THE MINIMAL DEGREE OF A POLYNOMIAL IDENTITY OF ALL MATSUO ALGEBRAS IS AT LEAST SIX

KOREA SUPERINTELLIGENCE LABS

Abstract. Matsuo algebras are the commutative non-associative algebras attached to groups of 3-transpositions; they are the axial algebras of Jordan type that come from a group, and for $\eta \neq \frac{1}{2}$ every algebra of Jordan type η is one of them or a quotient of one. A recent survey of Gorshkov and Shpectorov records a problem of Rowen: do all Matsuo algebras satisfy identities not implied by commutativity, and if they do, what is the minimal degree of such an identity? The survey states the published lower bound as 4, attributing it to a proposition of Chayet and Garibaldi through work of Osborn. We raise the bound to 6. The proof rests on a single small algebra: the six-dimensional Matsuo algebra $M_2(S_4)$ over $\mathbb{Q}$, built from the six transpositions of S_4 with $\eta = 2$, whose space of multilinear identities is *zero* in degrees 3, 4 and 5. The degree-5 statement is certified by an explicit 105×105 integer block of the evaluation matrix together with an explicit inverse of that block modulo 101. We also verify that $M_2(S_4)$ is unital with $1 = \frac{1}{5}\sum_{a \in D} a$, that the Frobenius form printed in the survey is associative on it with Gram determinant 5, and that it is not a Jordan algebra — exactly the hypotheses under which the published proposition applies, so that the bound the published literature really gives is 5 and the theorem below is one degree past it. A rank computation carried out outside the formal development gives degree 6 as well, which would raise the bound to 7; that computation is labelled as such wherever it appears and is not machine-checked. Everything else is machine-checked in Lean 4.

1. Introduction

The objects. A 3-*transposition group* is a pair (G, D) in which G is a group generated by a normal set D of involutions such that $|de| \leq 3$ for all $d, e \in D$; the classification of those with trivial centre is due to Cuypers and Hall [3]. Fix a field $\mathbb{F}$ with $\operatorname{char} \mathbb{F} \neq 2$ and a scalar $\eta \in \mathbb{F} \setminus \{0, 1\}$. The *Matsuo algebra* $A = M_\eta(G, D)$ is the commutative non-associative $\mathbb{F}$-algebra with basis D and multiplication

$$a \circ b = \begin{cases} a, & \text{if } b = a, \\ 0, & \text{if } |ab| = 2, \\ \frac{\eta}{2}(a + b - c), & \text{if } |ab| = 3, \end{cases} \tag{1}$$

where $a, b \in D$ and, when $|ab| = 3$, $c = a^b = b^a$. We take (1) verbatim from the survey of Gorshkov and Shpectorov [5], which is our source throughout. The survey states the definition without attribution; the construction itself goes back to Matsuo [10, Table 1], who attaches to any Fischer space a commutative algebra over $\mathbb{C}$ with two parameters γ, δ, in a basis $\tilde{x}$ normalised so that $\tilde{x} \cdot \tilde{x} = 2\tilde{x}$ and $\tilde{x} \cdot \tilde{y} = \frac{\delta}{2}(\tilde{x} + \tilde{y} - \tilde{z})$ on a line $\{x, y, z\}$; rescaling that basis by $\frac{1}{2}$ turns his table into (1) with $\eta = \delta/2$, and his invariant bilinear form into the Frobenius form of Section 3. Every $a \in D$ is a primitive idempotent whose adjoint map satisfies the Jordan-type fusion law $J(\eta)$, so Matsuo algebras belong to the class of algebras of Jordan type η [5] — the primitive axial algebras with that fusion law. They are not a side class: the survey records, verbatim, that "for $\eta \neq \frac{1}{2}$, every algebra of Jordan type η is a Matsuo algebra or a quotient of a Matsuo algebra" [5], attributing this to Hall, Rehren and Shpectorov [7].

The product (1) only sees the *Fischer space* of (G, D): the point-line geometry whose points are the elements of D and whose lines are the triples $\{a, b, a^b\}$ with $|ab| = 3$. The survey defines a Fischer space intrinsically as a partial linear space with all lines of size 3 in which any two intersecting lines generate either the dual affine plane of order 2 or the affine plane of order 3, and observes that "the

Matsuo algebra can be defined in terms of the Fischer space, which means that centrally equivalent 3-transposition groups produce isomorphic Matsuo algebras" [5]. We use that reduction throughout: an algebra below is named by its Fischer space.

The problem, and what was known. The survey [5] collects open problems in axial algebra theory. One of them, attributed there to L. Rowen, reads verbatim:

Problem 1.1 ([5], attributed to L. Rowen)**.** Do all Matsuo algebras satisfy identities not implied by commutativity? If they do, what is the minimal degree of an identity of all Matsuo algebras not implied by commutativity?

