

THE DEGREE FILTRATION OF THE ALPÖGE SUBALGEBRA: RETARGETING CANNOT LOWER THE MARKUS–YAMABE DIMENSION BELOW FOURTEEN

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ABSTRACT. A recent chain realization theorem of Castañeda, Honorato and Valenzuela-Henríquez turns a polynomial Keller map F of $\mathbb{R}^n$ with a multi-point fiber into a polynomial vector field on $\mathbb{R}^N$, $N = \sum_i \max(\deg F_i, 2) - n$, whose Jacobian has spectrum $\{-1\}$ everywhere and whose singularities are the fiber; fed with Alpöge’s three-dimensional counterexample to the Jacobian conjecture it refutes the weak Markus–Yamabe conjecture in dimension 14. The authors close by asking for the minimum of $\sum_i \max(\deg(A \circ F \circ B)_i, 2)$ over polynomial automorphisms A, B of $\mathbb{R}^3$, every unit removed being one dimension off the counterexample. We settle the target side of that minimisation, which is a question about the degree filtration of the subalgebra generated by the components of F . Among all polynomials of degree at most 8 in the three generators, those of total degree at most 3, 5 and 6 span spaces of dimensions exactly 1, 2 and 3: spanned by 1; by 1 and R ; and by 1, R and Q . This does not follow from the numerical semigroup $\langle 4, 6, 7 \rangle$: the three leading forms are algebraically dependent, the generators are not a SAGBI basis, and genuine subduction occurs. The consequence is that $\sum_i \max(\deg(A \circ F)_i, 2) \geq 4 + 6 + 7 = 17$ for every polynomial map A of degree at most 8 whose composite with F is dominant — a class wider than the automorphisms the question asks about — so no change of target coordinates lowers the realized dimension below 14. The certificate behind the filtration is a 162×162 integer identity, machine-checked in Lean 4.

1. INTRODUCTION

Let $X: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a C^1 map. Call X *Hurwitz* if every eigenvalue of $JX(x)$ has negative real part, at every $x \in \mathbb{R}^n$. The *weak Markus–Yamabe conjecture* asserts that a Hurwitz map is injective. It is the injectivity shadow of the global asymptotic stability question of Markus and Yamabe [7]; the stability statement itself is false for $n \geq 3$ by the polynomial counterexample of Cima, van den Essen, Gasull, Hubbers and Mañosas [4], whose map is nevertheless injective and so leaves the weak form untouched [3, §1]. The injectivity statement resisted longer.

On 5 August 2026 Castañeda, Honorato and Valenzuela-Henríquez [3] refuted the weak conjecture in dimension 14. Their engine is a *chain realization theorem*, which we quote as it is stated in the introduction of [3], namely as its Theorem 1.2 with the displayed formula for N inlined; it is proved there, in the sharper form that also writes the field down, as Theorem 4.1. All section, theorem and proposition numbers we attribute to [3] are those of its version v1, the only version posted.

“Let $F = I + H: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a polynomial Keller map, where H has no constant and no linear terms, and write $d_i = \deg F_i$. Fix $\beta \in \mathbb{R}^n$. Then there is an explicit polynomial vector field $X_{F,\beta}$ on $\mathbb{R}^N$, with $N = \sum_{i=1}^n \max(d_i, 2) - n$, such that $\text{Spec}(JX_{F,\beta}(Z)) = \{-1\}$ for every $Z \in \mathbb{R}^N$ and the singular set of $X_{F,\beta}$ is in polynomial bijection with the fiber $F^{-1}(\beta)$.”

Here a *Keller map* is a polynomial map whose Jacobian determinant is a nonzero constant. A field produced by the theorem is Hurwitz; if the chosen fiber has more than one point, the field has more than one singularity, and no Hurwitz map with two singularities is injective. So a Keller map of $\mathbb{R}^n$ with a multi-point fiber refutes the weak conjecture in dimension N and, by padding with $-\xi$, in every dimension above it.

The input is Alpöge’s counterexample [1] to the Jacobian conjecture in dimension three, in the form recorded in [3]: the map $F = (P, Q, R)$ of (1) below, with $\det JF \equiv -2$ and three distinct rational points sharing one image. Normalized as the theorem requires it becomes $\tilde{F} = (\frac{1}{2}R, Q, P)$ with $\det J\tilde{F} \equiv 1$

and component degrees 4, 6, 7, whence

$$N = \max(4, 2) + \max(6, 2) + \max(7, 2) - 3 = 17 - 3 = 14.$$

The dimension 14 is thus not intrinsic to the conjecture; it is the value of one combinatorial functional at one particular presentation of one particular map. The authors say so, and close [3] — in its concluding §10, “Dimensional remarks and concluding comments” — with the following problem, which we quote verbatim; the Theorem 4.1 it refers to is the chain realization theorem quoted above.

“The quantity $\sum_i \max(d_i, 2)$ is not invariant under composition with polynomial automorphisms. This suggests a concrete problem. Minimize $\sum_i \max(\deg(A \circ F \circ B)_i, 2)$ over automorphisms A, B of $\mathbb{R}^3$ normalized as in Theorem 4.1. Every unit removed from this sum lowers the dimension of the counterexample by one. In particular, the present argument does not settle the dimensions $3 \leq n \leq 13$.”

