SOLVABILITY AND MUTATION ORBITS OF THE GENERALIZED MARKOV EQUATIONS $x^2 + y^2 + z^2 + k_1yz + k_2xz + k_3xy = kxyz$

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ABSTRACT. Chen and Huang, in their study of mutation-preserving generalized cluster algebras, ask (Question 7.1) to classify the $k \in \mathbb{Z}_{>0}$ for which $x^2 + y^2 + z^2 + k_1yz + k_2xz + k_3xy = kxyz$ has a positive integer solution and to determine the orbits of the solutions under the generalized cluster mutation group $\hat{\Gamma}$; they record that for general k "we are even not clear about the existence of the solutions". We prove a Vieta descent bound valid for every (k_1, k_2, k_3, k): every positive solution descends to a *fundamental* one, and every fundamental solution lies in an explicit box. Two consequences hold for all coefficient triples: k admissible implies $k \le 3 + k_1 + k_2 + k_3$, a bound attained at $(1, 1, 1)$, and the admissible set is closed under divisors. We then show that each $\hat{\Gamma}$-orbit of positive solutions contains *exactly one* fundamental solution, so the number of orbits is the cardinality of an explicitly enumerable finite set; this makes the second half of Question 7.1 a finite computation for every coefficient triple. Carrying it out, we answer the existence half for every (k_1, k_2, k_3) with $k_i \le 8$ — the answer depends only on the multiset $\{k_1, k_2, k_3\}$, of which there are 165 in that range, all but two outside the cases settled in the literature — and the orbit half for each of those 165 multisets in its non-decreasing representative: 1430 orbit counts, of which the literature covered 169. We also exhibit a *unit family* of admissible values, one for each factorisation $ts = 2 + k_3$, and deduce that a triple with a positive even coefficient never has admissible set equal to the divisors of $3 + k_1 + k_2 + k_3$. The tabulated data show that the orbit count is not monotone in k, is not bounded by a small constant, and is not determined by the admissible set. Every numbered statement is formalized and machine-checked in Lean 4.

1. INTRODUCTION

The equations. Fix $k_1, k_2, k_3 \in \mathbb{N} = \{0, 1, 2, \dots\}$ and $k \in \mathbb{N}$, and consider the Diophantine equation

$$\mathcal{E}(k_1, k_2, k_3; k): \qquad x^2 + y^2 + z^2 + k_1\, yz + k_2\, xz + k_3\, xy \;=\; k\, xyz. \tag{1}$$

Throughout, the coefficient attached to a *pair* of variables is indexed by the *third* variable, so a permutation of (x, y, z) must be accompanied by the same permutation of (k_1, k_2, k_3). We call (k_1, k_2, k_3) the *coefficient triple*, write $K = k_1 + k_2 + k_3$, and are interested only in solutions $(x, y, z) \in \mathbb{Z}^3_{>0}$. Write

$$\mathcal{S}(k_1, k_2, k_3; k) = \{(x, y, z) \in \mathbb{Z}^3_{>0} : (1) \text{ holds}\}, \qquad \mathcal{K}(k_1, k_2, k_3) = \{k \ge 1 : \mathcal{S}(k_1, k_2, k_3; k) \ne \emptyset\},$$

and call the elements of $\mathcal{K}(k_1, k_2, k_3)$ the *admissible* values of k for that coefficient triple.

At $(k_1, k_2, k_3) = (0, 0, 0)$ equation (1) is the classical Markov equation $x^2 + y^2 + z^2 = kxyz$, solvable in positive integers exactly for $k = 1$ and $k = 3$ [4, 5]. The general family arises in cluster theory. Chen and Huang [1] attach to (1) the three *generalized cluster mutations*

$$\hat{\mu}_1(x, y, z) = \left(\tfrac{y^2 + k_1yz + z^2}{x},\ y,\ z\right), \qquad \hat{\mu}_2(x, y, z) = \left(x,\ \tfrac{x^2 + k_2xz + z^2}{y},\ z\right),$$
$$\hat{\mu}_3(x, y, z) = \left(x,\ y,\ \tfrac{x^2 + k_3xy + y^2}{z}\right), \tag{2}$$

and write $\hat{\Gamma} = \langle \hat{\mu}_1, \hat{\mu}_2, \hat{\mu}_3 \rangle$ for the group they generate, the *generalized cluster mutation group*. Each $\hat{\mu}_i$ preserves $\mathcal{S}(k_1, k_2, k_3; k)$ (Proposition 3.1 below), so $\hat{\Gamma}$ acts on the set of positive solutions, and $\hat{\Gamma}[(x, y, z)]$ denotes an orbit. The case $k = 3 + K$ was treated by Gyoda and Matsushita [2], who showed that all positive solutions then lie in the single orbit $\hat{\Gamma}[(1, 1, 1)]$; the case $(k_1, k_2, k_3) = (0, 2, 2)$, $k = 7$ goes back to Lampe [3].

The question. Chen and Huang [1, Question 7.1] ask:

> For the following class of Diophantine equations $x^2+y^2+z^2+k_1yz+k_2xz+k_3xy=kxyz$, where $k_1,k_2,k_3\in\mathbb{N}$, and $k\in\mathbb{N}_+$, classify all the choices of k such that this equation has positive integer solutions. Moreover, find out all the $\hat{\Gamma}$-orbits of the solutions.

Their own assessment, in the remark immediately following that question [1, Remark 7.2(1)], is that

> In fact, by [GM23], when $k=3+k_1+k_2+k_3$, all the positive integer solutions to $x^2+y^2+z^2+k_1yz+k_2xz+k_3xy=(3+k_1+k_2+k_3)xyz$ lie in the unique orbit $\hat{\Gamma}[(1,1,1)]$. However, for general k, it is still a difficult but interesting problem since we are even not clear about the existence of the solutions.

Before the present work, the first half of the question was answered at exactly one value of k per coefficient triple — the top value $k=3+K$, by [2] — together with two whole triples: $(0,0,0)$, which is Markov–Hurwitz, and $(2,0,0)$, where [1, Theorem 4.8] proves that $x^2+y^2+z^2+2yz=kxyz$ has positive solutions if and only if $k\in\{1,2,3,4,5\}$. For the second half, the orbit count was known at the top value of every triple [2, Theorem 1.1] and at the five admissible values of $(2,0,0)$ [1].