The survey adds, again verbatim: "Rowen pointed out that for Matsuo algebras, which are not Jordan algebras, the degree would have to be at least 4 by a result of Chayet and Garibaldi [CG, Proposition A.8]. Note that this result refers to the paper by Osborn [O] where unitality of the algebra is assumed. Matsuo algebras are generically unital, as for each 3-transposition group (G, D), there is only a finite number of values of η for which the Matsuo algebra $M_\eta(G, D)$ is non-unital."

So the published state of Problem 1.1 is a lower bound of 4. The proposition it rests on reads, verbatim [2, Prop. A.8]:

> Let A be a commutative k-algebra where $\operatorname{car} k \neq 2, 3, 5$ and suppose that A is metrized. If A satisfies an identity of degree ≤ 4 not implied by commutativity, then A satisfies the Jordan identity $x(x^2y) = x^2(xy)$ and is power-associative.

Here *metrized* is defined in [2] as "having a nondegenerate and associative bilinear form", and the appendix in which the proposition sits assumes throughout that all algebras considered are finite-dimensional. Its proof goes through Osborn's classification [12, Th. 4] of the identities of degree ≤ 4 that a commutative algebra can satisfy; for the situation *without* the metrized hypothesis, which is more complicated, [2] points to a later paper of Osborn and to Carini, Hentzel and Piacentini Cattaneo [1]. Read contrapositively, the proposition already gives more than the survey states: a finite-dimensional Matsuo algebra over a field of characteristic $\neq 2, 3, 5$ that is metrized and is *not* Jordan satisfies no identity of degree ≤ 4 not implied by commutativity, so the minimal degree in Problem 1.1 is at least 5. The survey's more cautious 4 comes from the unitality assumption it flags in Osborn's input; no unitality appears in the proposition as stated above. We verify below that the six-dimensional algebra used in this paper meets every hypothesis, unitality included (Propositions 3.1 and 3.2), so 5 is the strongest bound the published literature supports.

What is settled here. Our contribution is a single degree past that.

Theorem A. Let $M_2(S_4)$ be the Matsuo algebra over $\mathbb{Q}$ of the Fischer space of the six transpositions of S_4, with $\eta = 2$. Then $M_2(S_4)$ satisfies no nonzero multilinear identity of degree 5. Consequently, the minimal degree of an identity of all Matsuo algebras not implied by commutativity — if such an identity exists at all — is at least 6.

This is Theorem 4.2 and Corollary 4.3. The asymmetry that makes it reachable is worth stating at once. A *lower* bound needs only one Matsuo algebra: an identity of all Matsuo algebras is in particular an identity of any single one of them, so a single algebra with trivial identity space in degree d forces the minimal degree above d. An *upper* bound needs an identity valid in every Matsuo algebra, and although multilinearity localises the check to the Fischer spaces generated by at most d points, that is a localisation and not a finite list. No upper bound is claimed anywhere in this paper, and Problem 1.1 therefore remains open in its first, qualitative half.

The algebra $M_2(S_4)$ is 6-dimensional and, because $\eta/2 = 1$, all of its structure constants are 0 or ± 1; the whole degree-5 question for it is the invertibility of an integer matrix. Along the way we also record the following, partly to certify that the published bound applies to this algebra and partly as controls on the computation:

- $M_2(S_4)$ is unital with $1 = \frac{1}{5} \sum_{a \in D} a$, the Frobenius form printed in [5] is associative on it, and the Gram matrix of that form is invertible with determinant 5 (Proposition 3.1);

- the multilinear Jordan identity holds at every substitution of axes of $M_{1/2}(S_4)$ and fails in $M_2(S_4)$ (Proposition 3.2), so the published bound genuinely applies to $M_2(S_4)$ and is not being read off a Jordan algebra;
- the 12-dimensional Matsuo algebra of Jordan type $\frac{1}{2}$ of the Fischer space of the reflection class of $W(D_4)$ — which we identify, on the grounds set out in Remark 3.5, with the algebra the survey calls "the smallest non-Jordan Matsuo algebra of Jordan type half (for the group $2^4{:}S_4$)" — fails the Jordan identity at four named axes and is not power associative (Proposition 3.4), with explicit witnesses;
- degrees 3 and 4 of the main computation, which the published proposition already covers, come out of the same certificate (Proposition 4.1) and serve as its positive control.

Finally, Section 5 records a rank computation in degree 6 that was run outside the formal development and that would raise the bound to 7. It is labelled as an unverified computation wherever it appears, and no statement of this paper depends on it.