What is settled here. The minimisation is two-sided, and the two sides are of different natures. Post-composition by A changes which elements of the subalgebra

$$S := \mathbb{R}[\tilde{F}_1, \tilde{F}_2, \tilde{F}_3] = \mathbb{R}[R, Q, P] \subset \mathbb{R}[x, y, z]$$

are used as components; pre-composition by B changes the subalgebra itself. The target side is therefore a question about one fixed subalgebra of $\mathbb{R}[x, y, z]$: which *degrees* occur in it, and how many independent elements each degree buys. That is the bottom of the degree semigroup of S , the invariant of the Robbiano–Sweedler theory of subalgebra bases [8]. We compute it.

- (A) *The degree filtration* (Theorem 3.1). Among all polynomials of degree at most 8 in the three generators R, Q, P , those of total degree at most 3, at most 5 and at most 6 form $\mathbb{R}$ -vector spaces of dimensions exactly 1, 2 and 3, spanned respectively by 1; by 1, R ; and by 1, R, Q .
- (B) *Retargeting cannot beat 17* (Theorem 4.1). Let A be a polynomial map of $\mathbb{R}^3$ whose components have degree at most 8 and for which 1, $A_1 \circ \tilde{F}$, $A_2 \circ \tilde{F}$, $A_3 \circ \tilde{F}$ are linearly independent over $\mathbb{R}$ — which holds whenever $A \circ \tilde{F}$ is dominant, in particular whenever A is an automorphism. Then

$$\sum_{i=1}^3 \max(\deg(A \circ \tilde{F})_i, 2) \geq 4 + 6 + 7 = 17.$$

Consequently (Corollary 4.2) the chain realization theorem, applied to any retargeting of the Alpgöge map by such an A , still outputs a field on $\mathbb{R}^N$ with $N \geq 14$.

Two features of (B) deserve emphasis. First, the hypothesis is weaker than the problem’s: A need not be invertible, and need not even be a Keller map. This is not generosity but economy — the chain theorem needs of $G = A \circ \tilde{F} \circ B$ only that $\det JG$ be a nonzero constant and that some fiber of G have three points, and with $B = \text{id}$ the second holds for *every* A , because A collapses nothing that $\tilde{F}$ has not already collapsed. Second, the bound is sharp: it is attained by $A = \text{id}$, that is by $\tilde{F}$ itself.

What makes (A) a computation rather than an observation is that the answer is not the numerical semigroup $\langle 4, 6, 7 \rangle$ generated by the three degrees: the leading forms of $\tilde{F}_1, \tilde{F}_2, \tilde{F}_3$ are algebraically dependent, so the generators are not a SAGBI basis, and subduction genuinely occurs — $\deg(\tilde{F}_2^3 + 54\tilde{F}_1\tilde{F}_3^2) = 13$, down from the weighted value 18 (Proposition 3.3). So the filtration has to be computed rather than read off. Linear algebra does it, and (A) rests on a 162×162 integer certificate: an exhaustive check that a certain 162×162 submatrix of the coefficient table of the 165 products $R^a Q^b P^c$, $a+b+c \leq 8$, is invertible over $\mathbb{Q}$ with denominator dividing 3^{13} .

What is not settled. The source side. Pre-composition by an automorphism B replaces S by $\mathbb{R}[\tilde{F} \circ B]$, a different subalgebra with a different filtration, and no invariant we know bounds $\sum_i \max(d_i, 2)$ from below over all B . The two obvious candidates are far too weak. The geometric degree of $\tilde{F}$ — the number of points of a generic fiber — is 3 and is invariant under composition with automorphisms on either side, but Bézout turns that into $d_1 d_2 d_3 \geq 3$ only, which permits (1, 1, 3) and $N = 4$; and Wang’s theorem — a polynomial endomorphism of degree at most 2 with invertible Jacobian, over a field of characteristic $\neq 2$, is invertible [9], a statement reproduced with a short proof as [2, Thm. (2.4)]

— gives $N \geq 4$ unconditionally, since a map with all $d_i \leq 2$ is then injective and has no multi-point fiber. The gap between 4 and 14 is the whole problem, and it is untouched. Remark 5.1 reports what a finite search over elementary source-side shears found, which is nothing better than 18; that search is a measurement, outside the formal development, and is reported as such.

We have found no other work on this minimisation. The full-text searches reported here were run against a local mirror of the L^AT_EX sources of arXiv, complete through the identifier 2608.27451 and harvested on 28 August 2026. In that mirror the identifier 2608.05392 occurs exactly once, in the record of the paper itself: no paper in it cites [3]. Over the whole July–August 2026 window, “Markus–Yamabe” — in any of its hyphenations — occurs in exactly one paper, [3] itself, and a live arXiv query for the phrase over submissions between 1 June and 30 September 2026 returns the same single paper. Of the papers in that window that mention Alpöge’s map, then, none mentions Markus–Yamabe and so none discusses the dimensions $3 \leq n \leq 13$; and none computes a degree semigroup for the map, the few that use the language of subalgebra bases at all doing so in unrelated settings. Semantic Scholar and OpenAlex both report zero citations of [3], and its arXiv submission history lists v1 and nothing else. This is a not-found statement of 29 August 2026 and nothing more.

