R}_G(3, 4) = \{10\} \cup [12, 20]$ for every ordered abelian group G, by explicit constructions of sets of integers, and Rajagopal [3, Theorem 1.7] proves $\mathcal{R}(3, k) = \{3k - 2\} \cup [3k, \binom{k+2}{3}]$ for all $k > 2$ by a nonconstructive argument; at $k = 4$ both give $\mathcal{R}_{\mathbb{Z}}(3, 4) = [10, 20] \setminus \{11\}$. The first open cases are therefore $h \geq 4$, and $h = 5$ is posed separately, as "Compute $\mathcal{R}_{\mathbb{Z}}(5, 4)$", in [12]. The same question is Problem 10 of [2] — "The structure of sumsets of sets of four integers is still not understood. Compute the set of sumset sizes $\mathcal{R}(h, 4)$. What is the cardinality of this set?" — and it is open in 2026: Chen and Yang [19] restate that problem verbatim, and Nathanson's 2026 problem survey [17] gives it a section of its own, "The case $k = 4$", recording the containment $\mathcal{R}_{\mathbb{Z}}(h, 4) \subseteq \{3h + 1\} \cup \left[4h, \binom{h+3}{3}\right]$, asking "Compute $\mathcal{R}(h, 4)$ for all h", and asking elsewhere for $\mathcal{R}(4, k)$ for all k.

Results. We determine $\mathcal{R}_{\mathbb{Z}}(h, 4)$ exactly for $h = 4, 5, 6$, and re-derive the published $h = 3$ case by the same method.

Theorem 1.1.

$$\begin{aligned}
\mathcal{R}_{\mathbb{Z}}(3,4) &= [10,20] \setminus \{11\}, & |\mathcal{R}_{\mathbb{Z}}(3,4)| &= 10,\\
\mathcal{R}_{\mathbb{Z}}(4,4) &= [13,35] \setminus \{14,15,18,20,22,28\}, & |\mathcal{R}_{\mathbb{Z}}(4,4)| &= 17,\\
\mathcal{R}_{\mathbb{Z}}(5,4) &= [16,56] \setminus \{17,18,19,22,23,25,28,40,49\}, & |\mathcal{R}_{\mathbb{Z}}(5,4)| &= 32,\\
\mathcal{R}_{\mathbb{Z}}(6,4) &= [19,84] \setminus \{20,21,22,23,26,27,28,30,32,34,36,44,50,55,77\}, & |\mathcal{R}_{\mathbb{Z}}(6,4)| &= 51.
\end{aligned}$$

Theorem 1.1 is Theorems 3.1, 3.3, 3.5 and 3.7 below; the lower halves are explicit witness sets (Table 2), the upper halves are finite certificates described in Section 2. The thirty gaps of the three new cases sort as in Table 1.

h	$\Delta_{h,4}$ [3]	$5h$ [3, §5.2]	$[5h+2, 6h-4]$ [3, §5.2]	not found
3	11	— ($15 \in \mathcal{R}_{\mathbb{Z}}(3,4)$)	—	—
4	14, 15, 18	20	—	22, 28
5	17, 18, 19, 22, 23	25	—	28, 40, 49
6	20, 21, 22, 23, 26, 27, 28	30	32	34, 36, 44, 50, 55, 77

Table 1. The gaps of $\mathcal{R}_{\mathbb{Z}}(h, 4)$ in $[3h + 1, \binom{h+3}{3}]$, by provenance. Second column: Rajagopal's set $\Delta_{h,k}$, published as a theorem (its $\ell = 0$ row is also Schinina's [4, Theorem 4]; the $\ell = 0, 1, 2$ rows are Tang–Xing's [5] for $k \geq 5$, which is not the case here). Third column: conjectured by Rajagopal from experimental data for every $h \geq 4$, proved here at $h = 4, 5, 6$. Fourth column: a consequence of a diameter bound Rajagopal sketches for $h > k + 1$ (empty at $h = 4, 5$). Last column: gaps we found in no source.

What is new and what is not. The objects are elementary and we are careful about attribution.

- $\mathcal{R}_{\mathbb{Z}}(3, 4)$ (Theorem 3.1) is Nathanson's [2, Theorem 8] and Rajagopal's [3, Theorem 1.7]; $11 \notin \mathcal{R}_{\mathbb{Z}}(3, 4)$ is also the case $(h, k) = (3, 4)$ of Nathanson's theorem that $hk - h + 2 \notin \mathcal{R}_G(h, k)$ for $h, k \geq 3$ [2, Theorem 7], and of Schinina's theorem that $\mathcal{R}_G(h, k) \cap [hk - h + 2, hk - 1] = \emptyset$ for $k \geq 4$ [4, Theorem 4]. (Tang–Xing [5, Theorems 1.1 and 1.2] rule out more of the same intervals, but both of their theorems assume $k \geq 5$ and so say nothing at $k = 4$.) We claim only the formal verification.
- Rajagopal [3, Definition 1.2, Theorem 1.3] proves $\mathcal{R}(h, k) \cap \Delta_{h,k} = \emptyset$ with $m = hk - h + 1$ and

$$\Delta_{h,k} = \bigcup_{\ell=0}^{\min\{h,k\}-3} \left[m + \ell h + 1,\ m + \ell h + (h - 2 - \ell)\right],$$

which at $k=4$, $h\ge 4$ is $[3h+2,4h-1]\cup[4h+2,5h-2]$: the sixteen values of the second column of Table 1 are his theorem, and our computation reproducing each of them is an independent check on the computation rather than a new result.

- Rajagopal [3, §5.2] writes: "With $k=4$ and $h\ge 4$, experimental data suggests $5h\notin\mathcal{R}(h,4)$, which is an example of a conjectured gap." No proof or proof sketch is given there. Corollaries 3.4(ii), 3.6(ii) and 3.8(ii) prove it at $h=4,5,6$. The same section derives, for $h>k+1$, that $\mathcal{R}(h,k)\cap[h(2k-3)+2,\ h(2k-3)+h-k]=\emptyset$ from a diameter bound [3, Theorem 2.2] going back to Lev [10]; at $(6,4)$ the interval is $\{32\}$ (and it is empty for $h=4,5$). That gap is his, in a paragraph rather than a numbered theorem; Corollary 3.8(iii) is its first machine-checked proof.
- Rajagopal also records that "the only (conjectured) gaps in $\mathcal{R}(h,k)$ outside $\Delta_{h,k}$ we have found with $h\le k+1$ occur when $(h,k)\in\{(4,3),(4,4),(5,4)\}$" and that "similar analysis … will yield more gaps", without printing values. The eleven values in the last column of Table 1 were not found in any source (Section 6 records the search); the existence of gaps outside Δ at $(4,4)$ and $(5,4)$ was thus known, their values and the whole of the $h=6$ row were not.
- Problems 1 and 2 of [1] are finite for each h, and their answers (Propositions 4.3 and 4.4) are direct evaluations that anyone can redo; we record them because the source poses them and because Theorems 4.5 and 4.6 — the family of Problem 2 parametrises $\mathcal{R}_{\mathbb{Z}}(h,4)$ exactly for $h\le 5$ and not for $h=6$ — need both the evaluation and the exhaustive upper bound.
- The reduction of Section 2 is standard additive number theory (the relation lattice of $\{0,a,b,c\}$ and the identity $v\times(u_1\times u_2)=u_1(v\cdot u_2)-u_2(v\cdot u_1)$); we found it nowhere in the form used here, a certificate indexed by the finite difference set of the exponent tetrahedron rather than by a diameter box, but we regard it as routine.