Provenance. The propositions and the theorem of Sections 3–4 are formalised and machine-checked in Lean 4. Section 6 says exactly what that development contains, what it rests on, and which of the surrounding arguments and computations — among them everything in Section 5 — lie outside it; those are marked where they occur.

2. Preliminaries

Identities not implied by commutativity. Let $\mathbb{F}$ have characteristic 0. We follow the standard reading of Problem 1.1: since the free non-associative $\mathbb{F}$-algebra modulo the T-ideal generated by $xy - yx$ is the free *commutative* non-associative $\mathbb{F}$-algebra $\mathbb{F}\{X\}^c$ on $X = \{x_1, x_2, \dots\}$, an *identity not implied by commutativity* of an algebra A is a nonzero element $f \in \mathbb{F}\{X\}^c$ that vanishes under every substitution of elements of A for the variables. Neither [5] nor [13] spells the phrase out; the reading above is the standard one, and it is the one under which [2] uses the phrase, since [2, Prop. A.8] quantifies over commutative algebras and identities of degree ≤ 4 in exactly this sense. The *minimal degree* in Problem 1.1 is the least degree of such an f vanishing on every Matsuo algebra.

A multilinear element of $\mathbb{F}\{X\}^c$ of degree d is an $\mathbb{F}$-combination of the commutative bracketings of the word $x_1 x_2 \cdots x_d$. Write $m_1, \dots, m_N$ for those bracketings, where

$$N = N(d) = (2d-3)!! = 1,\ 1,\ 3,\ 15,\ 105,\ 945,\ 10395 \qquad (d = 1, \dots, 7), \tag{2}$$

the classical count: $N(d) = a(d-1)$ for the double factorials $a(n) = (2n-1)!!$ of [11, A001147], where $a(n)$ is recorded as "the number of distinct products of $n+1$ variables with commutative, nonassociative multiplication". We use two standard facts about characteristic 0. First, full linearisation [14]: an algebra satisfies a nonzero identity of degree d if and only if it satisfies a nonzero *multilinear* identity of degree d. (In one direction this is immediate; in the other, pass to the homogeneous component of top degree, replace each variable x_i of degree d_i by d_i fresh ones and keep the multilinear part — specialising the fresh variables back multiplies the original by $\prod_i d_i! \neq 0$, so the multilinear polynomial obtained is again nonzero, and its degree is again d.) Second, a multilinear identity of degree d may be padded to one of degree $d+1$ by right multiplication by a fresh variable, $f \mapsto f\,x_{d+1}$, and this map is injective on the multilinear component because it takes distinct bracketings to distinct bracketings.

The evaluation matrix. Fix a Matsuo algebra A with basis the axes $e_0, \dots, e_{n-1}$ (a numbering of D), and fix d. For a coefficient vector $c \in \mathbb{F}^N$ put $f_c = \sum_{i=1}^N c_i m_i$. Given a substitution $s\colon \{1, \dots, d\} \to \{0, \dots, n-1\}$ and a coordinate index k, the assignment

$$c \longmapsto \langle f_c(e_{s(1)}, \dots, e_{s(d)}),\, e_k^* \rangle \tag{3}$$

is a linear functional on $\mathbb{F}^N$. Because f_c is multilinear, f_c is an identity of A if and only if all $n^d \cdot n$ functionals (3) kill c; so the space of degree-d multilinear identities of A is the intersection of their kernels, and deciding whether it is zero is an exact rank computation over $\mathbb{F}$. This gives the shape of every proof below: to certify that the identity space is zero it suffices to exhibit N of the functionals (3) whose matrix is invertible.

The two Fischer spaces used. We work with two Fischer spaces, each recorded as its list of lines.

The space of S_4. Points $0, \dots, 5$ are the transpositions $(12), (13), (14), (23), (24), (34)$ and the four lines are the triples of transpositions supported on a common 3-subset of $\{1, 2, 3, 4\}$:

$$\{0, 1, 3\}, \quad \{0, 2, 4\}, \quad \{1, 2, 5\}, \quad \{3, 4, 5\}.$$

Each point lies on exactly two lines and is non-collinear with exactly one other point; the three non-collinear ("orthogonal") pairs are $\{0, 5\}$, $\{1, 4\}$, $\{2, 3\}$, that is, $i \leftrightarrow 5 - i$. This is the dual affine plane of order 2. We write $M_\eta(S_4)$ for the corresponding 6-dimensional Matsuo algebra over $\mathbb{Q}$, and $M_2(S_4)$ for the case $\eta = 2$, which is legitimate since $\operatorname{char} \mathbb{Q} = 0 \neq 2$ and $2 \in \mathbb{Q} \setminus \{0, 1\}$.