All references to [1] — the quotations above and the numbering of its results used below — are to arXiv version v3 of 19 August 2026, and the quotations are verbatim from the LaTeX source of that version. The pin matters: the section containing the question was renumbered between versions, so the question quoted above is Question 6.1 of v1 and Question 7.1 of v2 and v3, and Conjecture 7.4 acquired a hypothesis in v3. The numbering of Section 4, from which Proposition 4.5, Theorem 4.6, Proposition 4.7, Theorem 4.8 and Table 2 are cited below, is the same in all three versions.

What is proved here. The obstruction to answering either half is that for fixed (k_1,k_2,k_3,k) the search space is infinite, so "no solution" is not a search result and "these are all the orbits" is not an enumeration result. Sections 3 and 4 remove both obstructions, uniformly in the parameters.

- **A descent bound (§3).** Fixing y,z, equation (1) is a monic quadratic in x whose second root is again a positive integer; iterating the resulting Vieta jumps reduces $x+y+z$ until it reaches a *fundamental* solution, one from which no jump descends. A fundamental solution, sorted as $u\ge M\ge m\ge 1$, satisfies two inequalities (Theorem 3.3) that confine it to an explicit box. Consequently solvability is decidable (Theorem 3.5), the largest admissible k is exactly $3+K$ for every coefficient triple (Theorem 3.6), and $\mathcal{K}$ is closed under divisors (Proposition 3.8).
- **One fundamental solution per orbit (§4).** At a positive solution at most one coordinate can exceed its Vieta bound, so the descent is a well-defined map; a mutation that increases a coordinate is the inverse of one that decreases it, so the descent map is constant on $\hat{\Gamma}$-orbits. Hence distinct fundamental solutions lie in distinct orbits (Theorem 4.5), and the finite set of fundamental solutions is a complete irredundant set of orbit representatives (Theorem 4.6). This is what turns the second sentence of Question 7.1 into a finite computation.
- **Two structural theorems (§5).** Solvability depends only on the multiset $\{k_1,k_2,k_3\}$ (Proposition 5.1). For every factorisation $ts=2+k_3$ with $t\ge 1$, the point $(1,1,t)$ solves (1) at $k=t+k_1+k_2+s$ (Theorem 5.2); the case $t=1$ recovers the top value $3+K$ of [2], and the case $t=2$ shows that a coefficient triple with a positive even entry never has $\mathcal{K}=\mathrm{D}(3+K)$ (Corollary 5.3), where $\mathrm{D}(n)$ is the set of positive divisors of n.
- **Both halves of Question 7.1 for $k_i\le 8$ (§6).** Theorem 6.2 lists $\mathcal{K}(k_1,k_2,k_3)$ and Theorem 6.3 lists $\mathcal{N}(k_1,k_2,k_3;k)$, the number of $\hat{\Gamma}$-orbits, for all 165 non-decreasing coefficient triples with $k_i\le 8$ and all admissible k: 1430 orbit counts, of which 169 were covered by the literature. Table 1 is the answer. The admissible sets transport to the remaining orderings by Proposition 5.1; the orbit counts are asserted only for the non-decreasing triples, for the reason set out in Remark 6.7.

The classification statements of §6 rest on exhaustive computation, and the proof sketches there say exactly which finite search was run. The reasoning that makes those searches complete — everything in §3 and §4, and both theorems of §5 — involves no search at all. §7 records the previously known statements that the same machinery reproduces, as the check the new numbers pass through, and §8 the negative controls. §9 describes the formal verification.

What the answer looks like. The admissible sets are not intervals: $\mathcal{K}(0,1,1)=\{1,5\}$ omits $2,3,4$. They are not the divisors of the top value: $5\in\mathcal{K}(2,2,2)$ while $5\nmid 9$. Within $k_i\le 3$ the triples with

$\mathcal{K} = \mathrm{D}(3+K)$ are exactly those with no $k_i = 2$, and it is tempting to promote this to a rule; Corollary 5.3 proves one direction of it in the sharp form "no positive even coefficient", for all coefficient triples and with no search, and the extended table refutes the converse — $\mathcal{K}(0,0,4) = \{1,5,7\}$ while $\mathrm{D}(7) = \{1,7\}$. On the orbit side the counts are as irregular: $\mathcal{N}$ is not monotone in k, reaches 42 inside the tabulated range, and is not a function of $\mathcal{K}$ — the triples $(0,0,5)$ and $(1,1,3)$ have the same admissible set $\{1,2,4,8\}$ and different orbit counts at every k below the top. So the second sentence of Question 7.1 is not a corollary of the first.

2. Preliminaries

Definition 2.1. For $a,b,c,k \in \mathbb{N}$ and $x,y,z \in \mathbb{N}$ write

$$\mathcal{E}(a,b,c;k)(x,y,z): \qquad x^2+y^2+z^2+a\,yz+b\,xz+c\,xy = k\,xyz.$$

A *positive solution* is a solution with $x,y,z \geq 1$; $\mathcal{S}(a,b,c;k)$ is the set of these, and $\mathcal{K}(a,b,c) = \{k \geq 1 : \mathcal{S}(a,b,c;k) \neq \emptyset\}$.

Since a multiplies yz, b multiplies xz and c multiplies xy, transposing two variables and the two corresponding coefficients preserves the equation: $(x,y,z) \in \mathcal{S}(a,b,c;k)$ implies $(y,x,z) \in \mathcal{S}(b,a,c;k)$, $(x,z,y) \in \mathcal{S}(a,c,b;k)$ and $(z,y,x) \in \mathcal{S}(c,b,a;k)$. Three transpositions generate S_3, so for every permutation σ of the three coordinates, applying σ to both the variables and the coefficients is a bijection of solution sets. We use this constantly, and refer to the six ways of doing it as the six *coefficient frames.*

Definition 2.2. A positive solution (x,y,z) of $\mathcal{E}(a,b,c;k)$ is *fundamental* if

$$x^2 \leq y^2+z^2+a\,yz, \qquad y^2 \leq x^2+z^2+b\,xz, \qquad z^2 \leq x^2+y^2+c\,xy.$$

We write $\mathcal{F}(a,b,c;k)$ for the set of fundamental solutions.

Only the inequality at the largest coordinate is restrictive; the other two hold automatically. Stating all three avoids a case split on which coordinate is largest.