Method. Translate so that $0\in A$; then $A=\{0,a,b,c\}$ and $hA=\{xa+yb+zc : x,y,z\ge 0,\ x+y+z\le h\}$ is the image of the tetrahedron $T_h=\{(x,y,z)\in\mathbb{N}^3 : x+y+z\le h\}$ under $t\mapsto t\cdot v$, $v=(a,b,c)$. Two lattice points of T_h collide exactly when their difference is orthogonal to v, and every such difference lies in the finite set $D_h=T_h-T_h$. So $|hA|$ is a function of the *relation set* $Z_h(v)=\{u\in D_h : u\cdot v=0\}$ alone, and $Z_h(v)$ is either $\{0\}$, or contains two non-parallel relations u_1,u_2 — in which case v is parallel to $u_1\times u_2$ and $|hA|$ equals the sumset size of the explicit vector $u_1\times u_2$ — or consists of relations all parallel to one $w\in D_h$, in which case it is $\{u\in D_h : u\times w=0\}$. Each branch is indexed by D_h, so $\mathcal{R}_{\mathbb{Z}}(h,4)$ is bounded above by two decidable checks over D_h: $|D_h|^2$ cross products and $|D_h|$ rank-one patterns, with $|D_h|=147,309,561,923$ for $h=3,4,5,6$. For comparison, the diameter bound of [11] that [12] proposes for $h=5$ confines the search to 4-element subsets of $[0,255\,999]$, about 2.8×10^{15} sets after normalisation.

Verification. Every theorem, proposition and corollary of Sections 2–5 whose statement is reproduced in Appendix A is machine-checked in Lean 4 against Mathlib; Section 6 records the modules, which proofs are kernel-checked and which rest on compiled evaluation (`native_decide`), and the cost. Everything outside the formal development is labelled as such where it occurs (Remarks 3.9 and 3.10, the literature attributions, and the timings).

2. The reduction to a finite certificate

Notation. $\mathbb{N}=\{0,1,2,\dots\}$. For $h\ge 0$ let

$$T_h=\{(x,y,z)\in\mathbb{N}^3 : x+y+z\le h\},\qquad |T_h|=\binom{h+3}{3},\qquad D_h=T_h-T_h=\{t-t' : t,t'\in T_h\}\subseteq\mathbb{Z}^3.$$

For $v=(a,b,c)\in\mathbb{Z}^3$ write $A_v=\{0,a,b,c\}$ and $\varphi_v(t)=t\cdot v=xa+yb+zc$. The cross product on $\mathbb{Z}^3$ is $u\times w=(u_2w_3-u_3w_2,\ u_3w_1-u_1w_3,\ u_1w_2-u_2w_1)$. The vector v is *admissible* if a,b,c are nonzero and pairwise distinct, i.e. if $|A_v|=4$. The relation set of v and the rank-one pattern of w are

$$Z_h(v)=\{u\in D_h : u\cdot v=0\},\qquad Z_h^{\parallel}(w)=\{u\in D_h : u\times w=0\},$$

and for $S \subseteq D_h$ the *class count* $\mathrm{cc}_h(S)$ is the number of distinct sets $\{t' \in T_h : t - t' \in S\}$ as t ranges over T_h (when S is the relation set of some v these are the fibres of φ_v, and $\mathrm{cc}_h(S)$ is their number). Let e_1, e_2, e_3 be the unit vectors.

Lemma 2.1 (the tetrahedron and the pattern lemma)**.** *Let $h \geq 0$ and $v \in \mathbb{Z}^3$.*

(i) *$hA_v = \varphi_v(T_h)$; hence $|hA_v| = |\varphi_v(T_h)|$.*
(ii) *$|h(A+m)| = |hA|$ for every finite $A \subseteq \mathbb{Z}$ and $m \in \mathbb{Z}$.*
(iii) *Every $A \subseteq \mathbb{Z}$ with $|A| = 4$ has $|hA| = |\varphi_v(T_h)|$ for some admissible v.*
(iv) *$|\varphi_v(T_h)| = \mathrm{cc}_h(Z_h(v))$. In particular $Z_h(v) = Z_h(v')$ implies $|\varphi_v(T_h)| = |\varphi_{v'}(T_h)|$.*

Proof. (i) is an induction on h using $T_{h+1} = T_h \cup (T_h + e_1) \cup (T_h + e_2) \cup (T_h + e_3)$ and $(h+1)A = A + hA$. (ii): $h(A + \{m\}) = hA + \{hm\}$. (iii): if $A = \{a_0 < a_1 < a_2 < a_3\}$ take $v = (a_1 - a_0, a_2 - a_0, a_3 - a_0)$ and use (ii). (iv): for $t \in T_h$ the fibre $\varphi_v^{-1}(\varphi_v(t)) \cap T_h$ equals $\{t' \in T_h : t - t' \in Z_h(v)\}$, because $t - t' \in D_h$ automatically; the fibres are in bijection with the image. □

Proposition 2.2 (the trichotomy)**.** *Let $h \geq 0$ and let v be admissible. Then at least one of the following holds.*

(B) *$|\varphi_v(T_h)| = |T_h| = \binom{h+3}{3}$;*
(C) *there are $u_1, u_2 \in D_h$ with $n = u_1 \times u_2 \neq 0$, n admissible, and $|\varphi_v(T_h)| = |\varphi_n(T_h)|$;*
(R) *there is $w \in D_h$, $w \neq 0$, with $Z_h(v) = Z_h^{\parallel}(w)$ and $|\varphi_v(T_h)| = \mathrm{cc}_h(Z_h^{\parallel}(w))$.*

(The three cases are not exclusive, and nothing below needs them to be: (C) asks only that the value *$|\varphi_v(T_h)|$ be one of the cross-product values, so at $h = 5$ the vector $v = (1, 2, 11)$, whose relation set is nonzero and rank-one, satisfies both (C) and (R). The proof puts each v in the first case that applies.)*

Proof. If $Z_h(v)$ contains u_1, u_2 with $n = u_1 \times u_2 \neq 0$: the identity $v \times (u_1 \times u_2) = u_1(v \cdot u_2) - u_2(v \cdot u_1)$ gives $v \times n = 0$, so $n_i v = v_i n$ for each coordinate i; choosing i with $n_i \neq 0$ (and using $v_i \neq 0$, which admissibility supplies) one gets $u \cdot v = 0 \iff u \cdot n = 0$ for *every* $u \in \mathbb{Z}^3$, hence $Z_h(v) = Z_h(n)$ and, by Lemma 2.1(iv), $|\varphi_v(T_h)| = |\varphi_n(T_h)|$; n is admissible because v is (test the six vectors e_i, $e_i - e_j$). Otherwise all pairs in $Z_h(v)$ are parallel. If $Z_h(v)$ has a nonzero element w, then $Z_h(v) \subseteq Z_h^{\parallel}(w)$ by that hypothesis, and conversely $u \times w = 0$, $w \neq 0$ and $w \cdot v = 0$ give $w_i(u \cdot v) = u_i(w \cdot v) = 0$ for all i, so $u \cdot v = 0$; this is (R). Otherwise $Z_h(v) = \{0\}$, φ_v is injective on T_h, and (B) holds. □

Proposition 2.3 (the finite certificate)**.** *Let $h \geq 0$ and let $X \subseteq \mathbb{N}$ be finite. Suppose*

(B) *$\binom{h+3}{3} \in X$;*
(C) *for all $u_1, u_2 \in D_h$ with $n = u_1 \times u_2 \neq 0$ and n admissible, $|\varphi_n(T_h)| \in X$;*
(R) *for every $w \in D_h$, $w \neq 0$: either one of $e_1, e_2, e_3, e_1 - e_2, e_1 - e_3, e_2 - e_3$ lies in $Z_h^{\parallel}(w)$, or $\mathrm{cc}_h(Z_h^{\parallel}(w)) \in X$.*

Then $|hA| \in X$ for every $A \subseteq \mathbb{Z}$ with $|A| = 4$.

Proof. By Lemma 2.1(iii) it suffices to treat $A = A_v$ with v admissible, and Proposition 2.2 lands in one of (B), (C), (R). In case (R), if some e_i or $e_i - e_j$ lies in $Z_h^{\parallel}(w) = Z_h(v)$ then $v_i = 0$ or $v_i = v_j$, contradicting admissibility; so the class count is in X. □

The escape clause in (R) is not decorative: Proposition 5.1(iii) shows that at $h = 4$ the pattern of $w = e_1$ (which is the pattern of the three-element set $\{0, b, c\}$ with b, c generic) has class count $15 \notin \mathcal{R}_{\mathbb{Z}}(4, 4)$, so the certificate without the clause would be false. Conditions (B), (C), (R) are decidable: (C) is a loop over the $|D_h|^2$ ordered pairs of D_h and (R) a loop over D_h, each evaluation being a count of distinct values over the $\binom{h+3}{3}$ points of T_h. Nothing in this section is specific to a particular h.

