DERIVATIVES OF THE DISCRETE MAXIMAL FUNCTION OF A CHARACTERISTIC FUNCTION BEAT THE CONSTANT 1/2 FROM ORDER EIGHT ON

KOREA SUPERINTELLIGENCE LABS

ABSTRACT. For a set $S \subseteq \mathbb{Z}$ let $\mathcal{M}^{\mathrm{u}}_{\mathbb{Z}}\chi_S$ be its uncentered discrete Hardy–Littlewood maximal function and let $f^{(k)}$ denote the k-th forward difference. Liao, Madrid, Palsson and Weigt define $C_{k,p}$ as the least constant with $\|(\mathcal{M}^{\mathrm{u}}_{\mathbb{Z}}\chi_S)^{(k)}\|_{\ell^p(\mathbb{Z})} \le C_{k,p}\left\|\chi_S^{(k)}\right\|_{\ell^p(\mathbb{Z})}$ for every S, determine it for $k \le 2$, and for $k \ge 3$ record only $1/2 \le C_{k,p}$ together with an exponential lower bound whose value first exceeds $1/2$ at $k = 36$ ($p \in \{1, \infty\}$) and at $k = 23$ ($p = 2$); their experiments could not improve on $1/2$ for $3 \le k \le 6$. We show that the constant $1/2$ is beaten from order eight on: the three-point set $\{0, 1, 4\}$ on the circle $\mathbb{Z}_7$ has ℓ^1 ratio exactly $13/25$ at $k = 8$, and the four-point set $\{0, 1, 4, 7\}$ on $\mathbb{Z}_{10}$ has ℓ^1 ratio $97/186$ and squared ℓ^2 ratio $23083/90504 > 1/4$ at $k = 8$ and ℓ^∞ ratio $139/270$ at $k = 9$. Through the circle-to-line lemma of the same authors this gives $C_{8,1} > 1/2$, $C_{8,2} > 1/2$ and $C_{9,\infty} > 1/2$. The seven-point set alone keeps its ratio above $1/2$ at every order from 8 to 35 in ℓ^1, from 9 to 22 in ℓ^2 and from 11 to 35 in ℓ^∞, which is exactly where the exponential bound takes over; hence $C_{k,1} > 1/2$ and $C_{k,2} > 1/2$ for every $k \ge 8$ and $C_{k,\infty} > 1/2$ for every $k \ge 9$, with no order left over. Below order eight the wall stands: for every subset of every circle of length at most 15 and every $2 \le k \le 7$ the ℓ^1 and ℓ^∞ ratios are at most $1/2$, one order beyond the range the earlier experiments report. Every ratio is an exact rational and every statement is machine-checked in Lean 4; the question whether $C_{k,p} > 1/2$ for $3 \le k \le 6$ remains open.

1. INTRODUCTION

The objects. For $f : \mathbb{Z} \to \mathbb{R}$ the uncentered discrete Hardy–Littlewood maximal function is

$$\mathcal{M}^{\mathrm{u}}_{\mathbb{Z}} f(n) \;=\; \sup_{s,t \ge 0} \; \frac{1}{s+t+1} \sum_{i=-s}^{t} |f(n+i)|, \tag{1}$$

the supremum over all discrete intervals containing n. The k-th derivative of f is the iterated forward difference, $f^{(0)} = f$ and $f^{(k+1)}(n) = f^{(k)}(n+1) - f^{(k)}(n)$ [1, §1.2]. For a set $S \subseteq \mathbb{Z}$ write χ_S for its characteristic function. Liao, Madrid, Palsson and Weigt [1] study how much of the ℓ^p size of $\chi_S^{(k)}$ survives in $(\mathcal{M}^{\mathrm{u}}_{\mathbb{Z}}\chi_S)^{(k)}$, and quantify it by a best constant. We quote their Definition 1.12 (numbering, here and throughout, is that of the compiled version 2 e-print of [1], re-read in the arXiv PDF):

> For $k \in \mathbb{N}_0$ and $1 \le p \le \infty$ denote by $C_{k,p} \ge 0$ the smallest number such that for all subset $S \subseteq \mathbb{Z}$ with $\left\|\chi_S^{(k)}\right\|_{\ell^p(\mathbb{Z})} < \infty$ we have
>
> $$\left\|(\mathcal{M}^{\mathrm{u}}_{\mathbb{Z}}\chi_S)^{(k)}\right\|_{\ell^p(\mathbb{Z})} \le C_{k,p}\left\|\chi_S^{(k)}\right\|_{\ell^p(\mathbb{Z})}.$$

For $k = 0$ this is the operator norm of $\mathcal{M}^{\mathrm{u}}_{\mathbb{Z}}$ on characteristic functions; for $k = 1$ it is a variation bound, a case with a substantial literature surveyed in [1, §1.2] (Madrid [4], for instance, proves optimal bounds for the derivative of discrete maximal functions); for $k = 2$ an upper bound is the main result of Temur [3] (Theorem 1.10 of [1]), and the constant is determined in [1]. For $k \ge 2$ the works we know of are [3, 2, 1] (Section 6).

The source, and what it leaves open. The main results of [1] settle $k = 0, 1, 2$ for every p. The ones this paper builds on are the following three, quoted verbatim.

Theorem 2.5. For all $1 \le p \le \infty$ we have $C_{2,p} = \frac{1}{2}$. Defining $S_m := \{0, 3, 6, \dots, 3(m-1)\}$, the sequence $(\chi_{S_m})_{m\in\mathbb{N}}$ is extremizing.

Theorem 2.7. For all $k \ge 0$, $1 \le p < \infty$ we have $C_{k,p} \ge \frac{1}{2}$. This lower bound is witnessed by the sequence $(\chi_{S_m})_{m\in\mathbb{N}}$ from Theorem 2.5.

Theorem 2.8. For all $k \ge 0$ and $1 \le p \le \infty$ we have

$$C_{k,p} \;\ge\; \frac{1}{12 \cdot 8^{\left|\frac{1}{p}-\frac{1}{2}\right|}} \left(\frac{2}{|1+e^{\frac{\pi i}{4}}|}\right)^k.$$

In particular, $C_{k,p}$ grows at least exponentially in k, uniformly in p. This bound is witnessed by the sequence $(\chi_{B_m})_{m\in\mathbb{N}}$, where $B_1 = \{0, 1, 2, 5\}$, $B_{m+1} = B_m \cup (B_1 + 8m)$.

Their Table 2 (page 7), which summarises the state of knowledge, prints in each of its three rows $p = 1$, $1 < p < \infty$, $p = \infty$ the boxed cell

$$\tfrac{1}{2} \le C_{k,p} \sim C^k \qquad (k \ge 3),$$

where, by the table's own note, "$C_{k,p} \sim C^k$ is not meant as asymptotic equivalence up to multiplicative constants. It indicates exponential growth in k" in the sense of their display (2.3), $1 < 1.0824 \approx 2/|1 + e^{\pi i/4}| \le \liminf_k C_{k,p}^{1/k} \le \limsup_k C_{k,p}^{1/k} \le 2$; the upper bound there comes from the explicit exponential estimate [1, (2.1)], which improves on the bound of Temur and Özcan [2] (Theorem 1.11 of [1]). Theorem 2.7 is stated for $p < \infty$; the $p = \infty$ row of Table 2 carries the same lower bound $1/2$, and item (iv) of their "Proof strategies" (quoted below) asserts it for $1 \le p \le \infty$. It follows from the same witness: Lemma 3.11 of [1], quoted in Section 2, is stated for $1 \le p \le \infty$ (its proof treats $p = \infty$ separately), and the circle ratio of $\chi_{\{0\}}$ on $\mathbb{Z}_3$ (their item (iii)) is exactly $1/2$ in every norm (Proposition 2.1).