Finally, $\hat\mu_1, \hat\mu_2, \hat\mu_3$ are the maps (2) and $\hat\Gamma = \langle \hat\mu_1, \hat\mu_2, \hat\mu_3\rangle$; $\mathcal{N}(a,b,c;k)$ denotes the number of $\hat\Gamma$-orbits of $\mathcal{S}(a,b,c;k)$, and $\mathrm{D}(n)$ the set of positive divisors of n.

3. Vieta descent and the box

Proposition 3.1 (Vieta jumping)**.** *Let $(x,y,z) \in \mathcal{S}(a,b,c;k)$ and put $T_1 = y^2+z^2+a\,yz$. Then $x \mid T_1$, the integer $x' = T_1/x$ is positive, $x+x'+b\,z+c\,y = k\,yz$, and $(x',y,z) \in \mathcal{S}(a,b,c;k)$. The analogous statements hold at the second and third coordinates, with $T_2 = x^2+z^2+b\,xz$ and $T_3 = x^2+y^2+c\,xy$. In particular each $\hat\mu_i$ of (2) is defined on $\mathcal{S}(a,b,c;k)$ and maps it to itself.*

Proof sketch. Fixing y,z, equation (1) reads $x^2-(k\,yz-b\,z-c\,y)x+T_1 = 0$. Divisibility is read off the equation itself: x divides $x^2+b\,xz+c\,xy$ and $k\,xyz$, hence T_1. Both identities then give that the second root $x' = T_1/x$ is a positive integer and that (x',y,z) solves the same equation. No hypothesis beyond positivity is used. ☐

Theorem 3.2 (Descent)**.** *If $\mathcal{S}(a,b,c;k) \neq \emptyset$ then $\mathcal{F}(a,b,c;k) \neq \emptyset$.*

Proof sketch. If one of the three inequalities of Definition 2.2 fails at (x,y,z), say the first, then $x' = T_1/x < x$ by Proposition 3.1, so (x',y,z) is a positive solution with strictly smaller coordinate sum. Induction on $x+y+z$ terminates. ☐

The next theorem is the heart of the matter. It is stated in the *sorted frame*: the coordinates are $u \geq M \geq m \geq 1$ and the coefficients have been transported along, so that a multiplies Mm, b multiplies um and c multiplies uM.

Theorem 3.3 (Master inequalities)**.** *Let $a,b,c,k \in \mathbb{N}$ and let $u \geq M \geq m \geq 1$ be integers with*

$$u^2+M^2+m^2+a\,Mm+b\,um+c\,uM = k\,uMm \qquad \textit{and} \qquad u^2 \leq M^2+m^2+a\,Mm.$$

Then

$$(\star) \quad k\,Mm \leq (2+c)M+(2+2a+b)m \qquad \textit{and} \qquad (\star\star) \quad 3+c \leq k\,m.$$

Proof sketch. Multiply the equation by nothing and cancel one factor of u: substituting $u^2 \le M^2 + m^2 + a\,Mm$ into $u \cdot k\,Mm = u^2 + (M^2 + m^2 + a\,Mm) + b\,um + c\,uM$ and using $M \le u$ and $m \le u$ three times gives ($\star$) after dividing by u. For ($\star\star$), suppose $k\,m \le 2 + c$. Then $u \cdot k\,Mm \le (uM)(2+c) = 2uM + c\,uM$, while the same identity and $u^2 + M^2 \ge 2uM$ force $m^2 + a\,Mm + b\,um \le 0$, impossible for $m \ge 1$. $\square$

Inequality ($\star\star$) is the load-bearing half. Without it the region cut out by ($\star$) is infinite and every negative answer below would be an unsound search rather than a theorem. With it, ($\star$) and $m \le M$ give $k\,m \le 4 + 2a + b + c$, and then ($\star$) bounds M and the fundamentality inequality bounds u:

Corollary 3.4 (The box)**.** *In the situation of Theorem 3.3 with $k \ge 1$,*

$$m \le \left\lfloor \tfrac{4+2a+b+c}{k} \right\rfloor, \qquad M \le \left\lfloor \tfrac{(2+2a+b)\,m}{k\,m-(2+c)} \right\rfloor, \qquad u \le \left\lfloor \sqrt{M^2 + m^2 + a\,Mm} \right\rfloor,$$

where the middle denominator is at least 1 by ($\star\star$). Write $B(a,b,c;k) \subset \mathbb{Z}^3_{>0}$ for the finite set of triples (u,M,m) with $m \le M \le u$ satisfying these three bounds.

Theorem 3.5 (Decidability)**.** *Let $a,b,c \in \mathbb{N}$ and $k \ge 1$. Then $\mathcal{S}(a,b,c;k) \ne \emptyset$ if and only if there are a permutation $\sigma \in S_3$ and a triple $t \in B(\sigma \cdot (a,b,c);k)$ such that $\sigma^{-1} \cdot t$ is a solution of $\mathcal{E}(a,b,c;k)$. Consequently*

$$\mathcal{K}(a,b,c) = \{k : 1 \le k \le 3 + a + b + c \text{ and the above finite test succeeds}\}$$

is computable, and $k \in \mathcal{K}(a,b,c) \iff \mathcal{S}(a,b,c;k) \ne \emptyset$ for every $k \ge 1$.

Proof sketch. “If” is immediate. For “only if”, Theorem 3.2 produces a fundamental solution; sorting it decreasingly and transporting the coefficients along the same permutation produces a fundamental solution of the transported equation to which Theorem 3.3 and Corollary 3.4 apply. The range $k \le 3 + a + b + c$ is Theorem 3.6. The six frames are needed because sorting is a permutation of the coordinates and the coefficients travel with them. $\square$

Theorem 3.6 (The top value)**.** *For all $a,b,c \in \mathbb{N}$: if $k \ge 1$ and $\mathcal{S}(a,b,c;k) \ne \emptyset$ then $k \le 3 + a + b + c$; and $(1,1,1) \in \mathcal{S}(a,b,c;3+a+b+c)$. Hence* $\max \mathcal{K}(a,b,c) = 3 + a + b + c$ *for every coefficient triple.*

Proof sketch. The second assertion is a one-line substitution. For the first, take a fundamental solution sorted as $u \ge M \ge m \ge 1$ and split on m and M. If $m \ge 2$ then $2k \le k\,m \le 4 + 2a + b + c$. If $m = 1$ and $M \ge 2$ then ($\star$) gives $M\big(k - (2+c)\big) \le 2 + 2a + b$ together with $M \ge 2$. If $m = M = 1$ the equation reads $u^2 + (2+a) + (b+c)u = k\,u$, so $u \mid 2+a$; writing $ut = 2+a$ gives $k = u+t+b+c$ and $u+t \le 1+ut = 3+a$. In each case $k \le 3 + a + b + c$. $\square$

Remark 3.7. The $(2,0,0)$ instance of Theorem 3.6 is [1, Proposition 4.5]: “Let $k \in \mathbb{N}$ and the Diophantine equation be as follows: $x^2 + y^2 + z^2 + 2yz = kxyz$. Then, if $k \ge 6$, there is no positive integer solution.” The proof above is uniform in (k_1,k_2,k_3) and uses no search.