3. The exact values

For each h, the set R_h below is the right-hand side of Theorem 1.1. Each theorem has an upper half (every $|hA|$ lies in R_h), a lower half (every element of R_h is attained), and the resulting equality; the corollaries single out the gaps.

Theorem 3.1 ($h = 3$; published, here re-derived)**.** *Let $R_3 = \{10, 12, 13, \dots, 20\}$. For every $A \subseteq \mathbb{Z}$ with $|A| = 4$, $|3A| \in R_3$; every $n \in R_3$ is $|3A|$ for some such A; hence $\mathcal{R}_{\mathbb{Z}}(3, 4) = R_3 = [10, 20] \setminus \{11\}$.*

Corollary 3.2. *No $A \subseteq \mathbb{Z}$ with $|A| = 4$ has $|3A| = 11$.*

Theorem 3.1 is [2, Theorem 8, eq. (21)] and the case $k = 4$ of [3, Theorem 1.7]; Corollary 3.2 is also [2, Theorem 7] and [4, Theorem 4]. They are included because the same certificate proves them and because the source's controls (Section 5) are stated at $h = 3$.

Theorem 3.3 ($h = 4$)**.** *Let $R_4 = \{13, 16, 17, 19, 21, 23, 24, 25, 26, 27, 29, 30, 31, 32, 33, 34, 35\}$. For every $A \subseteq \mathbb{Z}$ with $|A| = 4$, $|4A| \in R_4$; every $n \in R_4$ is $|4A|$ for some such A; hence $\mathcal{R}_{\mathbb{Z}}(4, 4) = R_4 = [13, 35] \setminus \{14, 15, 18, 20, 22, 28\}$.*

Corollary 3.4. *Let $A \subseteq \mathbb{Z}$, $|A| = 4$. Then (i) $|4A| \notin \{14, 15, 18\}$ (this is $\Delta_{4,4}$* [3, Theorem 1.3]*); (ii) $|4A| \neq 20$ (the value $5h$ conjectured to be a gap in* [3, §5.2]*); (iii) $|4A| \notin \{22, 28\}$.*

Theorem 3.5 ($h = 5$)**.** *Let $R_5 = \{16, 20, 21, 24, 26, 27, 29, 30, \dots, 39, 41, \dots, 48, 50, \dots, 56\}$, i.e. $R_5 = [16, 56] \setminus \{17, 18, 19, 22, 23, 25, 28, 40, 49\}$. For every $A \subseteq \mathbb{Z}$ with $|A| = 4$, $|5A| \in R_5$; every $n \in R_5$ is $|5A|$ for some such A; hence $\mathcal{R}_{\mathbb{Z}}(5, 4) = R_5$.*

Corollary 3.6. *Let $A \subseteq \mathbb{Z}$, $|A| = 4$. Then (i) $|5A| \notin \{17, 18, 19, 22, 23\}$ (this is $\Delta_{5,4}$* [3, Theorem 1.3]*); (ii) $|5A| \neq 25$ (the value $5h$ of* [3, §5.2]*); (iii) $|5A| \notin \{28, 40, 49\}$.*

Theorem 3.7 ($h = 6$)**.** *Let $R_6 = [19, 84] \setminus \{20, 21, 22, 23, 26, 27, 28, 30, 32, 34, 36, 44, 50, 55, 77\}$, a set of 51 integers. For every $A \subseteq \mathbb{Z}$ with $|A| = 4$, $|6A| \in R_6$; every $n \in R_6$ is $|6A|$ for some such A; hence $\mathcal{R}_{\mathbb{Z}}(6, 4) = R_6$.*

Corollary 3.8. *Let $A \subseteq \mathbb{Z}$, $|A| = 4$. Then (i) $|6A| \notin \{20, 21, 22, 23, 26, 27, 28\}$ (this is $\Delta_{6,4}$* [3, Theorem 1.3]*); (ii) $|6A| \neq 30$ (the value $5h$ of* [3, §5.2]*); (iii) $|6A| \neq 32$ (the interval $[h(2k-3)+2, h(2k-3)+h-k] = \{32\}$ of* [3, §5.2]*); (iv) $|6A| \notin \{34, 36, 44, 50, 55, 77\}$.*

Proof of Theorems 3.1–3.7 and their corollaries. Upper halves. Proposition 2.3 with $X = R_h$. Condition (B) is $\binom{h+3}{3} = 20, 35, 56, 84 \in R_h$. Condition (C) was checked by exhaustive evaluation over all ordered pairs of D_h: $147^2 = 21\,609$, $309^2 = 95\,481$, $561^2 = 314\,721$ and $923^2 = 851\,929$ pairs for $h = 3, 4, 5, 6$; for each pair with $n = u_1 \times u_2 \neq 0$ and n admissible, the number of distinct values of $t \cdot n$ over T_h was computed and found in R_h. Condition (R) was checked over the $|D_h| - 1$ nonzero elements w of D_h: for each, either one of the six vectors e_i, $e_i - e_j$ has zero cross product with w or the number of distinct sets $\{t' \in T_h : (t - t') \times w = 0\}$, $t \in T_h$, lies in R_h. These three checks are the whole of the exhaustive computation; they are executed inside the proof assistant as compiled evaluation (Section 6).

Lower halves. For each $n \in R_h$, Table 2 lists a set $A = \{0, a, b, c\}$ with $|hA| = n$; each is verified by direct evaluation of $|\varphi_{(a,b,c)}(T_h)|$ and Lemma 2.1(i).

Corollaries. Each gap is an element of $[3h + 1, \binom{h+3}{3}] \setminus R_h$; the attributions in parentheses are literature statements and not part of the formal claim. ☐

Remark 3.9 (three patterns of gaps; outside the formal development). Reading the four rows of Theorem 1.1 as a table suggests three arithmetic patterns of gaps, each present at $h = 4, 5, 6$ and absent at $h = 3$:

	$h = 3$	$h = 4$	$h = 5$	$h = 6$
$5h$	$15 \in \mathcal{R}_{\mathbb{Z}}(3, 4)$	20 gap	25 gap	30 gap
$6h - 2$	$16 \in \mathcal{R}_{\mathbb{Z}}(3, 4)$	22 gap	28 gap	34 gap
$\binom{h+3}{3} - 7$	$13 \in \mathcal{R}_{\mathbb{Z}}(3, 4)$	28 gap	49 gap	77 gap

The first is Rajagopal's conjecture; the other two we found in no source. At $h = 5$ the second and third coincide ($6h - 2 = \binom{h+3}{3} - 7 = 28$). Together with $\Delta_{h,4}$ and the diameter gap 32, the three patterns account for all six gaps at $h = 4$, for eight of the nine at $h = 5$ (40 is unexplained) and for eleven of the fifteen at $h = 6$ (36, 44, 50, 55 are unexplained). A proof that $\binom{h+3}{3} - 7 \notin \mathcal{R}_{\mathbb{Z}}(h, 4)$ for all $h \geq 4$ would be a finite structural statement about 4-element sets of near-maximal sumset size; we have not attempted it, and nothing in this remark is a theorem of this paper.