Organisation. Section 2 fixes the map, the normalization and the reduction to a subalgebra question. Section 3 proves (A) and explains why it cannot be read off the degrees; Section 4 proves (B) and draws the consequence for the chain construction; Section 5 reports the source-side measurement. Section 6 records our reproduction of the published data the new results stand on, including the dimension-fourteen field itself, and Section 7 the formal verification.

2. THE MAP, THE NORMALIZATION, AND THE REDUCTION

The Alpöge map. Following [3, §3] — whose second component we write Q rather than Q_0 — put

$$(1) \quad \begin{aligned} P &= (1 + xy)^3 z + y^2(1 + xy)(4 + 3xy), \\ Q &= y + 3x(1 + xy)^2 z + 3xy^2(4 + 3xy), \\ R &= 2x - 3x^2 y - x^3 z, \end{aligned}$$

and $F = (P, Q, R): \mathbb{R}^3 \rightarrow \mathbb{R}^3$. Then $\det JF \equiv -2$, so F is a Keller map, and the three points

$$p_1 = (0, 0, -\tfrac{1}{4}), \quad p_2 = (1, -\tfrac{3}{2}, \tfrac{13}{2}), \quad p_3 = (-1, \tfrac{3}{2}, \tfrac{13}{2})$$

are distinct with the common image $F(p_1) = F(p_2) = F(p_3) = (-\tfrac{1}{4}, 0, 0)$. Both assertions are [3, Prop. 3.1]; we reproduced them (Section 6).

The normalization. The chain realization theorem wants $G = I + H$ with H free of constant and linear terms. Reordering and halving,

$$\tilde{F} := (\tfrac{1}{2}R, Q, P),$$

has exactly this shape, since expanding (1) gives

$$(2) \quad \begin{aligned} \tilde{F}_1 &= x - \tfrac{3}{2}x^2 y - \tfrac{1}{2}x^3 z, \\ \tilde{F}_2 &= y + 3xz + 12xy^2 + 6x^2 yz + 9x^2 y^3 + 3x^3 y^2 z, \\ \tilde{F}_3 &= z + 4y^2 + 3xyz + 7xy^3 + 3x^2 y^2 z + 3x^2 y^4 + x^3 y^3 z, \end{aligned}$$

with linear parts x, y, z and component degrees 4, 6, 7. Transposing the first and third components multiplies the Jacobian determinant by -1 and halving the first multiplies it by $\tfrac{1}{2}$, so $\det J\tilde{F} \equiv (-1) \cdot \tfrac{1}{2} \cdot (-2) = 1$. The collision becomes $\tilde{F}(p_1) = \tilde{F}(p_2) = \tilde{F}(p_3) = \beta$ with $\beta = (0, 0, -\tfrac{1}{4})$, and the chain realization theorem applied to $(\tilde{F}, \beta)$ outputs a degree-seven field on $\mathbb{R}^{14}$ with three singularities. That field, printed in [3, §5], is reproduced in Section 6.

Normalizing costs no degree. The problem quoted in Section 1 ranges over automorphisms “normalized as in” the chain realization theorem. The normalization is free:

Lemma 2.1. *Let $G: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a polynomial map with $\det JG$ a nonzero constant. Put $L := JG(0)$ and $G^\sharp := (G - G(0)) \circ L^{-1}$. Then L is invertible, $G^\sharp = I + H$ with H free of constant and linear terms, $\det JG^\sharp \equiv 1$, and $\deg G_i^\sharp = \deg G_i$ for each i . Moreover if q_1, q_2, q_3 are distinct points with $G(q_1) = G(q_2) = G(q_3)$ then Lq_1, Lq_2, Lq_3 are distinct with $G^\sharp(Lq_1) = G^\sharp(Lq_2) = G^\sharp(Lq_3)$.*

Proof. $\det L = \det JG(0) \neq 0$. Subtracting a constant leaves every component degree unchanged unless the component is constant, which it is not since JG is invertible; composing on the *source* with the linear automorphism L^{-1} cannot raise a degree, and cannot lower one either, as composing back with L would then raise it. By the chain rule $JG^\sharp(0) = JG(0)L^{-1} = I$, so $G^\sharp - I$ has neither constant nor linear part, and $\det JG^\sharp$, being a constant, equals its value $\det I = 1$ at the origin. The last sentence is immediate. $\square$

Note that Lemma 2.1 moves the correction to the *source* side, where it is degree-free. A linear change of *target* coordinates would mix the components and can change $\sum_i \max(d_i, 2)$; that is exactly the freedom the minimisation problem is about.