Proposition 3.8 (Divisor closure)**.** *If $j \mid k$ and $k \in \mathcal{K}(a,b,c)$ then $j \in \mathcal{K}(a,b,c)$. Equivalently, if $(x,y,z) \in \mathcal{S}(a,b,c;dk)$ with $d \ge 1$ then $(dx,dy,dz) \in \mathcal{S}(a,b,c;k)$.*

Proof. Both sides of (1) are homogeneous, of degrees 2 and 3: scaling (x,y,z) by d multiplies the left side by d^2 and the right side by d^3, so the admissible parameter is divided by d. Also $\mathcal{S}(a,b,c;0) = \emptyset$, which handles $k = 0$. $\square$

This is the scaling of [1, Proposition 4.7], for arbitrary (k_1,k_2,k_3) rather than only $(2,0,0)$. It explains the shape of every row of Table 1 but does not determine it: see Corollary 5.3 and Remark 6.5.

4. One fundamental solution per orbit

The maps $\hat{\mu}_i$ of (2) involve a division, so it is convenient to encode a mutation by the Vieta product instead.

Definition 4.1. For $p = (p_1,p_2,p_3)$ and $q = (q_1,q_2,q_3)$ in $\mathbb{Z}^3_{>0}$, say $p \sim_1 q$ if $q_2 = p_2$, $q_3 = p_3$ and $p_1 q_1 = T_1(p)$, where $T_1(p) = p_2^2 + p_3^2 + a\,p_2 p_3$; define $\sim_2$ and $\sim_3$ analogously with T_2 and T_3. Put $\mathrm{Adj} = \sim_1 \cup \sim_2 \cup \sim_3$ and let $\approx$ be the reflexive–transitive closure of Adj.

Swapping p and q maps each of the three clauses to itself, so Adj is symmetric by construction and $\approx$ is an equivalence relation.

Lemma 4.2. *Let $p \in \mathcal{S}(a,b,c;k)$. Then* $\mathrm{Adj}(p,q)$ *holds if and only if $q \in \{\hat{\mu}_1(p), \hat{\mu}_2(p), \hat{\mu}_3(p)\}$.*

Proof. Immediate from Proposition 3.1, which supplies the divisibility $p_i \mid T_i(p)$ that makes $\hat{\mu}_i(p)$ integral: $p_1 q_1 = T_1(p)$ with $p_1 \geq 1$ says exactly $q_1 = T_1(p)/p_1$. □

Remark 4.3 ($\approx$-classes and $\hat{\Gamma}$-orbits)*.* On $\mathcal{S}(a,b,c;k)$ the two partitions agree. Indeed, by Proposition 3.1 each $\hat{\mu}_i$ maps $\mathcal{S}(a,b,c;k)$ into itself; by Lemma 4.2 the graph of $\hat{\mu}_i$ on $\mathcal{S}(a,b,c;k)$ is the relation $\sim_i$, which is symmetric, so $\hat{\mu}_i$ restricted to $\mathcal{S}(a,b,c;k)$ is an involution and coincides there with $\hat{\mu}_i^{-1}$. Hence every element of $\hat{\Gamma}$, being a word in the generators and their inverses, acts on a positive integer solution as a word in the generators alone, and $\hat{\Gamma}[p]$ is exactly the $\approx$-class of p; in particular the $\hat{\Gamma}$-orbit of a positive integer solution contains no non-integral point.

This identification is the only step of §4 that is argued here in prose and is *not* part of the formal development described in §9. What is formalized is the relation: Lemma 4.2, Theorem 4.5 and Theorem 4.6 are proved for $\approx$, the reflexive–transitive closure of Adj, and every machine-checked orbit count in this paper is, verbatim, a count of $\approx$-classes. Readers who prefer to keep the two notions apart may read every occurrence of "$\hat{\Gamma}$-orbit" below as "$\approx$-class", with no change to any statement. Subject to that reading we use the two terms interchangeably from here on.

Lemma 4.4 (At most one coordinate is too large)**.** *Let $(x,y,z) \in \mathcal{S}(a,b,c;k)$. Then at most one of*

$$x^2 > y^2 + z^2 + a\,yz, \qquad y^2 > x^2 + z^2 + b\,xz, \qquad z^2 > x^2 + y^2 + c\,xy$$

holds.

Proof. Adding the first two gives $x^2 + y^2 > x^2 + y^2 + 2z^2 + a\,yz + b\,xz$, that is $0 > 2z^2 + a\,yz + b\,xz$, impossible for $z \geq 1$. The other two pairs are symmetric. □

So a non-fundamental positive solution has a unique too-large coordinate, and mutating there strictly decreases the coordinate sum. Let $d(p)$ denote the fundamental solution obtained from p by iterating this step; it is well defined by Lemma 4.4, and the iteration terminates by Theorem 3.2.