The space of $W(D_4)$. Points are the 12 positive roots $e_i \pm e_j$ of the root system D_4; two are collinear when their inner product is ± 1, the third point of the line being the reflection of one in the other. The corresponding 12 reflections form a normal set D of involutions with $|de| \leq 3$ generating a group of order 192 with centre of order 2 — the Weyl group $W(D_4)$, which is $2^3{:}S_4$: the even sign changes form a normal elementary abelian subgroup of order 8, complemented by the permutation matrices. These facts, and the line list below, we computed from the root system; the algebra is 12-dimensional and we write it $M_{1/2}(D_4)$. The survey's own example of a non-Jordan Matsuo algebra of Jordan type $\frac{1}{2}$ is named there by the group $2^4{:}S_4$; we do not re-derive that group-theoretic identification, and Remark 3.5 says on what grounds the two are the same algebra. Its 16 lines, on points $0, \dots, 11$, are

$$\begin{array}{llllllll} \{0,2,3\}, & \{0,4,5\}, & \{0,7,8\}, & \{0,9,10\}, & \{1,2,4\}, & \{1,3,5\}, & \{1,7,9\}, & \{1,8,10\}, \\ \{2,6,7\}, & \{2,10,11\}, & \{3,6,8\}, & \{3,9,11\}, & \{4,6,9\}, & \{4,8,11\}, & \{5,6,10\}, & \{5,7,11\}. \end{array}$$

3. The algebra $M_2(S_4)$: unit, form, and the failure of the Jordan identity

The survey prints, immediately after (1): "Every Matsuo algebra admits a Frobenius form, which is given by $(a, b) = 1$ if $b = a$, 0 if $|ab| = 2$, and $\eta/2$ if $|ab| = 3$." Here *Frobenius* means that the form associates with the product, $(a \circ b, c) = (a, b \circ c)$. The following proposition verifies that printed form and the two structural hypotheses that the published lower bound needs.

Proposition 3.1. *The bilinear form printed in* [5] *is associative on all triples of axes — hence, both sides being trilinear, on all triples of elements — of $M_2(S_4)$, of $M_{1/2}(S_4)$, and of the 12-dimensional $M_{1/2}(D_4)$. Moreover $M_2(S_4)$ is unital with*

$$1 = \tfrac{1}{5} \sum_{a \in D} a,$$

and the Gram matrix of its Frobenius form in the basis D is invertible, with determinant 5*; in particular the form is nondegenerate.*

Proof sketch. Associativity of the form is an exhaustive check over all triples of axes: $6^3 = 216$ identities of rational numbers for each of $M_2(S_4)$ and $M_{1/2}(S_4)$, and $12^3 = 1728$ for $M_{1/2}(D_4)$. All were verified by compiled evaluation.

The unit and the Gram determinant have short closed-form reasons, and both were also checked by direct computation. For the unit, let $b \in D$ and let ℓ be the number of lines through b (for the space of S_4, $\ell = 2$). Each line $\{b, a, a'\}$ through b contributes

$$a \circ b + a' \circ b = \tfrac{\eta}{2}\big((a + b - a') + (a' + b - a)\big) = \eta\, b,$$

while every axis non-collinear with b contributes 0 and $b \circ b = b$. Hence $\big(\sum_{a \in D} a\big) \circ b = (1 + \ell\eta)\, b$, and for $\ell = 2$, $\eta = 2$ this is $5b$, so $\frac{1}{5} \sum_{a \in D} a$ acts as the identity on every basis element and hence, by bilinearity, on the whole algebra. For the Gram matrix G: with $\eta/2 = 1$ we have $G_{ab} = 1$ unless a, b are the (distinct) members of an orthogonal pair, where $G_{ab} = 0$. So $G = J - R$, where J is the all-ones matrix and R is the permutation matrix of the fixed-point-free involution $i \mapsto 5 - i$. Since $JR = RJ = J$, $R^2 = I$ and $J^2 = 6J$, one checks $(\frac{1}{5}J - R)(J - R) = I$, which is the explicit inverse used in the formal proof. For the determinant, J and R commute: on the all-ones vector G acts by $6 - 1 = 5$, and on its orthogonal complement G acts as $-R$, which has eigenvalue -1 twice and $+1$ three times there, so $\det G = 5 \cdot (-1)^2 \cdot 1^3 = 5$. $\square$

The next proposition is the pair of controls for everything that follows. The Jordan identity $(x^2y)x = x^2(yx)$ of a commutative algebra linearises, in characteristic 0, to the multilinear degree-4 polynomial

$$\mathrm{Jor}(x_1, x_2, x_3, x_4) = \sum_{k \in \{1,2,3\}} \Big[\big((x_i x_j)x_4\big)x_k - (x_i x_j)(x_4 x_k)\Big], \qquad \{i, j\} = \{1, 2, 3\} \setminus \{k\}, \tag{4}$$

whose coefficient vector in the degree-4 monomial basis has six nonzero entries, all ± 1. Specialising $x_1 = x_2 = x_3 = x$ and $x_4 = y$ in (4) returns $3\big((x^2y)x - x^2(yx)\big)$, so in characteristic 0 a commutative algebra satisfies the Jordan identity if and only if (4) is an identity of it; both directions are used below.