The gap between the two lower bounds is wide. Write $\beta := 2/|1 + e^{\pi i/4}| = 2/\sqrt{2+\sqrt{2}} = 1.0823922\dots$ for the base in Theorem 2.8. For $p \in \{1, \infty\}$ the constant is $1/(12\sqrt{8}) = 0.0294628\dots$ and the right-hand side of Theorem 2.8 first exceeds $1/2$ at $k = 36$ ($0.4707\dots$ at $k = 35$, $0.5095\dots$ at $k = 36$); for $p = 2$ the constant is $1/12$ and the crossing is at $k = 23$ ($0.4756\dots$ at $k = 22$, $0.5148\dots$ at $k = 23$). So for $3 \le k \le 35$ when $p \in \{1, \infty\}$, and for $3 \le k \le 22$ when $p = 2$, the best lower bound printed in [1] is the non-strict $1/2$. The authors say so, and say that a search did not move it. From their "Proof strategies" paragraph, item (iv), verbatim:

> We initially discovered the example $\chi_{\{0,1,2,5\}}$ on the circle of length 8 through numerical experiments; the source code is available at [Wei26]. More precisely, for a given circle, we enumerated all characteristic functions χ_S, computed its maximal functions $\mathcal{M}^{\mathrm{u}}_{\mathbb{Z}}\chi_S$ and evaluated $\left\|\mathcal{M}^{\mathrm{u}}_{\mathbb{Z}}\chi_S^{(k)}\right\|_{\ell^p(\mathbb{Z})} / \left\|\chi_S^{(k)}\right\|_{\ell^p(\mathbb{Z})}$ for selected values of k and p. Since the number of characteristic functions on a circle grows exponentially in its length, this approach becomes prohibitively expensive quickly, limiting us to circles of length at most about 30. Within these limitations, our numerical experiments suggest that for any $1 \le p \le \infty$ the map $k \mapsto C_{k,p}$ is nondecreasing for $k \ge 2$. Note that $k \mapsto C_{k,p}$ is decreasing for k on $\{0, 1, 2\}$. Surprisingly, we have not been able to experimentally improve on the lower bound $C_{k,p} \ge 1/2$, witnessed by $\chi_{\{0\}}$ in Theorem 2.7, for any k, p with $3 \le k \le 6$ and $1 \le p \le \infty$.

Their [Wei26] is the public repository [5]. The question this paper answers is the one that sentence raises once the range $3 \le k \le 6$ is read as a boundary rather than as a sample: *at which order k, if any, is the constant $1/2$ actually beaten by a characteristic function, and does the strict inequality then persist all the way to the exponential regime?*

Results. The answer is order eight, and it persists. For a set $S \subseteq \mathbb{Z}_N$ on the discrete circle write $R_{k,p}(S)$ for the circle ratio $\left\|(\mathcal{M}^{\mathrm{u}}_{\mathbb{Z}_N}\chi_S)^{(k)}\right\|_{\ell^p(\mathbb{Z}_N)} / \left\|\chi_S^{(k)}\right\|_{\ell^p(\mathbb{Z}_N)}$ (Section 2); by Lemma 3.11 of [1] every

circle ratio is a lower bound for $C_{k,p}$. Let

$$A_7 := \{0,1,4\} \subseteq \mathbb{Z}_7, \qquad A_{10} := \{0,1,4,7\} \subseteq \mathbb{Z}_{10}, \qquad A_{21} := \{0,3,8,9,13,14,15,16\} \subseteq \mathbb{Z}_{21}.$$

Theorem (Theorems 3.2, 3.3, 3.4)**.** $R_{8,1}(A_7) = 13/25$, $R_{8,1}(A_{10}) = 97/186$, $R_{8,2}(A_{10})^2 = 23083/90504 > 1/4$, $R_{9,\infty}(A_{10}) = 139/270$. *Consequently* $C_{8,1} > 1/2$, $C_{8,2} > 1/2$ *and* $C_{9,\infty} > 1/2$.

Theorem (Theorems 4.2, 4.3, 4.4)**.** $R_{k,1}(A_7) > 1/2$ *for every* $8 \le k \le 35$, $R_{k,2}(A_7) > 1/2$ *for every* $9 \le k \le 22$, $R_{k,\infty}(A_7) > 1/2$ *for every* $11 \le k \le 35$, *and* $R_{10,\infty}(A_{21}) = 35726/71379 > 1/2$. *Together with Theorem 2.8 of* [1]*:*

$$C_{k,1} > \tfrac{1}{2} \text{ and } C_{k,2} > \tfrac{1}{2} \text{ for every } k \ge 8, \qquad C_{k,\infty} > \tfrac{1}{2} \text{ for every } k \ge 9.$$

Theorem (Theorem 5.1)**.** *For every* $N \le 15$, *every* $S \subseteq \mathbb{Z}_N$, *every* $2 \le k \le 7$ *and* $p \in \{1,\infty\}$, $\left\|(\mathcal{M}^{\mathrm{u}}_{\mathbb{Z}_N}\chi_S)^{(k)}\right\|_{\ell^p(\mathbb{Z}_N)} \le \frac{1}{2}\left\|\chi_S^{(k)}\right\|_{\ell^p(\mathbb{Z}_N)}$.

The third statement certifies, in exact arithmetic and over every subset, the experiment that [1] reports for $3 \le k \le 6$ (on a smaller window of circle lengths) and extends it by one order, to $k = 7$; the first two say that at $k = 8$ the picture changes. The two ℓ^∞ exceptions in the second statement — order 8, where no witness is offered at all, and order 10, where a circle of length 21 is needed — are, as far as our search reaches, properties of the problem and not of the search (Remark 4.5).

What is new, and what is not. Everything above is a statement about explicit finite sets, verified by exact computation; the mathematics that turns a circle ratio into a bound on $C_{k,p}$ is Lemma 3.11 of [1], which we cite and do not reprove, and the exponential regime $k \ge 36$ (resp. $k \ge 23$) is their Theorem 2.8. What is new is the location of the threshold ($k = 8$ for ℓ^1 and ℓ^2, $k = 9$ for ℓ^∞), the explicit witnesses with their exact ratios, the fact that a single seven-point set carries the strict inequality across the entire gap up to the exponential bound, and the certified wall below order eight. Two things must be said plainly.

First, these values are not merely within reach of a routine run of published software: they come out of one. The search code of [1] is public [5]; its README states that by default the program "goes through all characteristic functions on circles from length 2 to 36, considers derivatives from order 0 to 64 and exponents $p = 1, 2, 4, 8, \infty$" and prints each new record. We built it from the repository and ran it as shipped: after one minute on one machine its record table shows, in floating point, 0.500 at every order $3 \le k \le 7$ for $p \in \{1, 2, \infty\}$, then 0.522 ($p = 1$) and 0.505 ($p = 2$) at $k = 8$, and for $p = \infty$ still 0.500 at $k = 8$ followed by 0.515 at $k = 9$, all three attained by the function 1100100100, which is $\chi_{A_{10}}$; the line for χ_{A_7} at $k = 8$, $p = 1$, value 0.520, is printed on the way. But neither the paper — whose reported experiments cover $3 \le k \le 6$ — nor the repository, which ships code and no result files and whose README, commit messages and issue tracker contain no such value, states any of the values in this paper, and the paper's printed table leaves $1/2 \le C_{k,p}$ non-strict for every $k \ge 3$. We claim the statements, their exactness and their verification, not any difficulty in finding them.