Theorem 4.5 (Uniqueness in an orbit)**.** *Let $k \geq 1$ and $p, q \in \mathcal{F}(a,b,c;k)$ with $p \approx q$. Then $p = q$.*

Proof sketch. It suffices to show d is constant along a single mutation, since d fixes fundamental solutions. Let $\mathrm{Adj}(p,q)$ with p a positive solution, differing in the first coordinate say, so $p_1 q_1 = T_1(p)$. If $q_1 < p_1$ then $T_1(p) < p_1^2$, so the first coordinate is the too-large one at p; by Lemma 4.4 it is the only one, so the descent step at p is exactly q and $d(p) = d(q)$. If $q_1 = p_1$ then $q = p$. If $q_1 > p_1$, apply the previous case to $\mathrm{Adj}(q,p)$, using the symmetry of Adj and that q is again a positive solution. Induction along the reflexive–transitive closure gives $d(p) = d(q)$ whenever $p \approx q$; if both are fundamental this reads $p = q$. □

Theorem 4.6 (Transversal)**.** *Let $a, b, c \in \mathbb{N}$ and $k \geq 1$. Then $\mathcal{F}(a,b,c;k)$ is finite, is contained in the six boxes of Corollary 3.4, and is a complete and irredundant set of representatives for the $\hat{\Gamma}$-orbits on $\mathcal{S}(a,b,c;k)$: every positive solution is $\hat{\Gamma}$-equivalent to exactly one element of $\mathcal{F}(a,b,c;k)$. Consequently*

$$\mathcal{N}(a,b,c;k) = \#\mathcal{F}(a,b,c;k),$$

and this number is computable by the finite search of Theorem 3.5.

Proof sketch. Completeness of the enumeration is Corollary 3.4 together with the six-frame sort of Theorem 3.5; that every solution reaches a fundamental one in its own orbit is Theorem 3.2; irredundancy is Theorem 4.5. □

Theorem 4.6 is what makes the second sentence of Question 7.1 a finite computation: for each (k_1, k_2, k_3, k) the orbits are counted by enumerating an explicit finite box and filtering it, with no search of the orbits themselves. It is elementary: at $(k_1, k_2, k_3) = (0,0,0)$ it is the classical Markov tree of [4], and at $k = 3 + K$ it is the tree structure of [2]. What is used here is that the same argument goes through for an arbitrary quadruple (k_1, k_2, k_3, k).

5. Permutation invariance and the unit family

Proposition 5.1 (Permutation invariance)**.** *For every permutation σ of the three coordinates and every k, $\mathcal{S}(a,b,c;k) \neq \emptyset$ if and only if $\mathcal{S}(\sigma \cdot (a,b,c);k) \neq \emptyset$. Hence $\mathcal{K}(a,b,c)$ depends only on the multiset $\{a,b,c\}$, and for $k \geq 1$ membership $k \in \mathcal{K}(a,b,c)$ transports along σ as well.*

Proof. Apply σ to the coordinates of a solution; this is the symmetry recorded in §2, and the three transpositions generate S_3. □

In particular the $9^3 = 729$ coefficient triples with $k_i \leq 8$ collapse to the 165 non-decreasing ones tabulated below.

Theorem 5.2 (The unit family)**.** *Let $a,b,c \in \mathbb{N}$ and let $t,s \geq 0$ satisfy $t \geq 1$ and $ts = 2+c$. Then*

$$(1,1,t) \in \mathcal{S}\big(a,b,c;\, t+a+b+s\big), \qquad \text{so} \qquad t+a+b+s \in \mathcal{K}(a,b,c).$$

Proof. Substituting $(x,y,z) = (1,1,t)$ into (1) gives $1+1+t^2+at+bt+c = (t^2+at+bt)+(2+c)$ on the left and $(t+a+b+s)t = (t^2+at+bt)+ts$ on the right; these agree exactly when $ts = 2+c$. □

The choice $t=1$, $s=2+c$ gives $k = 3+a+b+c$, i.e. the top value of [2] and Theorem 3.6; so the unit family is a one-parameter extension of the case the literature had settled. The choice $t=2$ is available exactly when c is even and is what breaks the divisor pattern:

Corollary 5.3 (A positive even coefficient breaks the divisor pattern)**.** *Let $a,b \in \mathbb{N}$ and $e \geq 1$. Then $3+a+b+e \in \mathcal{K}(a,b,2e)$ and $(3+a+b+e) \nmid (3+a+b+2e)$. Consequently, if some k_i is positive and even then $\mathcal{K}(k_1,k_2,k_3) \neq \mathrm{D}(3+K)$.*

Proof. Take $t=2$, $s=1+e$ in Theorem 5.2, so that $k = 2+a+b+(1+e) = 3+a+b+e$ with witness $(1,1,2)$. Since $3+a+b+2e = (3+a+b+e)+e$, divisibility would force $(3+a+b+e) \mid e$, which is impossible as $3+a+b+e > e \geq 1$. The last sentence follows from Proposition 5.1. □

The hypothesis $e \geq 1$ is not decoration: for $e=0$ the conclusion fails, e.g. $\mathcal{K}(0,0,0) = \{1,3\} = \mathrm{D}(3)$.

6. Both halves of Question 7.1 for $k_i \leq 8$

Table 1 lists, for each of the 165 non-decreasing coefficient triples (k_1,k_2,k_3) with $k_i \leq 8$, the top value $3+K$, the admissible set $\mathcal{K}$ with each admissible k annotated by its orbit count in the form $k{:}\mathcal{N}$, and whether $\mathcal{K} = \mathrm{D}(3+K)$. By Proposition 5.1 the first column determines the admissible set of all 729 triples with $k_i \leq 8$; the orbit counts, by contrast, are asserted only for the non-decreasing triple shown in each row (Remark 6.7). The 20 rows marked $*$ are those with $k_i \leq 3$.

Theorem 6.1 (Existence, $k_i \leq 3$)**.** *For each of the 64 coefficient triples $(k_1,k_2,k_3) \in \{0,1,2,3\}^3$, the set $\mathcal{K}(k_1,k_2,k_3)$ is the one listed against its non-decreasing rearrangement in the rows of Table 1 marked $*$. There are 20 distinct answers.*

Theorem 6.2 (Existence, $k_i \leq 8$)**.** *For each of the 165 non-decreasing coefficient triples (k_1,k_2,k_3) with $k_i \leq 8$, the set $\mathcal{K}(k_1,k_2,k_3)$ is the set of values of k listed in the corresponding row of Table 1. Of these 165 triples, 145 lie outside the range of Theorem 6.1, and 163 lie outside the two coefficient triples settled in the literature.*

Theorem 6.3 (Orbits, $k_i \leq 8$)**.** *For each of the same 165 coefficient triples and each admissible k, the number $\mathcal{N}(k_1,k_2,k_3;k)$ of $\hat{\Gamma}$-orbits of positive integer solutions of (1) is the value listed after the colon in the corresponding entry of Table 1. The table records 1430 orbit counts in all.*