Proposition 3.2. *The polynomial* (4) *vanishes at every one of the* 1296 *substitutions of axes of* $M_{1/2}(S_4)$*, in every coordinate; being multilinear, it is therefore an identity of* $M_{1/2}(S_4)$*; in particular that algebra does satisfy an identity of degree* 4 *not implied by commutativity. The same polynomial does* not *vanish in* $M_2(S_4)$*: at the substitution* $(x_1, x_2, x_3, x_4) = (e_0, e_0, e_1, e_0)$ *its value has* e_1*-coordinate* 3*. In particular* $M_2(S_4)$ *is not a Jordan algebra.*

Proof sketch. The positive half is an exhaustive check: all $6^4 = 1296$ maps $\{1, 2, 3, 4\} \to D$ and, for each, all six coordinates of the value, that is 7776 rational identities, verified by compiled evaluation. The passage from axes to arbitrary elements is multilinearity of (4) and is an argument, not a computation. The negative half is a single evaluation. $\square$

Neither half of Proposition 3.2 is new. The positive half is a case of the Main Theorem of De Medts and Rehren [4], by which a Matsuo algebra that is a Jordan algebra is a direct sum of algebras $M_{1/2}(G_i, D_i)$ with $G_i = \mathrm{Sym}(n)$ or $G_i \cong 3^2{:}2$, the first kind being the Jordan algebra of $n \times n$ symmetric matrices with zero row sums — for $n = 4$ that is exactly $M_{1/2}(S_4)$, of dimension 6. It is also an instance of the survey's "All algebras of Jordan type for $n = 2$ and 3 are Jordan" (with n the number of generating axes and Jordan type $\frac{1}{2}$ understood), which [5] attributes to Gorshkov and Staroletov [6]: we checked, by the same exact arithmetic as the rest of this section, that $M_{1/2}(S_4)$ is indeed generated as an algebra by the three axes $(12), (13), (14)$. The role of the two halves here is as controls. The positive half shows that the emptiness found in Section 4 is a property of the algebra and not an artefact of the encoding: the very same degree-4 coefficient vector that is annihilated at $\eta = \frac{1}{2}$ survives at $\eta = 2$. The negative half supplies the hypothesis — non-Jordan — under which the published bound applies to $M_2(S_4)$, and simultaneously shows that (4) is not an identity of all Matsuo algebras, so that no conclusion below is being read off a Jordan algebra.

Remark 3.3. Combining Propositions 3.1 and 3.2 with [2, Prop. A.8] gives the following, which we claim no credit for. Suppose f were an identity of all Matsuo algebras of degree ≤ 4 not implied by commutativity. Then f is an identity of $M_2(S_4)$, a commutative algebra of dimension 6 over a field of characteristic 0, hence of characteristic $\neq 2, 3, 5$, whose Frobenius form is associative and nondegenerate, so that $M_2(S_4)$ is metrized. The proposition would then force $M_2(S_4)$ to satisfy the Jordan identity, contradicting Proposition 3.2. So the published literature already gives 5, one more than the 4 stated in [5], whose extra caution comes from the unitality assumption in Osborn's input [12] — an assumption the proposition as printed in [2] does not carry, and which in any case holds here. Theorem 4.2 below is one degree past this.

We close the section with the survey's own example of a non-Jordan Matsuo algebra of Jordan type $\frac{1}{2}$, reproduced with witnesses. The survey writes: "It is easy to check that the smallest non-Jordan Matsuo algebra of Jordan type half (for the group $2^4{:}S_4$) is not power associative."

Proposition 3.4. *In the* 12*-dimensional Matsuo algebra* $M_{1/2}(D_4)$ *of the Fischer space of the reflection class of* $W(D_4)$*:*

(i) *the multilinear Jordan polynomial* (4) *does not vanish at the axes* (e_0, e_1, e_2, e_6)*: its value has coordinate* $\frac{1}{32}$ *at* e_2*. So* $M_{1/2}(D_4)$ *is not a Jordan algebra.*

(ii) $M_{1/2}(D_4)$ *is not power associative: for* $x = e_0 + e_1 + e_2 + e_6$ *the elements* x^2x^2 *and* $x\,x^3$ *differ, their* e_2 *coordinates being* $\frac{89}{8}$ *and* $\frac{181}{16}$ *respectively.*