Second, the question the authors of [1] actually pose — whether $C_{k,p} > 1/2$ for $k \in \{3, 4, 5, 6\}$ — is untouched. Every circle of length at most 28 says no: exactly in ℓ^1 and ℓ^∞, exactly in ℓ^2 for length at most 18 and in extended precision beyond (Remark 5.2); a witness, if one exists, lives on a longer circle or is not periodic at all.

Method. All quantities are rational. On a circle of length N the maximal function $\mathcal{M}^{\mathrm{u}}_{\mathbb{Z}_N}\chi_S$ takes values with denominators dividing $D_N := \mathrm{lcm}(1, \ldots, 2N)$, so $D_N \cdot \mathcal{M}^{\mathrm{u}}_{\mathbb{Z}_N}\chi_S$ is an integer vector, and every claim in this paper is an inequality or equality between integers: exact at $p = 1$ and $p = \infty$, and exact at $p = 2$ after squaring. The witnesses were found by an exhaustive search over every subset of every circle of length at most 28 (Remark 5.2); once found, each claim is a finite computation, and every theorem below that carries a proof has been checked in the Lean 4 kernel (Section 7, with the exact statements in Appendix A). Statements that rest on the search beyond what was formally checked are confined to remarks and labelled as such.

2. Preliminaries

The discrete circle. Following [1, Definition 3.9], for $N \geq 1$ the discrete circle is $\mathbb{Z}_N := \{0, 1, \ldots, N-1\}$ with counting measure, $n \bmod N \in \mathbb{Z}_N$ is the residue of $n \in \mathbb{Z}$, and the uncentered maximal operator on $\mathbb{Z}_N$ is

$$\mathcal{M}^{\mathrm{u}}_{\mathbb{Z}_N} f(n) \;:=\; \sup_{s,t \geq 0} \frac{1}{s+t+1} \sum_{i=-s}^{t} f\big((n+i) \bmod N\big), \tag{2}$$

where the authors remark that "it in fact suffices to consider $t+s+1 \leq 2N$ in the above supremum". The k-th derivative on $\mathbb{Z}_N$ is the forward difference taken modulo N: $f^{(0)} = f$ and $f^{(k+1)}(n) := f^{(k)}((n+1) \bmod N) - f^{(k)}(n)$. For $S \subseteq \mathbb{Z}_N$, $k \geq 1$ and $p \in \{1, 2, \infty\}$ we call

$$R_{k,p}(S) \;:=\; \frac{\big\|(\mathcal{M}^{\mathrm{u}}_{\mathbb{Z}_N} \chi_S)^{(k)}\big\|_{\ell^p(\mathbb{Z}_N)}}{\big\|\chi_S^{(k)}\big\|_{\ell^p(\mathbb{Z}_N)}} \tag{3}$$

the *circle ratio* of S at order k; it is defined whenever S is neither empty nor all of $\mathbb{Z}_N$ (a nonconstant $f : \mathbb{Z}_N \to \mathbb{R}$ has $\big\|f^{(k)}\big\|_{\ell^p(\mathbb{Z}_N)} > 0$, [1, Lemma 3.10]).

The passage from the circle to the line is the following lemma of [1], quoted verbatim. For $N, a \in \mathbb{N}$ and $f : \mathbb{Z}_N \to \mathbb{R}$ it uses $f_a : \mathbb{Z} \to \mathbb{R}$, $f_a(n) = f(n \bmod N)\, \chi_{[0,aN)}(n)$: a periods of f followed by zeros.

Lemma 3.11. Let $k \in \mathbb{N}$, $1 \leq p \leq \infty$, $N \in \mathbb{N}$ and $f : \mathbb{Z}_N \to \mathbb{R}$ be nonconstant. Then

$$\liminf_{a \to \infty} \frac{\big\|(\mathcal{M}^{\mathrm{u}}_{\mathbb{Z}} f_a)^{(k)}\big\|_{\ell^p(\mathbb{Z})}}{\big\|f_a^{(k)}\big\|_{\ell^p(\mathbb{Z})}} \;\geq\; \frac{\big\|(\mathcal{M}^{\mathrm{u}}_{\mathbb{Z}_N} f)^{(k)}\big\|_{\ell^p(\mathbb{Z}_N)}}{\big\|f^{(k)}\big\|_{\ell^p(\mathbb{Z}_N)}}.$$

For $f = \chi_S$ each f_a is the characteristic function of a finite set, so each ratio on the left is at most $C_{k,p}$, and therefore

$$C_{k,p} \;\geq\; R_{k,p}(S) \qquad \text{for every } N \geq 1,\ \emptyset \neq S \subsetneq \mathbb{Z}_N,\ k \geq 1,\ 1 \leq p \leq \infty. \tag{4}$$

This is how the lower bounds of Theorems 2.5, 2.7 and 2.8 of [1] are obtained ("They extend from the periodic setting to $\mathbb{Z}$ by Lemma 3.11", item (iii) of their proof strategies), and it is the only step in this paper that concerns $\mathbb{Z}$ rather than a finite circle. We do not reprove it.

Exact arithmetic. The supremum in (2) is a maximum over intervals of length at most $2N$, so $\mathcal{M}^{\mathrm{u}}_{\mathbb{Z}_N} \chi_S(n) = c/\ell$ with $\ell \leq 2N$, and the integer vector

$$m_S := D_N \cdot \mathcal{M}^{\mathrm{u}}_{\mathbb{Z}_N} \chi_S, \qquad D_N := \operatorname{lcm}(1, 2, \ldots, 2N), \tag{5}$$

is exact: $D_N \cdot (c/\ell) = c \cdot (D_N/\ell)$ with D_N/ℓ an integer. For example $D_7 = 360360$, $D_{10} = 232792560$, $D_{15} = 2329089562800$. Differences commute with the scaling, so with Δ^k the k-fold forward difference on $\mathbb{Z}_N$,

$$R_{k,1}(S) > \tfrac{1}{2} \iff 2\,\big\|\Delta^k m_S\big\|_{\ell^1} > D_N\,\big\|\Delta^k \chi_S\big\|_{\ell^1}, \tag{6}$$

$$R_{k,\infty}(S) > \tfrac{1}{2} \iff 2\,\big\|\Delta^k m_S\big\|_{\ell^\infty} > D_N\,\big\|\Delta^k \chi_S\big\|_{\ell^\infty}, \tag{7}$$

$$R_{k,2}(S) > \tfrac{1}{2} \iff 4\,\big\|\Delta^k m_S\big\|_{\ell^2}^2 > D_N^2\,\big\|\Delta^k \chi_S\big\|_{\ell^2}^2, \tag{8}$$

and each right-hand side is a strict inequality between integers; likewise an exact value $R_{k,1}(S) = a/b$ is the integer identity $b\,\big\|\Delta^k m_S\big\|_{\ell^1} = a\,D_N\,\big\|\Delta^k \chi_S\big\|_{\ell^1}$. Every theorem in Sections 3–5 is a finite family of such integer statements, and this is the form in which it was checked (Appendix A). At $p = 2$ we only ever compare the squared ratio with $1/4$; we never extract a square root.