Proof sketch for Theorems 6.1, 6.2 and 6.3. All three rest on exhaustive computation, and the computation is exactly the one licensed by §3 and §4. For a fixed coefficient triple and each $k = 1,\dots,3+K$ — the range is complete by Theorem 3.6 — the six boxes $B(\sigma \cdot (k_1,k_2,k_3);k)$ of Corollary 3.4 are enumerated, each candidate triple is placed back into the variable order σ and tested against (1), and the survivors are tested for fundamentality (Definition 2.2) and de-duplicated. Theorem 3.5 makes the resulting list of k equal to $\mathcal{K}(k_1,k_2,k_3)$ — so the empty entries are theorems, not failed searches — and Theorem 4.6 makes the length of the surviving list at each k equal to $\mathcal{N}(k_1,k_2,k_3;k)$, rather than merely a count of representatives found. No orbit is ever traversed. The search performed for Theorems 6.2 and 6.3 is the union of these enumerations over the 165 non-decreasing triples with $k_i \le 8$ and all $k \le 3+K$; it examines on the order of 10^7 candidate triples and runs in about a minute. §9 describes how it was carried out and independently repeated. □

Corollary 6.4. *$\mathcal{K}(0,0,4) = \{1,5,7\}$ while $\mathrm{D}(7) = \{1,7\}$; $\mathcal{K}(1,1,4) = \{1,3,7,9\}$ while $\mathrm{D}(9) = \{1,3,9\}$; $\mathcal{K}(3,3,4) = \{1,11,13\}$ while $\mathrm{D}(13) = \{1,13\}$. In each case the extra value is witnessed by $(1,1,2)$, the $t = 2$ member of the unit family. At the opposite extreme $\mathcal{K}(2,4,6) = \{1,2,\ldots,15\}$ is the full interval up to its top value.*

Remark 6.5. Exactly 15 of the 165 rows of Table 1 satisfy $\mathcal{K} = \mathrm{D}(3+K)$. All 15 have no positive even coefficient, as Corollary 5.3 requires. The converse fails badly: the coefficients lying in $\{0,1,3,5,7\}$ give $\binom{5+3-1}{3} = 35$ non-decreasing triples with $k_i \le 8$ and no positive even coefficient, so 20 of them are counterexamples. The first of the 20 in table order is $\{0,0,7\}$, with $\mathcal{K} = \{1,2,3,5,6,10\}$ against $\mathrm{D}(10) = \{1,2,5,10\}$, and the smallest by coefficient sum is $\{0,1,5\}$, with $\mathcal{K} = \{1,3,5,9\}$ against $\mathrm{D}(9) = \{1,3,9\}$. Inside $k_i \le 3$ there are none, which is why the pattern "$\mathcal{K} = \mathrm{D}(3+K)$ if and only if no $k_i = 2$" looks like a rule there.

Remark 6.6. Let $U(k_1,k_2,k_3) \subseteq \mathcal{K}(k_1,k_2,k_3)$ be the set of values produced by Theorem 5.2 in all three coefficient frames and then closed under divisors (Proposition 3.8). Comparing U with the rows of Table 1, an arithmetic check on the tabulated sets, gives $U = \mathcal{K}$ for exactly 49 of the 165 triples and $U \subsetneq \mathcal{K}$ for the other 116. So the unit family, though it is what breaks the divisor pattern, is far from being the whole answer.

Remark 6.7 (The scope of the orbit column). The two halves of the table are asserted with different scopes, and the difference is deliberate.

For the admissible set, Proposition 5.1 is a theorem: $\mathcal{K}$ depends only on the multiset $\{k_1,k_2,k_3\}$, so a row of Table 1 settles the admissible set of all orderings of its coefficients, hence of all 729 triples with $k_i \le 8$. This is why the first column is written with multiset braces.

For the orbit counts no such transport is claimed. Every value $\mathcal{N}$ in the third column is computed for, and asserted only of, the *non-decreasing* ordered triple (k_1,k_2,k_3) shown — and likewise in Theorem 6.3 and Proposition 8.2, where a multiset $\{k_1,k_2,k_3\}$ named in an orbit statement always denotes its non-decreasing representative. We expect the counts to be permutation invariant, since permuting the variables and the coefficients simultaneously is a bijection of solution sets that commutes with the mutations (2) and so should carry $\hat{\Gamma}$-orbits to $\hat{\Gamma}$-orbits; but, unlike Proposition 5.1, that invariance is neither proved nor formalized here, and no statement in this paper uses it. Two kinds of orbit claim in this paper do reach beyond the tabulated rows, and neither goes through such a transport. The first assertion of Theorem 7.2, that $\mathcal{N}(k_1,k_2,k_3;3+K) = 1$, is proved directly for every ordered triple (k_1,k_2,k_3) and does not use the table at all. The ordered triple $(2,0,0)$, whose orbit data appear in Theorem 7.2 and Proposition 8.2, was enumerated directly rather than transported from the row $\{0,0,2\}$; that the two agree is a check on the expectation above and not a substitute for a proof of it.

Table 1: Both halves of Question 7.1 for all 165 non-decreasing coefficient triples $\{k_1,k_2,k_3\}$ with $k_i \le 8$ (Theorems 6.1, 6.2 and 6.3). An entry $k{:}\mathcal{N}$ means that k is admissible and that (1) has exactly $\mathcal{N}$ orbits of positive integer solutions under $\hat{\Gamma}$. The admissible set and the last column depend only on the multiset (Proposition 5.1); the orbit counts are asserted for the non-decreasing representative shown, and are not claimed to transport to its other orderings (Remark 6.7). Rows marked $*$ are those with all $k_i \le 3$. The last column says whether the admissible set is the set of divisors of the top value $3+K$.

$\{k_1,k_2,k_3\}$	$3{+}K$	admissible k with orbit count, $k{:}\mathcal{N}$	$\mathrm{D}(3{+}K)$?
$\{0,0,0\}^*$	3	1:1 3:1	yes
$\{0,0,1\}^*$	4	1:1 2:1 4:1	yes
$\{0,0,2\}^*$	5	1:4 2:1 3:2 4:1 5:1	–
$\{0,0,3\}^*$	6	1:1 2:1 3:1 6:1	yes
$\{0,0,4\}$	7	1:2 5:1 7:1	–
$\{0,0,5\}$	8	1:3 2:3 4:3 8:1	yes
$\{0,0,6\}$	9	1:4 2:1 3:4 6:1 9:1	–
$\{0,0,7\}$	10	1:4 2:4 3:1 5:1 6:1 10:1	–
$\{0,0,8\}$	11	1:8 5:2 7:1 11:1	–
$\{0,1,1\}^*$	5	1:1 5:1	yes

Table 1, continued.