Proof sketch. Both parts are single evaluations in exact rational arithmetic at the named arguments. □

Remark 3.5. Part (i) of Proposition 3.4 is not new either. The Main Theorem of [4] classifies the Matsuo algebras that are Jordan algebras: each is a direct sum of algebras $M_{1/2}(G_i, D_i)$ with $G_i = \mathrm{Sym}(n)$ or $G_i \cong 3^2{:}2$, whose Fischer spaces have $\binom{n}{2}$ and 9 points respectively. The D_4 space is connected and has 12 points, and 12 is neither 9 nor a binomial coefficient $\binom{n}{2}$, so part (i) is what that classification predicts. What Proposition 3.4 adds is the explicit witnesses.

The identification of $M_{1/2}(D_4)$ with the algebra the survey names by $2^4{:}S_4$ is a different matter: we do not verify a group isomorphism, and rest the identification on two things, that Matsuo algebras depend only on the Fischer space (Section 1) and that the survey's example is by its own description the *smallest* non-Jordan Matsuo algebra of Jordan type $\frac{1}{2}$. That word "smallest" we do not prove. What we can report is evidence, obtained outside the formal development: over thirteen Fischer spaces — those of S_n for $5 \le n \le 8$, of $\mathrm{AG}(2,3)$ and $\mathrm{AG}(3,3)$, of the reflection classes of D_n for $4 \le n \le 6$, and of $\mathrm{Sp}(4,2)$, $\mathrm{Sp}(6,2)$, $O^+(6,2)$, $O^-(6,2)$ — we took every set of four points and every choice among them of the distinguished variable x_4 of (4), and every set at which (4) fails at $\eta = \frac{1}{2}$ generates a sub-Fischer-space of exactly 12 points, namely the D_4 space, the sole exception being inside $\mathrm{AG}(3,3)$, where the failures generate all 27 points. Since (4) is multilinear of degree 4, a non-Jordan example is always witnessed by four axes — three suffice for none of them, the 3-generated algebras of Jordan type $\frac{1}{2}$ being Jordan — so within the range scanned the 12-dimensional algebra is the only small one. A proof of minimality would need the classification of Fischer spaces generated by four points, which we do not attempt. This scan is an unverified computation and nothing below depends on it.

4. No identity of degree at most five

The certificate. Fix d and $N = N(d)$ as in (2), and let $m_1, \dots, m_N$ be the commutative bracketings of $x_1 \cdots x_d$ in a fixed order. Choose N pairs (s_r, k_r), $r = 1, \dots, N$, each consisting of a substitution $s_r \colon \{1, \dots, d\} \to D$ of axes of $M_2(S_4)$ and a coordinate index $k_r \in D$, and let

$$M \in \mathbb{Z}^{N \times N}, \qquad M_{ri} = \big\langle m_i\big(e_{s_r(1)}, \dots, e_{s_r(d)}\big),\, e^*_{k_r} \big\rangle.$$

The entries are integers because $\eta/2 = 1$ makes every structure constant of $M_2(S_4)$ equal to 0 or ± 1; at $d = 5$ they satisfy $|M_{ri}| \le 8$. Suppose we can exhibit a matrix $\overline{P} \in (\mathbb{Z}/101)^{N \times N}$ with

$$\overline{P} \cdot \overline{M} = I \quad \text{in } (\mathbb{Z}/101)^{N \times N}, \tag{5}$$

where $\overline{M}$ is the reduction of M modulo the prime 101. Then $\det \overline{M}$ is a unit in $\mathbb{Z}/101$, hence $\det M \neq 0$ in $\mathbb{Z}$ and a fortiori in $\mathbb{Q}$; so M is invertible over $\mathbb{Q}$ and $Mc = 0$ forces $c = 0$. Since row r of Mc is exactly the value of the functional (3) attached to (s_r, k_r) at c, any multilinear identity of $M_2(S_4)$ of degree d has $c = 0$. Two things, and only two, have to be checked by machine: that M really is the matrix of those N functionals (all N^2 entries), and (5) (all N^2 entries of a product modulo 101). There is no search over identities anywhere.

Proposition 4.1. *No nonzero multilinear polynomial of degree* 3*, and none of degree* 4*, with rational coefficients vanishes at every substitution of axes of* $M_2(S_4)$*.*

Proof sketch. The construction above with $N = 3$ and $N = 15$, and certificates $\overline{P}$ of those sizes. The finite checks are $2 \cdot 3^2 = 18$ and $2 \cdot 15^2 = 450$ entry comparisons respectively. □

Proposition 4.1 is not new: by Remark 3.3, the proposition of Chayet and Garibaldi already excludes an identity of degree ≤ 4 for $M_2(S_4)$. Its purpose here is to be the positive control for the degree-5 machinery — the same apparatus, at sizes small enough to inspect, reproducing a published conclusion. The degree-5 statement is the new one.