The reduction. Fix $B = \text{id}$ and let $A = (A_1, A_2, A_3)$ be a polynomial map of $\mathbb{R}^3$. Then

$$(A \circ \tilde{F})_i = A_i(\tilde{F}_1, \tilde{F}_2, \tilde{F}_3) \in S := \mathbb{R}[\tilde{F}_1, \tilde{F}_2, \tilde{F}_3],$$

so the degrees available to the components of $A \circ \tilde{F}$ are exactly the degrees occurring in the subalgebra S , and the minimum of $\sum_i \max(d_i, 2)$ over all admissible A is read off the filtration of S by total degree. Since $\tilde{F}_1 = \frac{1}{2}R$, we have $S = \mathbb{R}[R, Q, P]$, and the substitution $u_1 \mapsto 2u_1$ is a degree-preserving bijection of the polynomials of degree at most K in three target variables; so nothing is lost by working with the integral generators R, Q, P , which is what we do from here on.

Definition 2.2. For $K \in \mathbb{N}$ let $S_K \subset \mathbb{R}[x, y, z]$ be the $\mathbb{R}$ -linear span of the products $R^a Q^b P^c$ with $a + b + c \leq K$, that is $S_K = \{A(R, Q, P) : A \in \mathbb{R}[u_1, u_2, u_3], \deg A \leq K\}$; and for $d \in \mathbb{N}$ let $S_K^{\leq d} := \{f \in S_K : \deg f \leq d\}$, where $\deg$ is total degree in x, y, z .

For $K = 8$ there are $\binom{11}{3} = 165$ products and $S_8^{\leq d}$ is a subspace of the 165-dimensional span; the point of the next section is that for $d \leq 6$ it is very small.

3. THE DEGREE FILTRATION

Theorem 3.1. *With the notation of Definition 2.2,*

$$S_8^{\leq 3} = \mathbb{R} \cdot 1, \quad S_8^{\leq 5} = \mathbb{R} \cdot 1 \oplus \mathbb{R} \cdot R, \quad S_8^{\leq 6} = \mathbb{R} \cdot 1 \oplus \mathbb{R} \cdot R \oplus \mathbb{R} \cdot Q,$$

of dimensions 1, 2 and 3 respectively. In particular the least degree of a nonconstant element of S_8 is 4; the least degree of an element of S_8 outside $\mathbb{R} \cdot 1 \oplus \mathbb{R} \cdot R$ is 6; and the least degree of an element outside $\mathbb{R} \cdot 1 \oplus \mathbb{R} \cdot R \oplus \mathbb{R} \cdot Q$ is 7.

Proof sketch. The inclusions $\supseteq$ are the degree computations $\deg R = 4$ and $\deg Q = 6$, read off (2); 1, R and Q have distinct degrees 0, 4, 6 and are therefore linearly independent, which gives the three dimensions once $\subseteq$ is known. We also record that $P \in S_8$ has degree 7 and coefficient 1 at $x^3 y^3 z$, so the third space is not larger.

For $\subseteq$, order the 165 exponent triples (a, b, c) with $a + b + c \leq 8$ by total degree then lexicographically, and expand each product $R^a Q^b P^c$ exactly in $\mathbb{Z}[x, y, z]$ by dynamic programming from (1). Choose 162 monomials of $\mathbb{Z}[x, y, z]$, each of total degree at least 7, whose columns in the resulting 165×162 coefficient table $\mathcal{A}$ are independent — the selection is made by Gaussian elimination modulo $2^{31} - 1$ and is only a choice, the certificate below being what is checked. The three rows indexed by 1, Q and R are identically

zero in this table, their degrees 0, 6, 4 all being below 7; write $\rho(1), \dots, \rho(162)$ for the remaining 162 row indices. Then there is an integer matrix W of size 162×162 with

$$(3) \quad \sum_{j=1}^{162} W_{kj} \mathcal{A}_{ij} = \begin{cases} D, & i = \rho(k), \\ 0, & \text{otherwise,} \end{cases} \quad \text{for all } 1 \leq i \leq 165, 1 \leq k \leq 162,$$

where $D = 1594323 = 3^{13}$. Concretely $W = D \cdot (\mathcal{A}_{\text{piv}}^T)^{-1}$ for the invertible 162×162 submatrix cut out by the chosen columns and rows; its entries are integers of at most 24 digits ($\max |W_{kj}| < 4 \cdot 10^{23}$) and 4.56% of them are nonzero.

Now let $f \in S_8^{\leq 6}$ and write $f = \sum_{i=1}^{165} c_i R^{a_i} Q^{b_i} P^{c_i}$ with $c \in \mathbb{R}^{165}$. Each of the 162 chosen monomials has degree at least 7, so its coefficient in f , namely $\sum_i c_i \mathcal{A}_{ij}$, vanishes. Hence for every k ,

$$0 = \sum_j W_{kj} \left(\sum_i c_i \mathcal{A}_{ij} \right) = \sum_i c_i \left(\sum_j W_{kj} \mathcal{A}_{ij} \right) = D c_{\rho(k)},$$

by (3) and $D \neq 0$, so $c_{\rho(k)} = 0$. The row indices $\rho(k)$ exhaust every index except those of 1, Q and R , so $f \in \mathbb{R} \cdot 1 \oplus \mathbb{R} \cdot R \oplus \mathbb{R} \cdot Q$. The two refinements then read one coefficient each. The monomial $x^3 y^2 z$ has degree 6 and occurs with coefficient 3 in Q and with coefficient 0 in 1 and in R ; so if $\deg f \leq 5$ the Q -coefficient of f vanishes and $f \in \mathbb{R} \cdot 1 \oplus \mathbb{R} \cdot R$. The monomial $x^3 z$ has degree 4 and occurs with coefficient -1 in R and with coefficient 0 in 1 and in Q ; so if $\deg f \leq 3$ the R -coefficient also vanishes and $f \in \mathbb{R} \cdot 1$. $\square$