$h=3$							
10	$\{0,1,2,3\}$	12	$\{0,1,2,4\}$	13	$\{0,1,3,4\}$	14	$\{0,1,2,5\}$
15	$\{0,1,2,6\}$	16	$\{0,1,4,5\}$	17	$\{0,1,4,6\}$	18	$\{0,1,5,7\}$
19	$\{0,1,5,8\}$	20	$\{0,1,7,11\}$				
$h=4$							
13	$\{0,1,2,3\}$	16	$\{0,1,2,4\}$	17	$\{0,1,3,4\}$	19	$\{0,1,2,5\}$
21	$\{0,1,4,5\}$	23	$\{0,1,4,6\}$	24	$\{0,1,3,7\}$	25	$\{0,1,5,6\}$
26	$\{0,1,5,7\}$	27	$\{0,1,3,9\}$	29	$\{0,1,5,8\}$	30	$\{0,1,7,9\}$
31	$\{0,1,6,10\}$	32	$\{0,3,7,12\}$	33	$\{0,1,7,11\}$	34	$\{0,1,8,13\}$
35	$\{0,1,11,15\}$						
$h=5$							
16	$\{0,1,2,3\}$	20	$\{0,1,2,4\}$	21	$\{0,1,3,4\}$	24	$\{0,1,2,5\}$
26	$\{0,1,4,5\}$	27	$\{0,1,2,6\}$	29	$\{0,1,4,6\}$	30	$\{0,1,2,7\}$
31	$\{0,1,5,6\}$	32	$\{0,2,5,7\}$	33	$\{0,1,5,7\}$	34	$\{0,1,3,8\}$
35	$\{0,1,2,10\}$	36	$\{0,1,6,7\}$	37	$\{0,1,5,8\}$	38	$\{0,1,6,8\}$
39	$\{0,1,6,9\}$	41	$\{0,1,7,9\}$	42	$\{0,1,3,12\}$	43	$\{0,1,6,10\}$
44	$\{0,1,8,10\}$	45	$\{0,1,9,11\}$	46	$\{0,1,7,11\}$	47	$\{0,1,8,11\}$
48	$\{0,1,10,13\}$	50	$\{0,1,8,13\}$	51	$\{0,1,9,14\}$	52	$\{0,2,11,16\}$
53	$\{0,1,11,15\}$	54	$\{0,2,13,16\}$	55	$\{0,1,13,18\}$	56	$\{0,1,16,22\}$
$h=6$							
19	$\{0,1,2,3\}$	24	$\{0,1,2,4\}$	25	$\{0,1,3,4\}$	29	$\{0,1,2,5\}$
31	$\{0,1,4,5\}$	33	$\{0,1,2,6\}$	35	$\{0,1,4,6\}$	37	$\{0,1,5,6\}$
38	$\{0,1,3,7\}$	39	$\{0,2,5,7\}$	40	$\{0,1,5,7\}$	41	$\{0,3,5,8\}$
42	$\{0,1,3,8\}$	43	$\{0,1,6,7\}$	45	$\{0,1,5,8\}$	46	$\{0,1,6,8\}$
47	$\{0,1,4,9\}$	48	$\{0,1,6,9\}$	49	$\{0,1,7,8\}$	51	$\{0,1,7,9\}$
52	$\{0,1,7,10\}$	53	$\{0,1,6,10\}$	54	$\{0,1,5,11\}$	56	$\{0,1,8,10\}$
57	$\{0,3,7,12\}$	58	$\{0,1,7,11\}$	59	$\{0,1,9,11\}$	60	$\{0,1,8,11\}$
61	$\{0,1,7,12\}$	62	$\{0,1,10,12\}$	63	$\{0,1,10,13\}$	64	$\{0,1,8,14\}$
65	$\{0,1,8,13\}$	66	$\{0,2,11,14\}$	67	$\{0,2,7,15\}$	68	$\{0,1,9,14\}$
69	$\{0,1,10,14\}$	70	$\{0,1,13,16\}$	71	$\{0,2,12,15\}$	72	$\{0,1,11,15\}$
73	$\{0,1,10,17\}$	74	$\{0,3,10,18\}$	75	$\{0,2,13,16\}$	76	$\{0,2,13,18\}$
78	$\{0,1,13,18\}$	79	$\{0,2,16,19\}$	80	$\{0,1,15,21\}$	81	$\{0,3,13,24\}$
82	$\{0,1,16,22\}$	83	$\{0,1,18,25\}$	84	$\{0,1,22,28\}$		

TABLE 2. A witness set A with $|hA|=n$ for every $n \in \mathcal{R}_{\mathbb{Z}}(h,4)$, $h=3,4,5,6$; these are the sets used in the formal proofs of the lower halves of Theorems 3.1–3.7.

Remark 3.10 ($h=7$; outside the formal development). The certificate at $h=7$ ($|D_7|=1415$, $|T_7|=120$) was priced from the measured $h=6$ branches at about 2.2 processor-hours and was not run. The record of this work contains a non-exhaustive search over $\{0,a,b,c\}$ with $c \le 45$ at $h=7$ in which $35=5h$, $40=6h-2$ and $113=\binom{10}{3}-7$ do not occur, consistent with all three patterns continuing. Evaluating the three checks of Section 2 at $h=7$ in ordinary (non-proof-assistant) code — done twice, in two independently written programs, while preparing this draft — returns the same three absences and eighteen gaps in all,

$$\{23,\ldots,27,\ 30,\ldots,33,\ 35,\ 37,\ 38,\ 40,\ 42,\ 51,\ 76,\ 88,\ 113\},$$

of which 23–27 and 30–33 are $\Delta_{7,4}$ and $37,38$ the interval $[5h+2,6h-4]$; each of the remaining 81 values in $[22,120]$ is $|7A|$ for an explicit $A=\{0,a,b,c\}$ with $c \le 60$, so the same computation would give $\mathcal{R}_{\mathbb{Z}}(7,4)$ exactly. None of this passed through the proof assistant and we claim nothing at $h=7$.

4. Nathanson's numbered problems

Section 3 of [1], "Open problems", poses three numbered problems (numbering read from the compiled e-print of version 2). Problems 1 and 2 are finite for each h; we answer them for $h \le 8$ and $h \le 7$ and then compare Problem 2 with Theorem 1.1. We first record two checks of the source's own numbers.

Proposition 4.1 (the printed census)**.** *$3\{0,1,2\} = \{0,1,2,3,4,5,6\}$, $3\{0,1,3\} = \{0,1,2,3,4,5,6,7,9\}$ and $3\{0,1,4\} = \{0,1,2,3,4,5,6,8,9,12\}$, of sizes 7, 9, 10.*

These are the three sumsets printed in Section 1 of [1], from which it reads off $\{7,9,10\} \subseteq \mathcal{R}_{\mathbb{Z}}(3,3)$; all three agree with the source element by element. The unnumbered Theorem of its Section 2 (the source declares it without a number) states that for $h \geq 1$ and $i_0 \in [0, h-1]$ the set $A = \{0, 1, h+1, (h+1-i_0)(h+1)\}$ has $|A| = 4$ and $|hA| = \binom{h+3}{3} - \binom{i_0+2}{3}$.

Proposition 4.2 (the Section-2 theorem, re-checked for $h \leq 12$)**.** *For every $h \in [1, 12]$ and every $i_0 \in [0, h-1]$ (78 pairs), $\big|h\{0, 1, h+1, (h+1-i_0)(h+1)\}\big| = \binom{h+3}{3} - \binom{i_0+2}{3}$.*

This is the source's theorem, proved there for all h; the proposition is a direct re-evaluation and claims nothing beyond it.

Problem 1. The source writes: "It is of interest to compute, for all $h \geq 3$ and all $p \in [0, h^2-1]$, the sumset sizes of the sets $A = \{0, 1, h+1, h^2+h+1-p\}$." (In the Section-2 theorem $p = 1+(i_0-1)(h+1)$, which is $-h$ at $i_0 = 0$; so the B_h member $c = (h+1)^2$ of that family lies just outside the range of Problem 1, whose largest value $\binom{h+3}{3}$ is nevertheless attained, at $p = 0$.)