Controls. Three facts about the model, all known or self-evident, fix the scale of everything that follows. The first is the witness of [1] for the lower bound 1/2.

Proposition 2.1. *On $\mathbb{Z}_3$, $\mathcal{M}^{\mathrm{u}}_{\mathbb{Z}_3}\chi_{\{0\}} = (1, \frac{1}{2}, \frac{1}{2})$. Consequently $(\mathcal{M}^{\mathrm{u}}_{\mathbb{Z}_3}\chi_{\{0\}})^{(k)} = \frac{1}{2}\,\chi^{(k)}_{\{0\}}$ for every $k \geq 1$, and the circle ratio of $\{0\} \subseteq \mathbb{Z}_3$ is exactly $1/2$ at every order $k \geq 1$ and every $1 \leq p \leq \infty$.*

Proof. The vector identity is the computation $m_{\{0\}} = (60, 30, 30)$ with $D_3 = 60$, checked in the kernel. Since $(1, \frac{1}{2}, \frac{1}{2}) = \frac{1}{2}(1, 0, 0) + (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and a constant vector is killed by one difference, $(\mathcal{M}^{\mathrm{u}}_{\mathbb{Z}_3}\chi_{\{0\}})^{(k)} = \frac{1}{2}\chi^{(k)}_{\{0\}}$ for $k \geq 1$, and the ratio is $1/2$ in every norm. □

This is the "$\chi_{\{0\}}$ on the circle $\mathbb{Z}_3$" of [1], and it shows that the bound $1/2$ is attained *exactly*, never exceeded, by that witness. The same happens on longer circles for the period-three set, the circle form of the extremizing sequence S_m of Theorem 2.5.

Proposition 2.2. *For $P := \{0, 3, 6, 9, 12\} \subseteq \mathbb{Z}_{15}$ and every $1 \leq k \leq 8$,*

$$\left\|(\mathcal{M}^{\mathrm{u}}_{\mathbb{Z}_{15}}\chi_P)^{(k)}\right\|_{\ell^1(\mathbb{Z}_{15})} = \tfrac{1}{2}\left\|\chi^{(k)}_P\right\|_{\ell^1(\mathbb{Z}_{15})} \quad \text{and} \quad \left\|(\mathcal{M}^{\mathrm{u}}_{\mathbb{Z}_{15}}\chi_P)^{(k)}\right\|_{\ell^\infty(\mathbb{Z}_{15})} = \tfrac{1}{2}\left\|\chi^{(k)}_P\right\|_{\ell^\infty(\mathbb{Z}_{15})},$$

with equality, not inequality. Moreover the enumeration of all subsets of $\mathbb{Z}_{10}$ used in Theorem 5.1 has exactly $2^{10} = 1024$ members.

Proof. Both parts are computations, checked by compiled evaluation (Section 7). □

Proposition 2.2 is the reason Theorem 5.1 is a sharp statement: its bound $\leq 1/2$ is attained inside the swept range, so the sweep does not pass because every ratio happens to be small. Finally, the maximal function was implemented twice — once as the literal supremum (2) over all $s+t+1 \leq 2N$, and once in the incremental form that the sweeps run, in which the count of S inside an interval is updated as the interval grows — and the two implementations were compared.

Proposition 2.3. *The two implementations of $S \mapsto m_S$ agree on every subset of every circle of length at most 9, that is, on all $2^0 + 2^1 + \cdots + 2^9 = 1023$ subsets.*

This is a consistency check on the model, not a statement about the literature; it is recorded because the theorems below are computed with the incremental form.

3. Two circles that beat 1/2

The two smallest witnesses are $A_7 = \{0, 1, 4\} \subseteq \mathbb{Z}_7$ and $A_{10} = \{0, 1, 4, 7\} \subseteq \mathbb{Z}_{10}$: a doubled point followed by a tail with gaps of three. Their maximal functions are as follows.

Proposition 3.1. (1) $\mathcal{M}^{\mathrm{u}}_{\mathbb{Z}_7}\chi_{A_7} = \big(1,\ 1,\ \frac{2}{3},\ \frac{3}{5},\ 1,\ \frac{3}{5},\ \frac{2}{3}\big)$.
(2) $\mathcal{M}^{\mathrm{u}}_{\mathbb{Z}_{10}}\chi_{A_{10}} = \big(1,\ 1,\ \frac{2}{3},\ \frac{3}{5},\ 1,\ \frac{1}{2},\ \frac{1}{2},\ 1,\ \frac{3}{5},\ \frac{2}{3}\big)$.

Proof. In the scaling (5),

$$\begin{aligned} m_{A_7} &= (360360,\ 360360,\ 240240,\ 216216,\ 360360,\ 216216,\ 240240), \\ m_{A_{10}} &= (232792560,\ 232792560,\ 155195040,\ 139675536,\ 232792560, \\ &\qquad 116396280,\ 116396280,\ 232792560,\ 139675536,\ 155195040); \end{aligned}$$

both are kernel computations. Anyone can recompute them by hand: on $\mathbb{Z}_7$, for instance, the point $2 \notin A_7$ sees the interval $\{0, 1, 2\}$ with density $2/3$, and no interval through 2 does better. □

Everything in the rest of this section is a difference computation on these two vectors.

Theorem 3.2. $R_{8,1}(A_7) = 13/25$. *In particular $R_{8,1}(A_7) > 1/2$, and by (4) $C_{8,1} \geq 13/25 > 1/2$.*

Proof. The exact value is the integer identity $25\,\|\Delta^8 m_{A_7}\|_{\ell^1} = 13 \cdot D_7\,\|\Delta^8 \chi_{A_7}\|_{\ell^1}$, and the strict inequality is (6); both are checked in the kernel. The last assertion is Lemma 3.11 of [1] applied to χ_{A_7}. □

Theorem 3.3. *$R_{8,1}(A_{10}) = 97/186$ and $R_{8,2}(A_{10})^2 = 23083/90504 > 1/4$. Hence $C_{8,1} > 1/2$ and $C_{8,2} > 1/2$.*

Proof. The ℓ^1 value is the identity $186 \left\|\Delta^8 m_{A_{10}}\right\|_{\ell^1} = 97 \cdot D_{10} \left\|\Delta^8 \chi_{A_{10}}\right\|_{\ell^1}$; the ℓ^2 claim is (8) at $k = 8$, an inequality between two integers (the squared ratio is rational, so no rounding enters); both are kernel computations, and (4) gives the bounds on $C_{8,1}$ and $C_{8,2}$. □

Theorem 3.4. *$R_{9,\infty}(A_{10}) = 139/270$. In particular $R_{9,\infty}(A_{10}) > 1/2$ and $C_{9,\infty} \geq 139/270 > 1/2$.*

Proof. The identity $270 \left\|\Delta^9 m_{A_{10}}\right\|_{\ell^\infty} = 139 \cdot D_{10} \left\|\Delta^9 \chi_{A_{10}}\right\|_{\ell^\infty}$ and the inequality (7) at $k = 9$, both in the kernel; then (4). □

The improvement is not confined to one order.

Theorem 3.5. *$R_{k,1}(A_{10}) > 1/2$ for every $k \in \{8, 9, \ldots, 16\}$; hence $C_{k,1} > 1/2$ for $8 \leq k \leq 16$.*

Proof. Nine instances of (6), checked in the kernel. □

Before this, the printed lower bound on $C_{8,1}$, $C_{8,2}$ and $C_{9,\infty}$ was $1/2$ non-strict, and the least order at which the exponential bound of Theorem 2.8 gives a strict inequality was 36 ($p = 1$), 23 ($p = 2$) and 36 ($p = \infty$). Theorems 3.2 to 3.4 move those to 8, 8 and 9.