$\{k_1, k_2, k_3\}$	$3{+}K$	admissible k with orbit count, k:$\mathcal{N}$	D($3{+}K$)?
$\{0,1,2\}^*$	6	1:6 2:3 3:2 4:1 5:1 6:1	–
$\{0,1,3\}^*$	7	1:1 7:1	yes
$\{0,1,4\}$	8	1:2 2:2 3:1 4:1 6:1 8:1	–
$\{0,1,5\}$	9	1:2 3:1 5:1 9:1	–
$\{0,1,6\}$	10	1:4 2:3 4:2 5:1 7:1 10:1	–
$\{0,1,7\}$	11	1:4 3:2 7:1 11:1	–
$\{0,1,8\}$	12	1:6 2:6 3:2 4:2 6:2 8:1 12:1	–
$\{0,2,2\}^*$	7	1:15 2:6 3:4 4:2 5:2 6:2 7:1	–
$\{0,2,3\}^*$	8	1:6 2:3 3:2 4:1 6:1 7:1 8:1	–
$\{0,2,4\}$	9	1:10 2:2 3:3 4:2 5:1 7:2 8:1 9:1	–
$\{0,2,5\}$	10	1:10 2:5 3:4 4:2 5:2 6:1 8:1 9:1 10:1	–
$\{0,2,6\}$	11	1:13 2:5 3:5 4:2 5:2 6:1 8:1 9:1 10:1 11:1	–
$\{0,2,7\}$	12	1:16 2:8 3:4 4:4 5:2 6:1 8:1 10:1 11:1 12:1	–
$\{0,2,8\}$	13	1:18 2:6 3:6 4:4 6:2 7:2 9:1 11:1 12:1 13:1	–
$\{0,3,3\}^*$	9	1:1 3:1 9:1	yes
$\{0,3,4\}$	10	1:2 2:2 4:1 5:1 8:1 10:1	–
$\{0,3,5\}$	11	1:2 7:1 11:1	–
$\{0,3,6\}$	12	1:3 2:2 3:3 4:1 6:2 9:1 12:1	–
$\{0,3,7\}$	13	1:2 3:1 9:1 13:1	–
$\{0,3,8\}$	14	1:4 2:4 4:2 5:1 7:1 8:1 10:1 14:1	–
$\{0,4,4\}$	11	1:3 3:2 9:2 11:1	–
$\{0,4,5\}$	12	1:4 2:4 3:2 4:2 5:1 6:2 8:1 10:1 12:1	–
$\{0,4,6\}$	13	1:5 2:1 5:2 7:1 10:1 11:1 13:1	–
$\{0,4,7\}$	14	1:5 2:5 3:1 4:3 5:1 6:1 7:1 10:1 12:1 14:1	–
$\{0,4,8\}$	15	1:7 3:4 5:1 7:1 9:1 11:1 13:1 15:1	–
$\{0,5,5\}$	13	1:5 3:2 5:2 9:2 13:1	–
$\{0,5,6\}$	14	1:8 2:6 4:4 5:1 7:2 8:1 10:1 11:1 14:1	–
$\{0,5,7\}$	15	1:5 3:3 5:1 11:2 15:1	–
$\{0,5,8\}$	16	1:8 2:8 3:3 4:4 5:1 6:3 8:2 10:1 12:2 16:1	–
$\{0,6,6\}$	15	1:7 2:2 3:7 4:2 5:1 6:2 9:2 12:2 15:1	–
$\{0,6,7\}$	16	1:11 2:8 3:2 4:2 5:3 6:2 8:1 10:1 12:1 13:1 16:1	–
$\{0,6,8\}$	17	1:14 2:2 4:1 5:3 7:1 8:1 11:2 13:1 14:1 17:1	–
$\{0,7,7\}$	17	1:9 5:2 13:2 17:1	–
$\{0,7,8\}$	18	1:8 2:8 3:3 4:4 6:3 7:2 9:1 12:1 14:2 18:1	–
$\{0,8,8\}$	19	1:13 3:8 5:4 9:2 13:2 15:2 19:1	–
$\{1,1,1\}^*$	6	1:1 2:1 3:1 6:1	yes
$\{1,1,2\}^*$	7	1:4 2:3 3:1 4:2 6:1 7:1	–
$\{1,1,3\}^*$	8	1:1 2:1 4:1 8:1	yes
$\{1,1,4\}$	9	1:2 3:1 7:1 9:1	–
$\{1,1,5\}$	10	1:1 2:1 5:1 10:1	yes
$\{1,1,6\}$	11	1:4 2:1 4:1 5:2 8:1 11:1	–
$\{1,1,7\}$	12	1:4 2:4 3:1 4:4 6:1 8:1 12:1	–
$\{1,1,8\}$	13	1:2 3:1 9:1 13:1	–
$\{1,2,2\}^*$	8	1:21 2:7 3:2 4:3 5:4 7:2 8:1	–
$\{1,2,3\}^*$	9	1:6 2:4 3:2 4:2 6:1 8:1 9:1	–
$\{1,2,4\}$	10	1:10 2:5 3:2 4:3 5:2 6:1 7:1 8:1 9:1 10:1	–
$\{1,2,5\}$	11	1:10 2:6 3:2 4:2 5:2 6:1 8:1 10:1 11:1	–
$\{1,2,6\}$	12	1:14 2:4 3:5 4:2 5:2 6:2 7:1 9:2 11:1 12:1	–
$\{1,2,7\}$	13	1:16 2:8 3:4 4:4 5:3 6:2 9:1 10:1 12:1 13:1	–
$\{1,2,8\}$	14	1:18 2:9 3:3 4:4 5:3 6:1 7:2 8:1 10:1 11:1 13:1 14:1	–
$\{1,3,3\}^*$	10	1:1 2:1 5:1 10:1	yes
$\{1,3,4\}$	11	1:2 3:1 9:1 11:1	–
$\{1,3,5\}$	12	1:1 2:1 3:1 4:1 6:1 12:1	yes
$\{1,3,6\}$	13	1:3 2:1 5:1 7:1 10:1 13:1	–
$\{1,3,7\}$	14	1:3 2:3 3:1 5:1 6:1 7:1 10:1 14:1	–
$\{1,3,8\}$	15	1:3 3:1 5:2 11:1 15:1	–
$\{1,4,4\}$	12	1:3 2:3 3:1 4:1 5:2 6:1 10:2 12:1	–
$\{1,4,5\}$	13	1:3 7:1 11:1 13:1	–
$\{1,4,6\}$	14	1:5 2:4 3:2 4:2 6:2 7:1 8:1 11:1 12:1 14:1	–
$\{1,4,7\}$	15	1:5 3:1 5:2 7:1 11:1 13:1 15:1	–
$\{1,4,8\}$	16	1:6 2:6 3:1 4:5 6:1 7:1 8:2 12:1 14:1 16:1	–
$\{1,5,5\}$	14	1:3 2:3 3:2 6:2 7:1 14:1	–
$\{1,5,6\}$	15	1:5 2:2 3:3 4:2 5:2 6:1 8:1 9:1 12:1 15:1	–
$\{1,5,7\}$	16	1:4 2:4 3:1 4:4 6:1 8:2 12:1 16:1	–
$\{1,5,8\}$	17	1:7 3:4 7:1 9:1 13:1 17:1	–
$\{1,6,6\}$	16	1:5 2:3 4:1 5:2 8:1 10:2 13:2 16:1	–
$\{1,6,7\}$	17	1:9 2:3 3:4 6:2 7:2 9:1 11:1 13:1 14:1 17:1	–
$\{1,6,8\}$	18	1:10 2:7 3:5 4:1 5:2 6:3 7:1 9:2 12:1 14:1 15:1 18:1	–
$\{1,7,7\}$	18	1:7 2:7 3:1 5:2 6:1 7:2 9:1 10:2 14:2 18:1	–
$\{1,7,8\}$	19	1:14 3:2 5:4 7:1 11:1 15:2 19:1	–
$\{1,8,8\}$	20	1:11 2:11 3:4 4:5 5:3 6:4 8:2 10:3 16:2 20:1	–
$\{2,2,2\}^*$	9	1:16 2:9 3:7 4:3 5:6 6:6 8:3 9:1	–
$\{2,2,3\}^*$	10	1:21 2:7 3:6 5:3 6:2 7:2 9:2 10:1	–
$\{2,2,4\}$	11	1:30 2:12 3:5 4:4 5:4 6:2 7:4 8:2 9:1 10:2 11:1	–
$\{2,2,5\}$	12	1:27 2:9 3:9 4:3 5:2 6:3 7:2 8:2 9:2 11:2 12:1	–
$\{2,2,6\}$	13	1:24 2:15 3:6 4:6 5:5 6:4 8:2 9:2 10:3 12:2 13:1	–
$\{2,2,7\}$	14	1:36 2:12 3:4 4:2 5:9 6:2 7:3 10:3 11:2 13:2 14:1	–
$\{2,2,8\}$	15	1:42 2:18 3:15 4:6 5:5 6:6 7:2 8:2 9:2 11:3 12:2 14:2 15:1	–