Two things in this proof rest on exhaustive computation, and they are exactly the two things that would be hopeless by hand. The first is the expansion of the 165 products $R^a Q^b P^c$ in $\mathbb{Z}[x, y, z]$, whose largest member has total degree 56; the second is the verification of (3), which is $165 \times 162 = 26\,730$ integer inner products of length 162 with entries up to $3.8 \cdot 10^{23}$. Both are carried out inside a proof assistant (Section 7) using a sparse polynomial layer built for the purpose; the entries of W are data supplied to the checker, not trusted, since (3) is verified as stated.

Remark 3.2. The truncation at generator degree $K = 8$ is a real hypothesis, not a formality: since subduction lowers degrees (Proposition 3.3), one cannot argue that a product with $a + b + c > 8$ is automatically of high degree. Outside the formal development we recomputed the filtration by modular rank at $K = 10$ (286 generators) and over the weighted range $4a + 6b + 7c \leq 44$ (147 generators, including exponent sums up to 11), obtaining in both cases the same profile as at $K = 8$: the degree- $\leq d$ part has dimension 1, 2, 2, 3, 4 for $d = 3, 4, 5, 6, 7$. We have no proof that the filtration is stable in K , and Theorem 3.1 claims nothing beyond $K = 8$.

Why the filtration is not the semigroup generated by the degrees. If the leading forms of $\tilde{F}_1, \tilde{F}_2, \tilde{F}_3$ were algebraically independent — equivalently, if $\{\tilde{F}_1, \tilde{F}_2, \tilde{F}_3\}$ were a SAGBI basis of S in the sense of Robbiano and Sweedler [8] — then the degrees occurring in S would be exactly the numerical semigroup $\langle 4, 6, 7 \rangle = \{0, 4, 6, 7, 8, 10, 11, \dots\}$ and Theorem 3.1 would be a one-line count. They are not.

Proposition 3.3. *The leading forms of $\tilde{F}_1, \tilde{F}_2, \tilde{F}_3$ for the total-degree grading are $h_1 = -\frac{1}{2}x^3z$, $h_2 = 3x^3y^2z$ and $h_3 = x^3y^3z$, and*

$$h_2^3 + 54 h_1 h_3^2 = 0.$$

Consequently $\{\tilde{F}_1, \tilde{F}_2, \tilde{F}_3\}$ is not a SAGBI basis of S , and subduction occurs:

$$\deg(\tilde{F}_2^3 + 54 \tilde{F}_1 \tilde{F}_3^2) = 13, \quad \deg(\tilde{F}_2^3 + 54 \tilde{F}_1 \tilde{F}_3^2 - 18 \tilde{F}_2 \tilde{F}_3) = 8,$$

where the weighted values of the two combinations are 18 and 13.

Proof. The leading forms are read off (2), and $h_2^3 + 54 h_1 h_3^2 = 27x^9y^6z^3 + 54 \cdot (-\frac{1}{2}x^3z) \cdot x^6y^6z^2 = 0$. A nontrivial algebraic relation among the leading forms is precisely the obstruction to the subalgebra of leading forms being generated by them, which is the SAGBI condition. The two degree statements are exact expansions in $\mathbb{Q}[x, y, z]$. $\square$

The two expansions in Proposition 3.3 were carried out in exact rational arithmetic outside the formal development; nothing in Theorem 3.1 or in Section 4 depends on them. Their role is to explain why the filtration had to be computed.

The degeneracy is structural rather than accidental. The three leading monomials x^3z , x^3y^2z , x^3y^3z are collinear in the exponent lattice — they differ only in the exponent of y — and collinear monomials are multiplicatively dependent: $(x^3y^2z)^3 = (x^3z)(x^3y^3z)^2$ is the relation of Proposition 3.3, up to the scalars. Moreover all three come from the z -linear part of $\tilde{F}$, namely

$$z \cdot \left(-\frac{1}{2}x^3, 3x(1+xy)^2, (1+xy)^3\right),$$

whose top monomials are $x^3z \cdot \{1, y^2, y^3\}$ for *every* positive weight vector, since within each factor consecutive monomials differ by the factor xy . So any grading whose leading form for each component comes from that part inherits the same dependence, and changing the weights does not repair the SAGBI failure. Theorem 3.1 is what linear algebra gives instead.