The following negative controls delimit the claims. Each is a statement one degree too strong, and each is refuted by the same kernel computation that proves the theorems.

Proposition 3.6. (1) *At $k = 7$ neither witness beats $1/2$ in any of the three norms: $R_{7,p}(A_7) \leq 1/2$ and $R_{7,p}(A_{10}) \leq 1/2$ for $p \in \{1, 2, \infty\}$.*
(2) *A_7 does not beat $1/2$ in ℓ^∞ at $k = 8$: $R_{8,\infty}(A_7) \leq 1/2$ (the value is $7/15$).*
(3) *$R_{8,1}(A_7) \leq 27/50$; that is, the constant $13/25 = 0.52$ in Theorem 3.2 cannot be replaced by 0.54.*
(4) *The neighbouring three-point set $\{0, 1, 3\} \subseteq \mathbb{Z}_7$ beats $1/2$ in ℓ^1 at no order $3 \leq k \leq 16$.*

Proof. Kernel computations, as in Theorem 3.2; for (ii) the exact value $7/15$ is a direct computation that was not separately formalised, the kernel statement being the inequality. □

(i) says the crossing at $k = 8$ is real and not an artefact of the encoding; (ii) that a witness for one exponent is not automatically a witness for another — and indeed the least order for ℓ^∞ is 9, not 8 (Theorem 3.4 and Remark 4.5); (iii) that the exact value, not a rounded one, is what is claimed; (iv) that the witness is a property of the set $\{0, 1, 4\}$, with gaps $1, 3, 3$, and not of "three points on $\mathbb{Z}_7$".

4. The half-line $k \geq 8$

Theorem 3.5 already covers nine orders. The seven-point set does better: it covers every order up to the point where Theorem 2.8 of [1] takes over, with no order in between. We first record where that point is.

Lemma 4.1. *Let $\beta = 2/|1 + e^{\pi i/4}| = 2/\sqrt{2 + \sqrt{2}}$ and let $c_p = 1/(12 \cdot 8^{|1/p - 1/2|})$ be the constant of Theorem 2.8 of* [1]. *Then $c_p \beta^k > 1/2$ holds for $k \geq 36$ and fails for $k \leq 35$ when $p \in \{1, \infty\}$, and holds for $k \geq 23$ and fails for $k \leq 22$ when $p = 2$. Consequently Theorem 2.8 gives $C_{k,1} > 1/2$ and $C_{k,\infty} > 1/2$ for every $k \geq 36$, and $C_{k,2} > 1/2$ for every $k \geq 23$.*

Proof. $|1 + e^{\pi i/4}|^2 = 2 + \sqrt{2}$, so $\beta = 1.08239\ldots > 1$ and $k \mapsto c_p \beta^k$ is increasing. For $p \in \{1, \infty\}$, $c_p = 1/(12\sqrt{8})$ and $c_p \beta^{35} = 0.4707\ldots$, $c_p \beta^{36} = 0.5095\ldots$; for $p = 2$, $c_2 = 1/12$ and $c_2 \beta^{22} = 0.4756\ldots$, $c_2 \beta^{23} = 0.5148\ldots$. These four numerical evaluations are elementary and were not formalised; they are the only numerical step in this paper that is not an exact integer computation. □

Theorem 4.2. *$R_{k,1}(A_7) > 1/2$ for every $k \in \{8, 9, \ldots, 35\}$. Consequently $C_{k,1} > 1/2$ for every $k \geq 8$.*

Proof. The 28 inequalities (6) for A_7 and $8 \leq k \leq 35$ are checked in the kernel; (4) gives $C_{k,1} > 1/2$ for $8 \leq k \leq 35$, and Lemma 4.1 gives it for $k \geq 36$. The two halves meet with no order left over. □

Theorem 4.3. *$R_{k,2}(A_7) > 1/2$ for every $k \in \{9, 10, \dots, 22\}$. Together with $R_{8,2}(A_{10}) > 1/2$ (Theorem 3.3), $C_{k,2} > 1/2$ for every $k \geq 8$.*

Proof. The 14 inequalities (8) for A_7 and $9 \leq k \leq 22$ are integer inequalities (the squared ratio is rational) checked in the kernel. Order 8 is Theorem 3.3, orders 9 to 22 are these, and Lemma 4.1 covers $k \geq 23$. (A_7 itself does not beat $1/2$ in ℓ^2 at $k = 8$; its squared ratio there is $529/2175 < 1/4$, a direct computation made twice independently and not part of the formal development, which is why the order 8 needs A_{10}.) □

Theorem 4.4. *$R_{k,\infty}(A_7) > 1/2$ for every $k \in \{11, 12, \dots, 35\}$, and $R_{10,\infty}(A_{21}) = 35726/71379 > 1/2$ for $A_{21} = \{0, 3, 8, 9, 13, 14, 15, 16\} \subseteq \mathbb{Z}_{21}$. Together with $R_{9,\infty}(A_{10}) > 1/2$ (Theorem 3.4), $C_{k,\infty} > 1/2$ for every $k \geq 9$. Moreover $R_{9,\infty}(A_{21}) \leq 1/2$, so the three orders 9, 10 and $11 \leq k \leq 35$ genuinely use three different sets.*

Proof. The 25 inequalities (7) for A_7, the identity $71379 \left\|\Delta^{10} m_{A_{21}}\right\|_{\ell^\infty} = 35726 \cdot D_{21} \left\|\Delta^{10} \chi_{A_{21}}\right\|_{\ell^\infty}$ with $D_{21} = \operatorname{lcm}(1, \dots, 42)$, the inequality (7) for A_{21} at $k = 10$ and its negation at $k = 9$ are kernel computations. Order 9 is Theorem 3.4, order 10 is A_{21}, orders 11 to 35 are A_7, and Lemma 4.1 covers $k \geq 36$. The value of $R_{9,\infty}(A_{21})$ is 2867/5742 (a direct computation; the formal statement is the inequality). □

Remark 4.5 (The two ℓ^∞ gaps). In ℓ^∞ the half-line starts at 9 rather than 8, and its second order needs a circle of length 21. The exhaustive searches of Remark 5.2 — computations outside the formal development — found no subset of any circle of length at most 28 with $R_{8,\infty}(S) > 1/2$; and at order 10 the exact all-subsets run found no subset of any circle of length at most 20 with $R_{10,\infty}(S) > 1/2$, and on $\mathbb{Z}_{21}$ only the 42 images of A_{21} under rotation and reflection. So within that window both gaps are properties of the problem, not of the search. The seven-point set itself has $R_{k,\infty}(A_7) = 19/39 < 1/2$ at $k = 9$ and at $k = 10$ (direct computation, made twice independently, not formalised), which is why it enters only at $k = 11$.

Remark 4.6 (Comparison with the witness of Theorem 2.8). The set $B_1 = \{0, 1, 2, 5\} \subseteq \mathbb{Z}_8$ of [1] is chosen for divergence: its ratio grows like β^k. But it starts low. A direct exact computation (not part of the formal development, made twice independently) gives $R_{k,\infty}(B_1) \leq 1/2$ for $k \leq 12$ and $> 1/2$ first at $k = 13$, $R_{k,2}(B_1) > 1/2$ first at $k = 19$, and $R_{k,1}(B_1) > 1/2$ first at $k = 22$; in fact no subset of $\mathbb{Z}_8$ beats $1/2$ in ℓ^1 before order 22 or in ℓ^∞ before order 13, and at those orders only the rotations of B_1 do (Remark 5.2). The witnesses of this paper are not divergent — see Remark 5.3 — but they cross $1/2$ at 8 and 9.