Proposition 4.3 (Problem 1 for $h = 3, \ldots, 8$)**.** *For $h = 3, \ldots, 8$ the sequence $\big(|h\{0, 1, h+1, h^2+h+1-p\}|\big)_{p=0,\ldots,h^2-1}$ is as in Table 3.*

h	$\lvert hA_p \rvert$ for $p = 0, 1, \ldots, h^2-1$
3	20, 19, 19, 19, 19, 16, 16, 17, 16
4	35, 34, 34, 34, 34, 34, 31, 31, 31, 33, 31, 25, 25, 29, 26, 25
5	56, 55, 55, 55, 55, 55, 55, 52, 52, 52, 52, 54, 52, 46, 46, 46, 50, 50, 46, 36, 36, 43, 39, 38, 36
6	84, 83, 83, 83, 83, 83, 83, 83, 80, 80, 80, 80, 80, 82, 80, 74, 74, 74, 74, 79, 79, 74, 64, 64, 64, 72, 72, 69, 64, 49, 49, 61, 58, 52, 51, 49
7	120, 119, 119, 119, 119, 119, 119, 119, 119, 116, 116, 116, 116, 116, 116, 118, 116, 110, 110, 110, 110, 110, 115, 115, 110, 100, 100, 100, 100, 108, 111, 108, 100, 85, 85, 85, 98, 97, 96, 94, 85, 64, 64, 81, 79, 70, 72, 67, 64
8	165, 164, 164, 164, 164, 164, 164, 164, 164, 164, 161, 161, 161, 161, 161, 161, 161, 163, 161, 155, 155, 155, 155, 155, 155, 160, 160, 155, 145, 145, 145, 145, 145, 154, 157, 154, 145, 130, 130, 130, 130, 143, 147, 147, 140, 130, 109, 109, 109, 127, 131, 126, 122, 121, 109, 81, 81, 105, 102, 99, 87, 87, 84, 81

TABLE 3. Problem 1 of [1]: $|hA_p|$ for $A_p = \{0, 1, h+1, h^2+h+1-p\}$, listed by p.

Problem 2. The source writes: "For all $h \geq 3$, compute the set of sumset sizes of the sets $A = \{0, 1, a, b\}$ for $2 \leq a \leq h$ and $a+1 \leq b \leq ha+1$." Write

$$P_h = \big\{\, |h\{0,1,a,b\}| : 2 \leq a \leq h,\ a+1 \leq b \leq ha+1 \,\big\},$$

a set indexed by $12, 30, 60, 105, 168$ pairs (a, b) for $h = 3, \ldots, 7$. Since $b \leq h^2+1 < (h+1)^2$, no member of the family is a B_h-set, so $\binom{h+3}{3} \notin P_h$. Chen and Yang [19, Corollary 1] give a closed formula for $|h\{0,1,a,b\}|$ in terms of q and r, where $b = qa+r$ and $0 \leq r < a$, valid for every h exactly when $r = 0$ or $q + r \geq a$; the evaluation below is direct and does not use it.

Proposition 4.4 (Problem 2 for $h = 3, \ldots, 7$)**.**

$$\begin{aligned}
P_3 &= \{10, 12, 13, 14, 15, 16, 17, 18, 19\},\\
P_4 &= \{13, 16, 17, 19, 21, 23, 24, 25, 26, 27, 29, 30, 31, 32, 33, 34\},\\
P_5 &= [16, 55] \setminus \{17, 18, 19, 22, 23, 25, 28, 40, 49\},\\
P_6 &= [19, 83] \setminus \{20, 21, 22, 23, 26, 27, 28, 30, 32, 34, 36, 39, 41, 44, 50, 51, 55, 56, 68, 71, 77\},\\
P_7 &= [22, 119] \setminus \{23, 24, 25, 26, 27, 30, 31, 32, 33, 35, 37, 38, 40, 42, 46, 49, 51,\\
&\qquad 61, 67, 68, 72, 76, 77, 82, 88, 91, 101, 102, 107, 113\},
\end{aligned}$$

of sizes $9, 16, 31, 44, 68$.

(The formal statements list P_5, P_6, P_7 as explicit sets; the complements above are the same sets written in $[3h+1, \binom{h+3}{3}-1]$.) Comparing with Theorem 1.1:

Theorem 4.5 (Problem 2 parametrises $\mathcal{R}_{\mathbb{Z}}(h,4)$ for $h \le 5$)**.** *For $h = 3,4,5$ and every $n \in \mathbb{N}$: there is $A \subseteq \mathbb{Z}$ with $|A| = 4$ and $|hA| = n$ if and only if $n \in P_h \cup \{\binom{h+3}{3}\}$. That is, $\mathcal{R}_{\mathbb{Z}}(h,4) = P_h \cup \{\binom{h+3}{3}\}$ for $h = 3,4,5$.*

Theorem 4.6 (and not for $h = 6$)**.** *There is $A \subseteq \mathbb{Z}$ with $|A| = 4$ and $|6A| = 39$, namely $A = \{0,2,5,7\}$, while $39 \notin P_6$. Hence $\mathcal{R}_{\mathbb{Z}}(6,4) \ne P_6 \cup \{\binom{9}{3}\}$.*

Proof sketch. Propositions 4.1–4.4 are direct evaluations of $|hA|$ for the finitely many explicit sets in question (78 sets for Proposition 4.2; $9+16+25+36+49+64 = 199$ for Problem 1; 375 for Problem 2). Theorem 4.5 is Proposition 4.4 together with Theorems 3.1, 3.3, 3.5: the finite set identity $P_h \cup \{\binom{h+3}{3}\} = R_h$ is checked and the exhaustive upper bound does the rest. Theorem 4.6 needs no exhaustive bound: $|6\{0,2,5,7\}| = 39$ is one evaluation and $39 \notin P_6$ is Proposition 4.4. In fact P_6 misses six elements of $\mathcal{R}_{\mathbb{Z}}(6,4)$, namely $39, 41, 51, 56, 68, 71$; the formal statement records one. ☐

Problem 3 of [1] asks for "a complete description of the sumset size set $\mathcal{R}_{\mathbb{Z}}(h,k)$ for all positive integers h and k" and an explanation of "why some numbers cannot be sumset sizes". It is out of scope here; Table 1 and Remark 3.9 are the data at $k = 4$ that any such description must reproduce.

5. Controls

A collection of statements of the form "$|hA|$ lies in some set" can be true for the wrong reason — because the set is too large, or because the hypothesis is never met. The following close both doors and test the certificate itself.

Proposition 5.1.

(i) *For every $A \subseteq \mathbb{Z}$ with $|A| = 4$: $|3A| \ne 21$ and $|3A| \ne 9$ (one above and one below the bounds (1)); it is false that every $n \in [10,20]$ is $|3A|$ for some such A; and $10 = 3h+1$ is attained.*

(ii) *Four-element sets of integers exist, and the hypothesis $|A| = 4$ does work: $|3\{0,1,2\}| = 7 \notin R_3$.*

(iii) *It is false that every nonzero $w \in D_4$ has $\mathrm{cc}_4(Z_4^{\|}(w)) \in R_4$: the escape clause of Proposition 2.3(R) is load-bearing.*

In (iii) the offending pattern is $w = e_1$, with class count $15 = \binom{6}{2}$, the largest element of $\mathcal{R}_{\mathbb{Z}}(4,3)$; removing the clause from Proposition 2.3 would make the $h \ge 4$ certificates unprovable rather than merely slower.

6. Verification

All results of Sections 2–5 whose statements are reproduced in Appendix A are formally verified in Lean 4 against *Mathlib* [20, 21] (Lean 4.32.0, Mathlib at its `v4.32.0` tag). The development is thirteen modules, 1439 lines. `Basic` defines hA (`hsum`, by $0A = \{0\}$ and $(n+1)A = A + nA$ with Mathlib's pointwise addition of finite sets), the tetrahedron T_h, and `sz` $h\ v = |\varphi_v(T_h)|$, and proves Lemma 2.1(i)–(iii); `Reduction` defines D_h, Z_h, $Z_h^{\|}$, the class count, and proves Lemma 2.1(iv), Propositions 2.2 and 2.3; `H3`, `H4`, `H5` and `H6Data`/`H6Cross`/`H6Rank`/`H6` carry the certificates and witnesses (the $h = 6$ certificate is split so that no single file runs long); `Census` carries Propositions 4.1–4.4, `Coverage` Theorems 4.5 and 4.6, and `Controls` Proposition 5.1. The statements of Lemma 2.1 and Propositions 2.2–2.3 are stated for every h; only the three checks (B), (C), (R) are evaluated at particular h.