4. RETARGETING CANNOT BEAT SEVENTEEN

Theorem 4.1. *Let $A = (A_1, A_2, A_3): \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a polynomial map whose components have total degree at most 8, and suppose that*

$$1, A_1 \circ \tilde{F}, A_2 \circ \tilde{F}, A_3 \circ \tilde{F}$$

are linearly independent over $\mathbb{R}$. Write $d_i = \deg(A_i \circ \tilde{F})$. Then

$$\sum_{i=1}^3 \max(d_i, 2) = d_1 + d_2 + d_3 \geq 4 + 6 + 7 = 17,$$

and 17 is attained, by $A = \text{id}$.

Proof sketch. Each $A_i \circ \tilde{F}$ lies in S_8 (Definition 2.2; the halving is cleared as in Section 2). Suppose some $d_i \leq 3$. By Theorem 3.1, $A_i \circ \tilde{F} \in \mathbb{R} \cdot 1$, so 1 and $A_i \circ \tilde{F}$ are linearly dependent, contradicting the hypothesis. Hence $d_i \geq 4$ for every i , and in particular $\max(d_i, 2) = d_i$.

Suppose next that two of the three, say $A_i \circ \tilde{F}$ and $A_j \circ \tilde{F}$, had $d_i, d_j \leq 5$. By Theorem 3.1 both lie in the two-dimensional space $\mathbb{R} \cdot 1 \oplus \mathbb{R} \cdot R$, so 1, $A_i \circ \tilde{F}$, $A_j \circ \tilde{F}$ are three vectors in a two-dimensional space and are linearly dependent — again a contradiction. So at most one of the three degrees is at most 5, and the second smallest is at least 6.

Finally, if all three had $d_i \leq 6$ then 1, $A_1 \circ \tilde{F}$, $A_2 \circ \tilde{F}$, $A_3 \circ \tilde{F}$ would be four vectors in the three-dimensional space $\mathbb{R} \cdot 1 \oplus \mathbb{R} \cdot R \oplus \mathbb{R} \cdot Q$, hence dependent. So the largest degree is at least 7. The sorted triple therefore dominates (4, 6, 7) componentwise, and the sum is at least 17. Attainment is $A = \text{id}$, for which $(d_1, d_2, d_3) = (4, 6, 7)$. $\square$

The formal statement is slightly stronger than the one above in two respects, both harmless and both worth recording. The hypothesis is imposed on the coefficient vectors of the A_i in the 165 products rather than on the polynomials $A_i \circ \tilde{F}$ themselves — linear independence of the polynomials implies linear independence of the coefficient vectors, so the formal hypothesis is the weaker one — and d_i may be any upper bound for the degree of $A_i \circ \tilde{F}$ rather than the degree itself.

Corollary 4.2. *Let A be a polynomial automorphism of $\mathbb{R}^3$ whose components have degree at most 8, put $G := A \circ \tilde{F}$ and let $G^\sharp$ be its normalization as in Lemma 2.1. Then $G^\sharp$ is a Keller map of the form $I + H$ with H free of constant and linear terms, some fiber of $G^\sharp$ contains three distinct points, and the chain realization theorem of [3] applied to $G^\sharp$ outputs a polynomial vector field on $\mathbb{R}^N$ with*

$$N = \sum_{i=1}^3 \max(\deg G_i^\sharp, 2) - 3 \geq 17 - 3 = 14.$$

The same holds for any polynomial map A of degree at most 8 with $\det JA$ a nonzero constant. In particular no change of target coordinates of degree at most 8 lowers the dimension of the counterexample below 14.

Proof. $\det JG = ((\det JA) \circ \tilde{F}) \cdot \det J\tilde{F}$ is a nonzero constant, being a nonzero constant times 1; so Lemma 2.1 applies and $\deg G_i^\# = \deg G_i$. The three points p_1, p_2, p_3 of Section 2 satisfy $\tilde{F}(p_i) = \beta$, hence $G(p_i) = A(\beta)$ for all three, and Lemma 2.1 carries the collision over to $G^\#$. For the degree bound we must check the hypothesis of Theorem 4.1. Since $\det J\tilde{F} \neq 0$ the components of $\tilde{F}$ are algebraically independent over $\mathbb{R}$, so $\tilde{F}$ is dominant; composing with the automorphism A (or with any map of nonzero constant Jacobian determinant) keeps the composite dominant, so the components $A_i \circ \tilde{F}$ are algebraically independent, and in particular satisfy no nontrivial affine relation: $1, A_1 \circ \tilde{F}, A_2 \circ \tilde{F}, A_3 \circ \tilde{F}$ are linearly independent. Theorem 4.1 then gives $\sum_i \max(\deg G_i, 2) \geq 17$. $\square$

Remark 4.3. The Jacobian criterion used in the last proof — that three polynomials in three variables over a field of characteristic zero with nonvanishing Jacobian determinant are algebraically independent — and the passage from a bound on $\sum_i \max(d_i, 2)$ to a bound on the output dimension of the chain realization theorem are classical, and are the two steps of Corollary 4.2 carried out by hand rather than mechanically (Section 7).

5. THE SOURCE SIDE: A MEASURED NEGATIVE

Everything above fixes $B = \text{id}$. A nonlinear B replaces S by $\mathbb{R}[\tilde{F} \circ B]$, whose filtration must be recomputed from scratch; linear B change nothing, since composing with a linear automorphism preserves every component degree.