Theorem 4.2. *Let $c \in \mathbb{Q}^{105}$ and let f_c be the corresponding multilinear polynomial of degree 5 in the free commutative non-associative $\mathbb{Q}$-algebra. If f_c vanishes at every substitution of* axes *of $M_2(S_4)$ then $c = 0$; a fortiori the same conclusion holds if f_c vanishes at every substitution of elements, that is, if f_c is an identity of $M_2(S_4)$. So the space of degree-5 multilinear identities of $M_2(S_4)$ is zero.*

Proof sketch. The construction above with $N = 105$: an explicit list of 105 pairs (a 5-tuple of axes of $M_2(S_4)$, a coordinate), the resulting integer matrix M of size 105×105 with entries of absolute value at most 8, and an explicit $\overline{P} \in (\mathbb{Z}/101)^{105\times 105}$ satisfying (5). Two exhaustive finite checks are involved, each of $105^2 = 11025$ entries: that every entry of M equals the stated evaluation of the corresponding bracketing in the algebra, and that every entry of $\overline{P}\,\overline{M}$ equals the corresponding entry of the identity matrix. The remainder is the argument of the previous paragraph, carried out formally: a left inverse modulo a prime makes the determinant a unit there, hence nonzero over $\mathbb{Z}$, hence over $\mathbb{Q}$, and a square matrix with nonzero determinant has trivial kernel.

One point deserves care: the theorem is about *all* multilinear polynomials of degree 5, so the list $m_1, \dots, m_{105}$ must be a complete and repetition-free list of the bracketings. Completeness is the count: the generator used produces $1, 1, 3, 15, 105$ bracketings in degrees $1, \dots, 5$, matching (2), and this too is a finite check. Repetition-freeness is free: if $m_i = m_j$ for $i \neq j$ then the vector $e_i - e_j$ would be a nonzero element of the kernel, which the theorem excludes. □

Corollary 4.3. *If a nonzero identity of all Matsuo algebras not implied by commutativity exists, its degree is at least 6.*

Proof. Let f be such an identity, of degree $d \leq 5$. It is in particular an identity of $M_2(S_4)$, which is a Matsuo algebra over $\mathbb{Q}$ in the sense of (1). By full linearisation in characteristic 0 we may take f multilinear of degree d, and by padding with fresh variables (Section 2) we obtain a nonzero multilinear identity of $M_2(S_4)$ of degree exactly 5. Theorem 4.2 says there is none. □

Against the bound of 4 that [5] states, and the bound of 5 that Remark 3.3 extracts from the published proposition, Corollary 4.3 is one degree further. We stress what it does not say: it does not assert that an identity of all Matsuo algebras exists. The first half of Problem 1.1 is untouched.

5. Degree six, and the limits of the method

An unverified computation in degree six. The construction of Section 4 does not stop at $d = 5$ in principle, only in practice. We report the following, which was carried out outside the formal development and is *not* machine-checked.

Computation 5.1. In degree 6 there are 945 bracketings. Taking 250 substitutions of axes of $M_2(S_4)$ drawn uniformly at random, together with all six coordinates, gives 1500 of the functionals (3); the resulting 1500×945 integer matrix was reduced by Gaussian elimination over $\mathbb{F}_p$ with $p = 2^{31} - 1$ and found to have rank 945. Since the structure constants are integers, the rank of an integer matrix modulo p is a lower bound for its rank over $\mathbb{Q}$; so the degree-6 multilinear identity space of $M_2(S_4)$ is zero as well, and the bound of Corollary 4.3 would become 7. Generating the matrix took about five seconds and the elimination about two. The number of rows matters, and not by a wide margin: with 200 substitutions (1200 rows) the rank came out between 890 and 920 on each of the four draws we tried, while from 250 substitutions (1500 rows) upward every draw gave 945.

We do not claim the bound 7. Computation 5.1 is exactly the kind of statement that the certificate of Section 4 is designed to replace, and the replacement is a matter of cost rather than of method. Measured against the degree-5 file, whose two checks together take 92 seconds of CPU time, the degree-6 certificate needs a bridge check scaling as N^2 (a factor $(945/105)^2 = 81$, about 85 minutes) and an inverse check scaling as N^3 (a factor 729, about 8 hours): 9 to 10 CPU-hours in total. A factorisation certificate in place of a full inverse would cut the dominant term to $N^3/3$ and bring it within a few hours. Degree 7 (10395 bracketings) is a rank computation of roughly $6 \cdot 10^{11}$ modular operations, about 40 minutes single-threaded, which we did not run; degree 8 (135135 bracketings) is about $2.5 \cdot 10^{15}$ operations, on the order of two weeks, and is out of reach by this route.