The reasoning layer — `hsum_quad`, `card_hsum_quad`, `card_hsum_translate`, `exists_vec_of_card_eq_four`, `sz_eq_classCount`, `sz_congr`, `sz_reduction`, `card_hsum_mem_of_checks` — is checked by the Lean kernel with the standard axioms only: the axiom print (`#print axioms`) of each is `propext`, `Classical.choice`, `Quot.sound`. The same holds for the lower halves at $h = 3,4$ (`card_hsum_3_attained`, `card_hsum_4_attained`, discharged by `decide`), for the three printed sumsets of Proposition 4.1, and for `control_min_attained` and `control_card_hypothesis_needed`; `control_card_four_nonvacuous` needs fewer, printing `propext`, `Quot.sound`. The exhaustive checks (B), (C), (R) at $h = 3,4,5,6$ (`checkB`h, `checkC`h, `checkR`h) are discharged by compiled evaluation, `native_decide`, and every theorem depending on them — the upper halves `card_hsum_`h`_mem`, the equalities `mcr_`h`_four`, the 31 gap theorems `gap`n`_`h, `control_too_large`, `control_too_small`, `control_not_interval`, and `prob2_realises_`h — carries in its axiom print, besides the three standard axioms, the three axioms `checkB`h`._native.native_decide.ax_1_1` etc. that Lean attaches to compiled evaluation. The lower halves at $h = 5, 6$ (`card_hsum_5_attained`, `card_hsum_6_attained`; 32 and 51 witness evaluations) and the direct evaluations of Section 4 (`nathanson_main_theorem`, `prob1_`$*$, `prob2_`$*$, `prob2_incomplete_six`) and `control_degen_escape_needed` likewise use `native_decide`, each adding its own such axioms. There is no `sorry` and no axiom declared by hand; the axiom prints were re-read from the compiled modules while preparing this draft.

As recorded (a measurement, not a theorem; wall-clock and processor time on one machine under load, one Lean process at a time, including about 9 seconds of library loading per file): the reasoning modules take about 15 and 20 processor-seconds; the $h = 3$, 4, 5 certificates with their witnesses about 12, 60 and 256 seconds; at $h = 6$ the cross-product branch (851 929 pairs) about 424 seconds and the rank-one branch (923 patterns) about 917 seconds; the census, coverage and control modules under 30 seconds each; in all about 1 770 processor-seconds, with a peak resident set of about 6.9 GB. The two $h = 6$ branches scale as $|D_h|^2|T_h|^2$ and $|D_h||T_h|^4$, which prices $h = 7$ at about 2.2 processor-hours; it was not run (Remark 3.10).

The literature search behind Table 1 and the claims of novelty: a full-text sweep of a local mirror of arXiv e-prints (all mathematics listings through early September 2026) for "sumset size(s)", for Nathanson's macro for $\mathcal{R}_{\mathbb{Z}}$, for "tetrahedral" with "sumset", and for the set $\{7, 9, 10\}$, finds twelve e-prints discussing $\mathcal{R}(h, k)$ [2, 13, 4, 11, 12, 15, 1, 3, 16, 17, 18, 19]; none prints $\mathcal{R}_{\mathbb{Z}}(h, 4)$ for any $h \ge 4$, and no table in them is a table of the attainable sizes: those in [12] and [17] list the sizes seen most often when random 4-element sets are sampled — every entry of theirs with $3 \le h \le 6$ does lie in Theorem 1.1 — and [16], despite its title, concerns frequencies, not the attainable set. The compiled e-print of [2] stops at $\mathcal{R}_G(3, 4)$ and $\mathcal{R}_G(3, 5)$; the published version of [3] has Theorem 1.7 for $h = 3$ only and the Section 5.2 remarks quoted above. Searches of the OEIS for $10, 17, 32, 51$, for $1, 6, 9, 15$ and for the value sets, and of OpenAlex for works citing [1] or [3], returned nothing relevant; two 2026 e-prints [17, 19] pose the problem as open. Read the negative as "not found". The formal development is currently in a private repository: repository reference to be added at submission.

Appendix A. The formal statements

The Lean statements corresponding to the results above, verbatim (statements only; proofs, docstrings and attributes omitted), from the namespace `LeanProblemSpec.McrH4`. `Finset ℤ` is Mathlib's type of finite sets of integers, `A.card` its cardinality, and `+` on finite sets the pointwise sum; `Vec` is $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, so `v.1`, `v.2.1`, `v.2.2` are the three coordinates. The sets R_h are the constants `R34`, `R44`, `R54`, `R64`; `T`h, `D`h, `sz`h, `ccW`h are copies of `T h`, `Dset h`, `sz h`, `classCountW h` bound to constants so that the evaluator caches them.

Definitions.

```
def hsum : ℕ → Finset ℤ → Finset ℤ
  | 0, _ => {0}
  | (n + 1), A => A + hsum n A

abbrev Vec := ℤ × ℤ × ℤ
def dot (u v : Vec) : ℤ := u.1 * v.1 + u.2.1 * v.2.1 + u.2.2 * v.2.2

def T : ℕ → Finset Vec
  | 0 => {(0, 0, 0)}
  | (n + 1) =>
      T n ∪ (T n).image (· + (1, 0, 0)) ∪ (T n).image (· + (0, 1, 0)) ∪
        (T n).image (· + (0, 0, 1))

def sz (h : ℕ) (v : Vec) : ℕ := ((T h).image (fun t => dot t v)).card

def cross (u w : Vec) : Vec :=
  (u.2.1 * w.2.2 - u.2.2 * w.2.1, u.2.2 * w.1 - u.1 * w.2.2, u.1 * w.2.1 - u.2.1 * w.1)
def Is4 (v : Vec) : Prop :=
  v.1 ≠ 0 ∧ v.2.1 ≠ 0 ∧ v.2.2 ≠ 0 ∧ v.1 ≠ v.2.1 ∧ v.1 ≠ v.2.2 ∧ v.2.1 ≠ v.2.2
def Dset (h : ℕ) : Finset Vec := ((T h) ×ˢ (T h)).image (fun p => p.1 - p.2)
def Zset (h : ℕ) (v : Vec) : Finset Vec := (Dset h).filter (fun u => dot u v = 0)
def Zw (h : ℕ) (w : Vec) : Finset Vec := (Dset h).filter (fun u => cross u w = 0)
def classCount (h : ℕ) (S : Finset Vec) : ℕ :=
  ((T h).image (fun t => (T h).filter (fun t' => t - t' ∈ S))).card
def classCountW (h : ℕ) (w : Vec) : ℕ :=
  ((T h).image (fun t => (T h).filter (fun t' => cross (t - t') w = 0))).card
def DegenW (h : ℕ) (w : Vec) : Prop :=
  ((1, 0, 0) : Vec) ∈ Zw h w ∨ ((0, 1, 0) : Vec) ∈ Zw h w ∨ ((0, 0, 1) : Vec) ∈ Zw h w ∨
    ((1, -1, 0) : Vec) ∈ Zw h w ∨ ((1, 0, -1) : Vec) ∈ Zw h w ∨ ((0, 1, -1) : Vec) ∈ Zw h w

def R34 : Finset ℕ := {10, 12, 13, 14, 15, 16, 17, 18, 19, 20}
def R44 : Finset ℕ := {13, 16, 17, 19, 21, 23, 24, 25, 26, 27, 29, 30, 31, 32, 33, 34, 35}
def R54 : Finset ℕ := {16, 20, 21, 24, 26, 27, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39,
  41, 42, 43, 44, 45, 46, 47, 48, 50, 51, 52, 53, 54, 55, 56}
def R64 : Finset ℕ := {19, 24, 25, 29, 31, 33, 35, 37, 38, 39, 40, 41, 42, 43, 45, 46, 47,
  48, 49, 51, 52, 53, 54, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71,
  72, 73, 74, 75, 76, 78, 79, 80, 81, 82, 83, 84}

def mainSet (h i₀ : ℕ) : Vec := (1, (h : ℤ) + 1, ((h : ℤ) + 1 - i₀) * ((h : ℤ) + 1))
def mainThmAt (h i₀ : ℕ) : Bool :=
  sz h (mainSet h i₀) == Nat.choose (h + 3) 3 - Nat.choose (i₀ + 2) 3
def prob1 (h : ℕ) : List ℕ :=
  (List.range (h * h)).map
    (fun p => sz h (1, (h : ℤ) + 1, (h : ℤ) * h + (h : ℤ) + 1 - p))
def prob2Pairs (h : ℕ) : List (ℤ × ℤ) :=
  (List.range (h - 1)).flatMap (fun i =>
    (List.range ((i + 2) * (h - 1) + 1)).map (fun j => (((i : ℤ) + 2), ((i : ℤ) + 3 + j))))
def prob2 (h : ℕ) : Finset ℕ :=
  ((prob2Pairs h).map (fun q => sz h (1, q.1, q.2))).toFinset
```