Remark 5.1 (a measurement, outside the formal development). We ran a finite search over elementary shears: B the identity in two coordinates and $v \mapsto v + \lambda m$ in the third, with m one of the 7 monomials of degree 2 or 3 in the other two variables (21 shapes) and λ in a fixed set of 46 rationals — 966 automorphisms, each evaluated by computing the filtration of $\mathbb{R}[\tilde{F} \circ B]$ by *transcendence* degree at generator degree 6, as a rank computation modulo $2^{31} - 1$. None of the 966 gives $\sum_i \max(d_i, 2) < 17$. The best value found is 18, so $N = 15$, attained only by $B = (x, y - \frac{1}{3}xz, z)$, the one shear in the sweep producing a genuine degree drop: $\tilde{F}_1 \circ B = x - \frac{3}{2}x^2y$ has degree 3, the term $-\frac{3}{2}x^2 \cdot (-\frac{1}{3}xz)$ cancelling $-\frac{1}{2}x^3z$ exactly. The next value is 20, attained by the whole family $z \mapsto z + \lambda y^2$, whose transcendence filtration is $(5, 7, 8)$ for all 46 nonzero λ tested — a degree-invariance that presumably has a structural explanation. Total cost 259.6 seconds of one core. No statement elsewhere in this paper depends on this paragraph.

That the filter must be transcendence degree and not dimension is the trap of the source side. For the same shear B the filtration of $\mathbb{R}[\tilde{F} \circ B]$ by *dimension* is $(3, 6, 7)$, of sum 16, which would give $N = 13$ and would be an improvement; it is not one, because the degree-6 element realizing the second step is $(\tilde{F}_1 \circ B)^2$, algebraically dependent on the degree-3 element, whereas three components of a dominant map must be algebraically independent. The true filtration is $(3, 7, 8)$, of sum 18. The distinction is invisible on the target side: in Theorem 4.1 linear independence suffices, because there the three smallest available degrees already belong to algebraically independent elements.

6. THE PUBLISHED DATA, REPRODUCED

None of the above would mean anything about the counterexample of [3] if the map we work with were not its map. Before formulating any new statement we recomputed, in exact rational arithmetic and then again inside a proof assistant, every printed value of [3, §3 and §5] that the argument touches. The following are assertions of [3], not new results; what is ours is the verification.

Proposition 6.1. *For P, Q, R as in (1) and over any commutative ring, the 3×3 determinant of the nine partial derivatives of (P, Q, R) equals -2 ; the same constant is obtained as the determinant of the formal Jacobian matrix of the sparse representation built from the defining expressions $P = (1 + xy)B$, $Q = y + 3xB$, $R = 2x - 3x^2y - x^3z$ with $B = (1 + xy)^2z + y^2(4 + 3xy)$. The three points p_1, p_2, p_3 are pairwise distinct and $F(p_1) = F(p_2) = F(p_3) = (-\frac{1}{4}, 0, 0)$.*

Both determinant computations are done, because the first is an identity in nine polynomials that a transcription error would silently preserve, while the second differentiates the defining expression itself.

These facts were already machine-checked once, in Isabelle/HOL by Freitas Ramos, Barros Hulak and de Queiroz [5], which [3] cites as its exact-arithmetic check; ours is an independent formalization in a different system.

Proposition 6.2. *The expansions (2) hold identically on $\mathbb{R}^3$; the component degrees of $\tilde{F}$ are 4, 6, 7 with 3, 6, 7 terms respectively; and $\max(4, 2) + \max(6, 2) + \max(7, 2) - 3 = 14$. Moreover the polynomial $\lambda^3 - 58\lambda^2 - \frac{289}{2}\lambda - 1$, which [3, §3] computes as the characteristic polynomial of $J\tilde{F}(1, 1, 0)$, has a root in $(-1, 0)$; so $-\tilde{F}$ is not a Hurwitz map on $\mathbb{R}^3$.*

The last clause is the source’s own reason for going up in dimension: the Keller property in dimension three does not by itself produce a Hurwitz map, and the chain construction is what manufactures one. Our proof of the root is the intermediate value theorem on $[-1, 0]$, the polynomial being positive at -1 and negative at 0 .

Proposition 6.3. *Let $r = x/(1 + xy)$. At every point of $\mathbb{R}^3$ with $1 + xy \neq 0$,*

$$2Pr^3 - Qr^2 + 2r - R = 0.$$

Over the target $(-\frac{1}{4}, 0, 0)$ this cubic becomes $-\frac{1}{2}r^3 + 2r$, whose real roots are exactly $r \in \{0, 2, -2\}$, and these are the values of r at p_1, p_3, p_2 respectively.

Proposition 6.3 exhibits the collision of [3] as the totally split fiber of an explicit cubic, and in particular re-proves that the fiber over $(-\frac{1}{4}, 0, 0)$ meets $\{1 + xy \neq 0\}$ in at most three points. It is not new: the same cubic appears, in the disguise $w^3 - 2w^2 + Xw - 2Y$ with $X = QR$, $Y = PR^2$ and $w = R/r$, in Gao [6], whose Theorem on the Alpöge map states that “its generic fiber consists of exactly 3 points”; we verified by exact polynomial arithmetic that the two cubics agree under that substitution. The derivation in [3] stops one step earlier, at $Q = y + 3Pr$ and $R = 2r - yr^2 - Pr^3$, without eliminating y ; [6], five days older, is not cited there.