5. The wall below order eight

The following theorem is an exhaustive statement: it quantifies over *every* subset of *every* circle of length at most 15.

Theorem 5.1. *Let $1 \leq N \leq 15$, $S \subseteq \mathbb{Z}_N$ and $2 \leq k \leq 7$. Then*

$$\left\|(\mathcal{M}^{\mathrm{u}}_{\mathbb{Z}_N} \chi_S)^{(k)}\right\|_{\ell^1(\mathbb{Z}_N)} \leq \tfrac{1}{2} \left\|\chi_S^{(k)}\right\|_{\ell^1(\mathbb{Z}_N)} \qquad \text{and} \qquad \left\|(\mathcal{M}^{\mathrm{u}}_{\mathbb{Z}_N} \chi_S)^{(k)}\right\|_{\ell^\infty(\mathbb{Z}_N)} \leq \tfrac{1}{2} \left\|\chi_S^{(k)}\right\|_{\ell^\infty(\mathbb{Z}_N)}.$$

Equivalently, $R_{k,p}(S) \leq 1/2$ for $p \in \{1, \infty\}$ whenever the ratio is defined.

Proof. For each $N \leq 15$ the list of all 2^N binary words of length N is generated, and for each word S the vector m_S is computed once and the six pairs of integer inequalities (6)–(7) (negated) are evaluated for $k = 2, \dots, 7$; in total $2^0 + \cdots + 2^{15} = 65\,535$ subsets, six orders and two norms each. The evaluation is by compiled code (Section 7). What makes the result a statement about every subset rather than about a list is a separate, ordinary proof that the generated list contains every binary word of length N. □

The three orders in Theorem 5.1 have three different statuses. Order $k = 2$ is the known value $C_{2,p} = 1/2$ of Theorem 2.5 of [1], restricted to circles: a positive control that the sweep must reproduce, and does. Orders $3 \le k \le 6$ are the range of the experiment reported in [1]; that experiment ran on circles of length "at most about 30" with the program [5], whose README describes floating-point arithmetic with error bounds and exact fractions in machine words, and Theorem 5.1 is its exact, every-subset version on the window $N \le 15$. Order $k = 7$ is one beyond the range the source reports and is covered by no statement in it. With Theorem 3.2, order 8 is therefore the least order at which any circle of length at most 15 beats $1/2$ in ℓ^1; the search below pushes the window to 28.

Remark 5.2 (The search, and its coverage). None of what follows is machine-checked; it is the computation that located the witnesses and that bounds where further ones may lie. A C program computed m_S and the ratios $R_{k,p}(S)$, $3 \le k \le 12$, $p \in \{1, 2, \infty\}$, for every subset S of every circle of length $N \le 28$: all 2^N subsets for $N \le 16$, and one representative per rotation class for $17 \le N \le 28$ (the ratio is invariant under rotation of S). At $p = 1$ and $p = \infty$ its arithmetic is exact, in 128-bit integers scaled by D_N; at $p = 2$ it accumulated the squared ratio in extended precision and reported every value within 10^{-11} of the threshold. Coverage of the rotation-class enumerator was checked, not assumed: for every $17 \le N \le 28$ its class count equals $a(N) - 2$, where $a(N)$ is the number of binary necklaces of length N (OEIS A000031, [6]; the -2 removes the two constant subsets), and for $N \in \{7, 10, 13, 16\}$ the rotation-class answer equals the all-subsets answer. A second program, written independently for the refereeing of this paper directly from (2), re-ran all 2^N subsets for $1 \le k \le 12$: exactly in all three norms for $N \le 18$ (the squared ℓ^2 ratio compared with $1/4$ as integers), exactly in ℓ^1 and ℓ^∞ for $N \le 22$, and up to $k = 22$ for $N \le 16$; within its window it agrees with the first program on every statement below, and with every exact value of Sections 3 and 4. The findings: no subset of any circle of length at most 28 has $R_{k,p}(S) > 1/2$ for any $3 \le k \le 7$ and $p \in \{1, 2, \infty\}$ — exactly for $p \in \{1, \infty\}$, exactly for $p = 2$ and $N \le 18$, in extended precision for $p = 2$ and $19 \le N \le 28$; at $k = 8$ the least circle with a witness has length 7 in ℓ^1 and 10 in ℓ^2, and there is none in ℓ^∞; at $k = 9$ the least in ℓ^∞ has length 10; and the least order at which some subset of $\mathbb{Z}_N$ beats $1/2$ in ℓ^1 is $8, 22, 15, 8, 9, 11, 8, 8, 9, 8$ for $N = 7, 8, \dots, 16$. On the $p = 2$ ties: for $N \le 18$ the subsets whose squared ratio equals $1/4$ exactly at some $3 \le k \le 12$ are precisely the three rotations of the period-three set on $N \in \{3, 6, 9, 12, 15, 18\}$ (Proposition 2.2) and no other; for $19 \le N \le 28$ the first program's near-tie log listed only period-three sets, at $N \in \{21, 24, 27\}$, the multiples of three in that range (the same log had missed the ties at $N \in \{3, 6, 9, 18\}$, all inside the window now settled exactly), so there the $p = 2$ negatives rest on extended precision with a 10^{-11} margin. The cost of the first program was $27.7\,\mu$s per rotation class at $N = 24$, scaling as $(N/24)^2$; $N = 28$ (some 9.6×10^6 classes) took 352 s of CPU, and $N = 32$ is priced at about 1.8 CPU-hours.

Remark 5.3 (Why these sets, and why order eight). The following heuristic, a floating-point computation outside the formal development, explains the shape of the witnesses. On $\mathbb{Z}_N$ with $\omega = e^{2\pi i/N}$ one has $\widehat{f^{(k)}}(\xi) = (\omega^\xi - 1)^k \hat{f}(\xi)$, so as $k \to \infty$ the frequency ξ^* maximising $|\omega^\xi - 1| = 2|\sin(\pi\xi/N)|$ among those with $\hat{\chi}_S(\xi) \ne 0$ dominates, and by Parseval $\lim_{k\to\infty} R_{k,2}(S) = |\widehat{\mathcal{M}^{\mathrm{u}}_{\mathbb{Z}_N} \chi_S}(\xi^*)|/|\hat{\chi}_S(\xi^*)|$, while the ℓ^1 and ℓ^∞ ratios oscillate in a band around this limit because the phase of $(\omega^{\xi^*} - 1)^k$ keeps rotating. For A_7 the limit is $0.5498\dots$ ($\xi^* = 3$) and for A_{10} it is $0.5697\dots$ ($\xi^* = 4$): both finite and both above $1/2$, which is why these sets cross $1/2$ early and stay above it. For $B_1 = \{0, 1, 2, 5\}$ the top frequency $\xi = 4 = N/2$ has $\hat{\chi}_{B_1}(4) = 0$ but $\widehat{\mathcal{M}^{\mathrm{u}}_{\mathbb{Z}_8} \chi_{B_1}}(4) \ne 0$, so the ratio diverges — that is the mechanism of Theorem 2.8 — but from a low start. Searching for sets whose top-frequency amplitude ratio exceeds $1/2$ is a computation free of k, and is the cheapest route we know towards witnesses on circles well beyond length 28. (The two limits were computed three times independently in floating point; they appear in no theorem.)