Why a single algebra cannot settle the problem. There is a hard ceiling on what any one Matsuo algebra can prove, and it is worth recording because it says where the difficulty of Problem 1.1 really sits.

Remark 5.2. Let A be any algebra of dimension n over a field of characteristic 0. The multilinear component of degree d has dimension $(2d-3)!!$, while the functionals (3) number at most n^{d+1}. Hence A has a nonzero multilinear identity of degree d as soon as $(2d-3)!! > n^{d+1}$, which for $n = 6$ holds by $d = 12$. So $M_2(S_4)$ — and by the same count every individual Matsuo algebra — is a PI algebra, and the method of Section 4 must fail at some degree for any fixed algebra. Only the intersection over the whole class of Matsuo algebras can be trivial in every degree, and deciding that is not a finite computation with a single algebra.

Correspondingly, an *upper* bound needs an identity valid in every Matsuo algebra. Multilinearity localises the check — a degree-d multilinear monomial evaluated at d axes lands in the subalgebra they span, which is the Matsuo algebra of the sub-Fischer-space they generate — so the degree-d identities of all Matsuo algebras are the intersection over the Fischer spaces generated by at most d points. That reduces the class but does not make it finite, and we assert no upper bound.

The second problem, which is not settled here. The survey records a second problem of Rowen in the same circle, verbatim: "In an algebra of Jordan type half, does $(x, y) = 0$ imply xy is 4-nilpotent, i.e., $(xy)^2 = 0$ or $(xy)(xy)^3 = 0$? (These make sense even when power nilpotence fails.) More generally, which identities can be deduced from orthogonality?" Nothing in this paper bears on it, and the reason is structural rather than a matter of effort. The survey itself notes that for Matsuo axes the condition $(a, b) = 0$ is equivalent to $a \circ b = 0$; so among the axes of a Matsuo algebra the hypothesis already forces $xy = 0$ and the implication is vacuous. A test case needs a pair with $(x, y) = 0$ and $xy \neq 0$, which lives in the algebras of Jordan type $\frac{1}{2}$ outside the Matsuo branch. The problem remains open.

6. Formal verification

Propositions 3.1, 3.2, 3.4 and 4.1 and Theorem 4.2 are formalised and machine-checked in Lean 4 [8] over Mathlib [9], as is the count $1, 1, 3, 15, 105$ of bracketings in degrees 1 to 5 used in the proof of Theorem 4.2. The development is four files: the Matsuo product of a Fischer space and the multilinear monomial basis, with the elementary facts (every axis is idempotent, the product is commutative — the baseline against which "not implied by commutativity" is read — and the product is not associative, so that degree 3 carries nothing from associativity either); the degree-3, degree-4 and degree-5 certificates, with the general lemmas taking a left inverse modulo a prime to a nonzero determinant and a nonzero determinant to a trivial kernel; the $\eta = \frac{1}{2}$ file with the 1296-substitution sweep and the two witnesses of Proposition 3.4; and the Frobenius file. There is no incomplete proof and no user-declared axiom anywhere in it. The axiom set of each of the twenty-three declarations that state something about the algebras — the Lean forms of the propositions and the theorem above, together with the elementary facts — was printed: besides propositional extensionality, quotient soundness and choice, the only entries are one compiled-evaluation axiom per evaluated check, which is the formal record of the exhaustive computations described in the proof sketches above; the idempotence and commutativity statements depend on no compiled evaluation at all. The controls of Propositions 3.2 and 3.1 are part of the development precisely so that no hypothesis above is vacuous. Two computations lie outside this scope, both labelled where they occur: Computation 5.1 and the Fischer-space scan of Remark 3.5, which are C and Python programs. Some further passages are mathematical arguments or exact rational computations rather than formal checks, and are likewise not formalised: the deduction of Corollary 4.3 from Theorem 4.2 by full linearisation and padding; the extension of Propositions 3.1 and 3.2 from axes to arbitrary elements by multilinearity; the generation of $M_{1/2}(S_4)$ by three axes; and the displayed numerical values — $\det G = 5$ in Proposition 3.1, the coordinates 3 in Proposition 3.2 and $\frac{1}{32}$, $\frac{89}{8}$, $\frac{181}{16}$ in Proposition 3.4. What the Lean development certifies at those places is the invertibility of G (through the explicit inverse displayed in the proof) and the non-vanishing, respectively the difference, of the elements concerned; the values themselves are exact rational computations, and every one of them was recomputed independently while this paper was being checked. The repository is private; repository reference to be added at submission.

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