Proposition 6.4. *Let X be the degree-seven polynomial vector field on $\mathbb{R}^{14}$ printed in [3, §5], entered from its display, and let $M := JX + I_{14}$ be formed from the formal partial derivatives of that entry. Then $M^{14} = 0$ as a matrix of polynomials in the fourteen coordinates, and $M^{13} \neq 0$ — it has 33 nonzero entries — so M is nilpotent of index exactly 14, the largest possible for a 14×14 matrix. Hence $\text{Spec}(JX(Z)) = \{-1\}$ for every $Z \in \mathbb{R}^{14}$. The field has degree 7, and the three rational points S_1, S_2, S_3 listed in [3, §5] are pairwise distinct zeros of X .*

This is the content of the main theorem of [3] in dimension 14, whose own verification is the Wolfram Language block of its appendix (“the nilpotency of $JX + I_{14}$ is checked symbolically”). What our reproduction adds is that the check runs inside a proof assistant, from the field’s own printed display by formal differentiation, and that the nilpotency index is *exactly* 14 — a value the source does not state. As a control on the transcription we also verified the nonzero-entry counts 41 of M and 187 of M^7 against an independent implementation; a mis-entered coefficient would surface there rather than hide behind the vanishing of M^{14} .

7. VERIFICATION

The computational content of the statements above has been formally verified in Lean 4 against *Mathlib*, in three modules of a repository that is private at the time of writing; repository reference to be added at submission. The development contains no `sorry`. What is proved in the kernel with no axioms beyond `propext`, `Classical.choice` and `Quot.sound` is: the determinant identity of Proposition 6.1 as a `ring` identity in the nine written-out partial derivatives, over an arbitrary commutative ring; the collision and the distinctness of p_1, p_2, p_3 ; the expansions (2); the resolvent identity and the root set $\{0, 2, -2\}$ of Proposition 6.3; the intermediate-value argument of Proposition 6.2; and the entire deduction of Sections 3 and 4 from the certificate — that is, the passage from (3) to the three filtration statements, the linear-dependence lemmas for three vectors in a plane and four in a 3-space, and the assembly into Theorem 4.1. What is delegated to compiled evaluation inside the kernel’s `native_decide` is the finite data: the 165 exact expansions $R^a Q^b P^c$ in a sparse polynomial layer over $\mathbb{Z}$ (three variables,

exponents packed in base 128, like terms collected through a hash map, formal partial derivatives, cofactor expansion of 3×3 determinants) written for the purpose; the 26 730 integer inner products of (3); the degree and term counts (4, 6, 7) and (3, 6, 7) and the two witness coefficients; the formal Jacobian determinant of Proposition 6.1; and, in a separate fourteen-variable layer over $\mathbb{Q}$, the degree of the field, the matrix powers M^{14} , M^{13} , M^7 , M and the three singularity evaluations of Proposition 6.4. What is *not* formalised and is done by hand here is the mathematics around the certificates: Lemma 2.1; the Jacobian criterion and the implication “dominant $\Rightarrow$ linearly independent with 1” used in Corollary 4.2; the one-line passage from the nilpotency of M to $\text{Spec}(JX) = \{-1\}$; the chain realization theorem itself, which we quote from [3] and do not reprove; Proposition 3.3, whose two expansions are exact rational arithmetic outside the development; and the modular rank checks of Remark 3.2 and the sweep of Remark 5.1, both explicitly labelled as measurements.

The two halves of Theorem 3.1 sit in different places, and it is worth saying where. The inclusions $\subseteq$ — one kernel-checked statement per threshold — are the deduction from (3) just described. The reverse inclusions, which are what make the three dimensions *exactly* 1, 2 and 3, rest instead on the evaluated degrees $\deg R = 4$ and $\deg Q = 6$ and on the coefficient of P at x^3y^3z ; those are the negative controls listed next, and are checked together with the attainment half of Theorem 4.1 rather than with the certificate. The linear independence of $1, R, Q$ is then read off their distinct degrees 0, 4, 6 by hand.

Negative controls accompany the new statements, since a filtration certificate that proves nothing looks much like one that proves something: the bound 17 is attained, so $\sum_i \max(d_i, 2) \geq 18$ is false and the theorem is not vacuously strong; Q itself has degree 6, so the degree-6 space really is 3-dimensional and the hypothesis of the main filtration step is not empty; P has coefficient 1 at the degree-7 monomial x^3y^3z , so that space is not larger either and the filtration is exact at all three thresholds; and for Proposition 6.4 the entry counts 41 and 187 pin the assembled matrix against an independent build. The three modules check in 61, 45 and 71 seconds. Finally, every numerical value quoted in this paper was first produced by an independent implementation and the formal statement was then written against it, and the reproduction of Section 6 was run in full before any new statement was formulated.

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