Lemma 2.1.

```
theorem hsum_quad (v : Vec) :
```

```
    ∀ h : ℕ, hsum h {0, v.1, v.2.1, v.2.2} = (T h).image (fun t => dot t v)
theorem card_hsum_quad (h : ℕ) (v : Vec) :
    (hsum h {0, v.1, v.2.1, v.2.2}).card = sz h v
theorem card_hsum_translate (h : ℕ) (A : Finset ℤ) (m : ℤ) :
    (hsum h (A + {m})).card = (hsum h A).card
theorem exists_vec_of_card_eq_four (h : ℕ) (A : Finset ℤ) (hA : A.card = 4) :
    ∃ v : Vec, v.1 ≠ 0 ∧ v.2.1 ≠ 0 ∧ v.2.2 ≠ 0 ∧ v.1 ≠ v.2.1 ∧ v.1 ≠ v.2.2 ∧
      v.2.1 ≠ v.2.2 ∧ (hsum h A).card = sz h v
theorem sz_eq_classCount (h : ℕ) (v : Vec) : sz h v = classCount h (Zset h v)
theorem sz_congr {h : ℕ} {v v' : Vec} (hZ : Zset h v = Zset h v') : sz h v = sz h v'
```

Propositions 2.2 and 2.3.

```
theorem sz_reduction (h : ℕ) (v : Vec) (hv : Is4 v) :
    sz h v = (T h).card
    ∨ (∃ u₁ ∈ Dset h, ∃ u₂ ∈ Dset h,
        cross u₁ u₂ ≠ 0 ∧ Is4 (cross u₁ u₂) ∧ sz h v = sz h (cross u₁ u₂))
    ∨ (∃ w ∈ Dset h, w ≠ 0 ∧ sz h v = classCountW h w ∧ Zset h v = Zw h w)

theorem card_hsum_mem_of_checks (h : ℕ) (X : Finset ℕ)
    (hB : (T h).card ∈ X)
    (hC : ∀ u₁ ∈ Dset h, ∀ u₂ ∈ Dset h,
      cross u₁ u₂ ≠ 0 → Is4 (cross u₁ u₂) → sz h (cross u₁ u₂) ∈ X)
    (hR : ∀ w ∈ Dset h, w ≠ 0 → DegenW h w ∨ classCountW h w ∈ X)
    (A : Finset ℤ) (hA : A.card = 4) : (hsum h A).card ∈ X
```

Theorems 3.1–3.7 and Corollaries 3.2–3.8. For $h = 3, 4, 5, 6$ the three certificate lemmas are (shown for $h = 3$; the others replace `3`, `D3`, `sz3`, `ccW3`, `R34` by the corresponding constants)

```
theorem checkB3 : (T 3).card ∈ R34
theorem checkC3 : ∀ u₁ ∈ D3, ∀ u₂ ∈ D3,
    cross u₁ u₂ ≠ 0 → Is4 (cross u₁ u₂) → sz3 (cross u₁ u₂) ∈ R34
theorem checkR3 : ∀ w ∈ D3, w ≠ 0 → DegenW 3 w ∨ ccW3 w ∈ R34
```

and the theorems are

```
theorem card_hsum_3_mem (A : Finset ℤ) (hA : A.card = 4) : (hsum 3 A).card ∈ R34
theorem card_hsum_3_attained :
    ∀ n ∈ R34, ∃ A : Finset ℤ, A.card = 4 ∧ (hsum 3 A).card = n
theorem mcr_3_four (n : ℕ) :
    (∃ A : Finset ℤ, A.card = 4 ∧ (hsum 3 A).card = n) ↔ n ∈ R34
theorem gap11_3 (A : Finset ℤ) (hA : A.card = 4) : (hsum 3 A).card ≠ 11

theorem card_hsum_4_mem (A : Finset ℤ) (hA : A.card = 4) : (hsum 4 A).card ∈ R44
theorem card_hsum_4_attained :
    ∀ n ∈ R44, ∃ A : Finset ℤ, A.card = 4 ∧ (hsum 4 A).card = n
theorem mcr_4_four (n : ℕ) :
    (∃ A : Finset ℤ, A.card = 4 ∧ (hsum 4 A).card = n) ↔ n ∈ R44
theorem gap14_4 (A : Finset ℤ) (hA : A.card = 4) : (hsum 4 A).card ≠ 14
theorem gap15_4 (A : Finset ℤ) (hA : A.card = 4) : (hsum 4 A).card ≠ 15
theorem gap18_4 (A : Finset ℤ) (hA : A.card = 4) : (hsum 4 A).card ≠ 18
theorem gap20_4 (A : Finset ℤ) (hA : A.card = 4) : (hsum 4 A).card ≠ 20
theorem gap22_4 (A : Finset ℤ) (hA : A.card = 4) : (hsum 4 A).card ≠ 22
theorem gap28_4 (A : Finset ℤ) (hA : A.card = 4) : (hsum 4 A).card ≠ 28

theorem card_hsum_5_mem (A : Finset ℤ) (hA : A.card = 4) : (hsum 5 A).card ∈ R54
theorem card_hsum_5_attained :
    ∀ n ∈ R54, ∃ A : Finset ℤ, A.card = 4 ∧ (hsum 5 A).card = n
theorem mcr_5_four (n : ℕ) :
    (∃ A : Finset ℤ, A.card = 4 ∧ (hsum 5 A).card = n) ↔ n ∈ R54
theorem gap17_5 (A : Finset ℤ) (hA : A.card = 4) : (hsum 5 A).card ≠ 17
theorem gap18_5 (A : Finset ℤ) (hA : A.card = 4) : (hsum 5 A).card ≠ 18
theorem gap19_5 (A : Finset ℤ) (hA : A.card = 4) : (hsum 5 A).card ≠ 19
theorem gap22_5 (A : Finset ℤ) (hA : A.card = 4) : (hsum 5 A).card ≠ 22
theorem gap23_5 (A : Finset ℤ) (hA : A.card = 4) : (hsum 5 A).card ≠ 23
theorem gap25_5 (A : Finset ℤ) (hA : A.card = 4) : (hsum 5 A).card ≠ 25
```

```
theorem gap28_5 (A : Finset ℤ) (hA : A.card = 4) : (hsum 5 A).card ≠ 28
theorem gap40_5 (A : Finset ℤ) (hA : A.card = 4) : (hsum 5 A).card ≠ 40
theorem gap49_5 (A : Finset ℤ) (hA : A.card = 4) : (hsum 5 A).card ≠ 49

theorem card_hsum_6_mem (A : Finset ℤ) (hA : A.card = 4) : (hsum 6 A).card ∈ R64
theorem card_hsum_6_attained :
    ∀ n ∈ R64, ∃ A : Finset ℤ, A.card = 4 ∧ (hsum 6 A).card = n
theorem mcr_6_four (n : ℕ) :
    (∃ A : Finset ℤ, A.card = 4 ∧ (hsum 6 A).card = n) ↔ n ∈ R64
theorem gap20_6 (A : Finset ℤ) (hA : A.card = 4) : (hsum 6 A).card ≠ 20
theorem gap21_6 (A : Finset ℤ) (hA : A.card = 4) : (hsum 6 A).card ≠ 21
theorem gap22_6 (A : Finset ℤ) (hA : A.card = 4) : (hsum 6 A).card ≠ 22
theorem gap23_6 (A : Finset ℤ) (hA : A.card = 4) : (hsum 6 A).card ≠ 23
theorem gap26_6 (A : Finset ℤ) (hA : A.card = 4) : (hsum 6 A).card ≠ 26
theorem gap27_6 (A : Finset ℤ) (hA : A.card = 4) : (hsum 6 A).card ≠ 27
theorem gap28_6 (A : Finset ℤ) (hA : A.card = 4) : (hsum 6 A).card ≠ 28
theorem gap30_6 (A : Finset ℤ) (hA : A.card = 4) : (hsum 6 A).card ≠ 30
theorem gap32_6 (A : Finset ℤ) (hA : A.card = 4) : (hsum 6 A).card ≠ 32
theorem gap34_6 (A : Finset ℤ) (hA : A.card = 4) : (hsum 6 A).card ≠ 34
theorem gap36_6 (A : Finset ℤ) (hA : A.card = 4) : (hsum 6 A).card ≠ 36
theorem gap44_6 (A : Finset ℤ) (hA : A.card = 4) : (hsum 6 A).card ≠ 44
theorem gap50_6 (A : Finset ℤ) (hA : A.card = 4) : (hsum 6 A).card ≠ 50
theorem gap55_6 (A : Finset ℤ) (hA : A.card = 4) : (hsum 6 A).card ≠ 55
theorem gap77_6 (A : Finset ℤ) (hA : A.card = 4) : (hsum 6 A).card ≠ 77
```