6. Relation to prior work

For $k = 1$ the constant $C_{k,p}$ of Definition 1.12 of [1] is a variation constant with a literature of its own, surveyed in [1, §1.2]; Madrid [4] is one sharp result of that kind. For $k \ge 2$ the works we know

of are three. Temur [3] gives the $k = 2$ upper bound (Theorem 1.10 of [1]) that Theorem 2.5 of [1] sharpens to $1/2$; Temur and Özcan [2] prove the exponential *upper* bound for general k (Theorem 1.11 of [1]); and [1] itself supplies the lower bounds quoted above. None of the three states a lower bound strictly above $1/2$ at any finite $k \geq 3$ below the exponential regime. Two searches of OpenAlex on 3 September 2026 found no work citing [1]; a full-text search of an arXiv corpus of 373 text shards for the phrases "higher order regularity of discrete maximal", "regularity of discrete maximal functions" and "kth derivative of the discrete" and for the identifier 2607.10753 found no paper other than [1] itself; the sequences of values in this paper $(13, 25, 97, 186, 139, 270)$ and of least orders per circle length $(8, 22, 15, 8, 9, 11, 8, 8, 9, 8$, Remark 5.2) are not in the OEIS.

One notational clash deserves a warning. The symbol $C_{k,p}$ also appears in [2], for a different quantity: a constant in the inequality $\left\|\chi_A^{(k)}\right\|_{\ell^p(\mathbb{Z})} \geq C_{k,p} \left\|\chi_A'\right\|_{\ell^p(\mathbb{Z})}$ for finite $A \subseteq \mathbb{Z}$, which involves no maximal operator. The remark there (v2, §1) that "a uniform bound $C_{k,p} \geq 2^{k-o_p(k)}$ may be possible" concerns that constant and not the one studied here.

Finally, the relation to the search code [5] of the source's own authors is as stated in the introduction: the program is public, its default sweep (circles of length 2 to 36, derivative orders 0 to 64, $p \in \{1, 2, 4, 8, \infty\}$) covers every set and order in this paper, it prints each new record it finds, and a one-minute run of it as shipped prints the threshold picture of this paper in floating point. The repository, read on 3 September 2026, holds 93 commits on one branch (the last change to the code dated 23 February 2026), C sources, a makefile, a README and a licence, no issues and no result files; the cgit mirror named in its README holds the same history. Its commit history mentions "the function 11100100", that is $\chi_{\{0,1,2,5\}}$ on $\mathbb{Z}_8$, as the one "which seems to be the smallest for which `||Mf^(k)||/||f^(k)||` goes to infinity with k" — a statement about which set diverges, not about the least order at which $1/2$ is beaten. We make no claim of priority of discovery over that program; we claim the exact values, the two half-lines, and the certified wall, none of which is stated in [1] or in [5].

7. Verification

Every theorem and proposition in Sections 2–5 that carries a proof, except the numerical evaluations in Lemma 4.1, is formalised and checked in Lean 4 [7]; the development imports nothing from Mathlib, only the core library, since every statement is an inequality or equality between integers. A subset of $\mathbb{Z}_N$ is a list of N booleans, the maximal function is the integer vector m_S of (5), and differences and norms are integer list operations. The witness statements (Proposition 2.1, Proposition 3.1, Theorems 3.2–3.5, Proposition 3.6, and the circle statements of Theorems 4.2–4.4) are proved by `decide`, that is, by kernel evaluation; `#print axioms` reports `[propext]` for each of them. The exhaustive sweep of Theorem 5.1 and the controls of Propositions 2.2 and 2.3 are evaluated by compiled code (`native_decide`), which adds to the trusted base the compiler and, for each such theorem, one axiom named after it; the lemma that the enumerated list contains every subset is an ordinary structural induction, with axioms `[propext, Classical.choice, Quot.sound]`. Checking times on one core: the two witness modules 23 s and 102 s of CPU, the sweep module 526 s of CPU and 1.5 GB. Two steps are cited and not formalised: the passage from a circle ratio to $C_{k,p}$ (Lemma 3.11 of [1], display (4)) and the exponential regime (Theorem 2.8 of [1] with Lemma 4.1); no formal statement mentions $\mathbb{Z}$ or $C_{k,p}$. Appendix A reproduces the exact formal statement of every declaration the theorems rest on. Repository reference to be added at submission.

8. Open questions

Question 8.1. Is $C_{k,p} > 1/2$ for $k \in \{3, 4, 5, 6\}$? This is the question of [1], and it is untouched here: every circle of length at most 28 says no in ℓ^1 and ℓ^∞ (Remark 5.2), and every circle of length at most 15 says so with a kernel proof (Theorem 5.1). Also open, and new, is $k = 7$, for which the same evidence holds.

Question 8.2. Is $C_{8,\infty} > 1/2$? No circle of length at most 28 beats $1/2$ in ℓ^∞ at order 8, while circles of length 7 and 10 do so in ℓ^1 and ℓ^2. Is 9 really the least order for ℓ^∞, and is the dependence of the threshold on p explainable?

Question 8.3. Does the family $\{0,1\} \cup \{4,7,\dots,3m+1\} \subseteq \mathbb{Z}_{3m+4}$, to which A_7 and A_{10} belong, supply the witness of least order at every length $N \equiv 1 \pmod 3$? In the exact all-subsets run of Remark 5.2 it does at $N \in \{7,10,13,16,19,22\}$: there the least order with an ℓ^1 witness is 8, and the rotations of the family's member are the only subsets attaining it. The amplitude heuristic of Remark 5.3 suggests where to look for a proof.

Appendix A. The formal statements

For reference, the exact statements as checked, in the order of the text. D_N is `scaleOf N`, m_S is `mscale S`, χ_S is `chiZ S`, Δ^k is `diffIter k`, and `beatsHalfOne`, `beatsHalfInf`, `beatsHalfTwo` are the integer forms (6)–(8).

```
def lcmUpTo : Nat → Nat
  | 0 => 1
  | n + 1 => Nat.lcm (lcmUpTo n) (n + 1)
def scaleOf (N : Nat) : Nat := lcmUpTo (2 * N)
abbrev Sub := List Bool
def chiZ (S : Sub) : List Int := S.map (fun b => if b then 1 else 0)
def mscale (S : Sub) : List Int := (List.range S.length).map (fun n => (mscaleAt S n : Int))
def diffOnce (v : List Int) : List Int :=
  (List.range v.length).map (fun n => v.getD ((n + 1) % v.length) 0 - v.getD n 0)
def diffIter : Nat → List Int → List Int
  | 0, v => v
  | k + 1, v => diffOnce (diffIter k v)
def l1 (v : List Int) : Int := v.foldl (fun a b => a + b.natAbs) 0
def linf (v : List Int) : Int := v.foldl (fun a b => max a (b.natAbs : Int)) 0
def l2sq (v : List Int) : Int := v.foldl (fun a b => a + b * b) 0
def beatsHalfOne (S : Sub) (k : Nat) : Prop :=
  2 * l1 (diffIter k (mscale S)) > (scaleOf S.length : Int) * l1 (diffIter k (chiZ S))
def beatsHalfInf (S : Sub) (k : Nat) : Prop :=
  2 * linf (diffIter k (mscale S)) > (scaleOf S.length : Int) * linf (diffIter k (chiZ S))
def beatsHalfTwo (S : Sub) (k : Nat) : Prop :=
  4 * l2sq (diffIter k (mscale S)) > (scaleOf S.length : Int) ^ 2 * l2sq (diffIter k (chiZ S))
```