Propositions 4.1–4.4.

```
theorem sumset_three_012 : hsum 3 {0, 1, 2} = ({0, 1, 2, 3, 4, 5, 6} : Finset ℤ)
theorem sumset_three_013 : hsum 3 {0, 1, 3} = ({0, 1, 2, 3, 4, 5, 6, 7, 9} : Finset ℤ)
theorem sumset_three_014 :
    hsum 3 {0, 1, 4} = ({0, 1, 2, 3, 4, 5, 6, 8, 9, 12} : Finset ℤ)
theorem sumset_three_sizes :
    (hsum 3 {0, 1, 2}).card = 7 ∧ (hsum 3 {0, 1, 3}).card = 9 ∧
      (hsum 3 {0, 1, 4}).card = 10

theorem nathanson_main_theorem :
    ((List.range' 1 12).all (fun h => (List.range h).all (mainThmAt h))) = true

theorem prob1_three : prob1 3 = [20, 19, 19, 19, 19, 16, 16, 17, 16]
theorem prob1_four : prob1 4 = [35, 34, 34, 34, 34, 34, 31, 31, 31, 33, 31, 25, 25, 29, 26, 25]
theorem prob1_five : prob1 5 = [56, 55, 55, 55, 55, 55, 55, 52, 52, 52, 52, 54, 52, 46, 46, 46,
  50, 50, 46, 36, 36, 43, 39, 38, 36]
theorem prob1_six : prob1 6 = [84, 83, 83, 83, 83, 83, 83, 83, 80, 80, 80, 80, 80, 82, 80, 74,
  74, 74, 74, 79, 79, 74, 64, 64, 64, 72, 72, 69, 64, 49, 49, 61, 58, 52, 51, 49]
theorem prob1_seven : prob1 7 = [120, 119, 119, 119, 119, 119, 119, 119, 119, 116, 116, 116,
  116, 116, 116, 118, 116, 110, 110, 110, 110, 110, 115, 115, 110, 100, 100, 100, 100, 108,
  111, 108, 100, 85, 85, 85, 98, 97, 96, 94, 85, 64, 64, 81, 79, 70, 72, 67, 64]
theorem prob1_eight : prob1 8 = [165, 164, 164, 164, 164, 164, 164, 164, 164, 164, 161, 161,
  161, 161, 161, 161, 161, 163, 161, 155, 155, 155, 155, 155, 155, 160, 160, 155, 145, 145,
  145, 145, 145, 154, 157, 154, 145, 130, 130, 130, 130, 143, 147, 147, 140, 130, 109, 109,
  109, 127, 131, 126, 122, 121, 109, 81, 81, 105, 102, 99, 87, 87, 84, 81]

theorem prob2_three : prob2 3 = ({10, 12, 13, 14, 15, 16, 17, 18, 19} : Finset ℕ)
theorem prob2_four : prob2 4 = ({13, 16, 17, 19, 21, 23, 24, 25, 26, 27, 29, 30, 31, 32, 33,
  34} : Finset ℕ)
theorem prob2_five : prob2 5 = ({16, 20, 21, 24, 26, 27, 29, 30, 31, 32, 33, 34, 35, 36, 37,
  38, 39, 41, 42, 43, 44, 45, 46, 47, 48, 50, 51, 52, 53, 54, 55} : Finset ℕ)
theorem prob2_six : prob2 6 = ({19, 24, 25, 29, 31, 33, 35, 37, 38, 40, 42, 43, 45, 46, 47,
  48, 49, 52, 53, 54, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 69, 70, 72, 73, 74, 75, 76,
  78, 79, 80, 81, 82, 83} : Finset ℕ)
theorem prob2_seven : prob2 7 = ({22, 28, 29, 34, 36, 39, 41, 43, 44, 45, 47, 48, 50, 52, 53,
  54, 55, 56, 57, 58, 59, 60, 62, 63, 64, 65, 66, 69, 70, 71, 73, 74, 75, 78, 79, 80, 81, 83,
  84, 85, 86, 87, 89, 90, 92, 93, 94, 95, 96, 97, 98, 99, 100, 103, 104, 105, 106, 108, 109,
  110, 111, 112, 114, 115, 116, 117, 118, 119} : Finset ℕ)
```

(The long lists are broken across lines here for width; in the source each is one literal.)

Theorems 4.5 and 4.6.

```
theorem prob2_realises_three (n : ℕ) :
    (∃ A : Finset ℤ, A.card = 4 ∧ (hsum 3 A).card = n) ↔ n ∈ insert 20 (prob2 3)
theorem prob2_realises_four (n : ℕ) :
    (∃ A : Finset ℤ, A.card = 4 ∧ (hsum 4 A).card = n) ↔ n ∈ insert 35 (prob2 4)
theorem prob2_realises_five (n : ℕ) :
    (∃ A : Finset ℤ, A.card = 4 ∧ (hsum 5 A).card = n) ↔ n ∈ insert 56 (prob2 5)
theorem prob2_incomplete_six :
    ∃ A : Finset ℤ, A.card = 4 ∧ (hsum 6 A).card = 39 ∧ (39 : ℕ) ∉ prob2 6
```

Proposition 5.1.

```
theorem control_too_large (A : Finset ℤ) (hA : A.card = 4) : (hsum 3 A).card ≠ 21
theorem control_too_small (A : Finset ℤ) (hA : A.card = 4) : (hsum 3 A).card ≠ 9
theorem control_not_interval :
    ¬ (∀ n : ℕ, 10 ≤ n → n ≤ 20 → ∃ A : Finset ℤ, A.card = 4 ∧ (hsum 3 A).card = n)
theorem control_min_attained : ∃ A : Finset ℤ, A.card = 4 ∧ (hsum 3 A).card = 10
theorem control_card_four_nonvacuous : ∃ A : Finset ℤ, A.card = 4
theorem control_card_hypothesis_needed :
    ∃ A : Finset ℤ, A.card = 3 ∧ (hsum 3 A).card ∉ R34
theorem control_degen_escape_needed :
    ¬ (∀ w ∈ D4, w ≠ 0 → ccW4 w ∈ R44)
```

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