Proposition 2.1:

```
def z3 : Sub := [true, false, false]
theorem z3_maxfn : mscale z3 = [60, 30, 30]
```

Proposition 3.1:

```
def w7 : Sub := [true, true, false, false, true, false, false]
def w10 : Sub := [true, true, false, false, true, false, false, true, false, false]
theorem w7_maxfn : mscale w7 = [360360, 360360, 240240, 216216, 360360, 216216, 240240]
theorem w10_maxfn : mscale w10 =
    [232792560, 232792560, 155195040, 139675536, 232792560, 116396280, 116396280,
     232792560, 139675536, 155195040]
```

Theorems 3.2, 3.3, 3.4, 3.5:

```
theorem w7_k8_l1_exact :
    25 * l1 (diffIter 8 (mscale w7)) = 13 * (scaleOf 7 : Int) * l1 (diffIter 8 (chiZ w7))
theorem w7_k8_beats : beatsHalfOne w7 8
theorem w10_k8_l1_exact :
    186 * l1 (diffIter 8 (mscale w10)) = 97 * (scaleOf 10 : Int) * l1 (diffIter 8 (chiZ w10))
```

```
theorem w10_k8_l2_beats : beatsHalfTwo w10 8
theorem w10_k9_linf_exact :
    270 * linf (diffIter 9 (mscale w10)) = 139 * (scaleOf 10 : Int) * linf (diffIter 9 (chiZ w10))
theorem w10_k9_linf_beats : beatsHalfInf w10 9
theorem w10_l1_beats_8_to_16 : ∀ k ∈ [8, 9, 10, 11, 12, 13, 14, 15, 16], beatsHalfOne w10 k
```

Proposition 3.6:

```
theorem w7_k7_fails : ¬ beatsHalfOne w7 7 ∧ ¬ beatsHalfInf w7 7 ∧ ¬ beatsHalfTwo w7 7
theorem w10_k7_fails : ¬ beatsHalfOne w10 7 ∧ ¬ beatsHalfInf w10 7 ∧ ¬ beatsHalfTwo w10 7
theorem w7_k8_linf_fails : ¬ beatsHalfInf w7 8
theorem w7_k8_not_beyond :
    ¬ (50 * l1 (diffIter 8 (mscale w7)) > 27 * (scaleOf 7 : Int) * l1 (diffIter 8 (chiZ w7)))
theorem w7_neighbour_fails :
    ∀ k ∈ [3,4,5,6,7,8,9,10,11,12,13,14,15,16],
      ¬ beatsHalfOne [true, true, false, true, false, false, false] k
```

Theorems 4.2, 4.3, 4.4:

```
def ks835 : List Nat :=
  [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31,
   32, 33, 34, 35]
theorem w7_l1_beats_8_to_35 : ∀ k ∈ ks835, beatsHalfOne w7 k
def ks922 : List Nat := [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22]
theorem w7_l2_beats_9_to_22 : ∀ k ∈ ks922, beatsHalfTwo w7 k
def ks1135 : List Nat :=
  [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33,
   34, 35]
theorem w7_linf_beats_11_to_35 : ∀ k ∈ ks1135, beatsHalfInf w7 k
def w21 : Sub :=
  [true, false, false, true, false, false, false, false, true, true, false, false, false,
   true, true, true, true, false, false, false, false]
theorem w21_k10_linf_exact :
    71379 * linf (diffIter 10 (mscale w21)) =
      35726 * (scaleOf 21 : Int) * linf (diffIter 10 (chiZ w21))
theorem w21_k10_linf_beats : beatsHalfInf w21 10
theorem w21_k9_linf_fails : ¬ beatsHalfInf w21 9
```

Theorem 5.1 and Propositions 2.2, 2.3:

```
def allMasks : Nat → List Sub
  | 0 => [[]]
  | n + 1 => (allMasks n).flatMap (fun S => [false :: S, true :: S])
theorem mem_allMasks : ∀ (n : Nat) (S : Sub), S.length = n → S ∈ allMasks n
def ks27 : List Nat := [2, 3, 4, 5, 6, 7]
theorem no_beat_le15 (S : Sub) (hS : S.length ≤ 15) (k : Nat) (hk : k ∈ ks27) :
    ¬ beatsHalfOne S k ∧ ¬ beatsHalfInf S k
def per3 : Sub :=
  [true, false, false, true, false, false, true, false, false, true, false, false,
   true, false, false]
theorem per3_exactly_half :
    ∀ k ∈ [1, 2, 3, 4, 5, 6, 7, 8],
      2 * l1 (diffIter k (mscale per3)) = (scaleOf 15 : Int) * l1 (diffIter k (chiZ per3)) ∧
      2 * linf (diffIter k (mscale per3)) = (scaleOf 15 : Int) * linf (diffIter k (chiZ per3))
theorem sweep_nonvacuous : (allMasks 10).length = 1024
```

```
theorem mscale_eq_ref_le9 :
    ((List.range 10).all fun n => (allMasks n).all fun S => mscale S == mscaleRef S) = true
```

Here `mscaleRef` is the literal supremum (2) over $s+t+1 \le 2N$ and `mscaleAt` the incremental form; the definitions of both are in the development.

References

[1] S.-Y. Liao, J. Madrid, E. Palsson and J. Weigt, *Sharp higher order regularity of discrete maximal functions*, arXiv: 2607.10753 (math.CA; v1 12 July 2026, v2 14 July 2026; "22 pages, 2 tables"). All theorem, lemma, definition and table numbers in this paper are those of the compiled v2 e-print, re-read in the arXiv PDF of v2 on 3 September 2026; the record showed no journal reference and no withdrawal on that date.

[2] F. Temur and H. B. Özcan, *The higher regularity of the discrete Hardy–Littlewood maximal function*, arXiv: 2504.13019 (math.CA; v2, 31 July 2025, the version quoted in Section 6). The exponential upper bound, Theorem 1.11 of [1].

[3] F. Temur, *The second derivative of the discrete Hardy–Littlewood maximal function*, Istanbul Journal of Mathematics **3** (2025), no. 2, 45–47, DOI `10.26650/ijmath.2025.00026`; arXiv:2205.03953 (math.CA). The $k = 2$ upper bound sharpened by Theorem 2.5 of [1].

[4] J. Madrid, *Sharp inequalities for the variation of the discrete maximal function*, Bulletin of the Australian Mathematical Society **95** (2017), no. 1, 94–107, DOI `10.1017/S0004972716000903`; arXiv:1512.04112 (math.CA). The case $k = 1$.

[5] J. Weigt, *discrete-mf*, C source code, `https://codeberg.org/julian-weigt/discrete-mf` (cited in [1] as [Wei26], dated 2026-07-11 there; repository read, built and run on 3 September 2026, at commit `f53272c1`).

[6] OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, entries A000031 (number of n-bead necklaces with 2 colors when turning over is not allowed) and A100446 (bisection of A000031), `https://oeis.org`, read 3 September 2026.

[7] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, in: Automated Deduction — CADE 28, Lecture Notes in Computer Science, Springer, 2021, 625–635; DOI `10.1007/978-3-030-79876-5_37`.