Table 1, continued.

$\{k_1, k_2, k_3\}$	$3+K$	admissible k with orbit count, k:$\mathcal{N}$	D($3+K$)?
$\{2,3,3\}^*$	11	1:4 2:3 3:2 5:1 6:2 10:1 11:1	–
$\{2,3,4\}$	12	1:12 2:6 3:3 4:3 5:1 6:2 7:2 8:1 10:1 11:1 12:1	–
$\{2,3,5\}$	13	1:10 2:6 3:3 4:3 6:2 7:1 8:2 12:1 13:1	–
$\{2,3,6\}$	14	1:12 2:3 3:4 5:1 6:1 7:2 9:2 11:1 13:1 14:1	–
$\{2,3,7\}$	15	1:14 2:8 3:3 4:2 5:2 6:2 7:2 8:1 10:1 11:1 14:1 15:1	–
$\{2,3,8\}$	16	1:18 2:9 3:7 4:3 5:2 6:5 7:2 8:1 9:1 10:1 11:1 12:1 15:1 16:1	–
$\{2,4,4\}$	13	1:12 2:3 3:3 4:3 5:2 6:1 7:2 8:2 9:2 11:2 12:1 13:1	–
$\{2,4,5\}$	14	1:18 2:9 3:5 4:5 5:1 6:1 7:2 8:3 9:2 10:1 12:1 13:1 14:1	–
$\{2,4,6\}$	15	1:20 2:8 3:6 4:3 5:4 6:2 7:3 8:1 9:1 10:2 11:1 12:1 13:1 14:1 15:1	–
$\{2,4,7\}$	16	1:20 2:10 3:6 4:5 5:3 6:3 7:3 8:2 9:1 10:1 11:1 12:2 14:1 15:1 16:1	–
$\{2,4,8\}$	17	1:28 2:8 3:5 4:6 5:4 6:1 7:3 8:3 9:1 10:1 11:2 12:1 13:2 15:1 16:1 17:1	–
$\{2,5,5\}$	15	1:12 2:7 3:3 4:4 5:3 7:3 8:2 9:2 10:2 14:1 15:1	–
$\{2,5,6\}$	16	1:24 2:8 3:8 4:3 5:4 6:1 8:2 9:3 10:1 11:2 13:1 15:1 16:1	–
$\{2,5,7\}$	17	1:20 2:12 3:3 4:6 5:2 6:1 8:3 9:1 10:2 11:1 12:1 13:1 16:1 17:1	–
$\{2,5,8\}$	18	1:28 2:14 3:8 4:7 5:2 6:4 7:1 8:3 9:3 10:1 11:2 12:2 13:1 14:1 17:1 18:1	–
$\{2,6,6\}$	17	1:14 2:9 3:8 4:3 5:2 6:4 7:2 8:1 9:2 10:2 12:2 14:2 16:1 17:1	–
$\{2,6,7\}$	18	1:28 2:9 3:6 4:2 5:5 6:1 7:3 8:1 9:3 10:2 11:2 13:1 14:1 15:1 17:1 18:1	–
$\{2,6,8\}$	19	1:32 2:16 3:11 4:6 5:4 6:5 7:3 8:1 9:3 10:3 11:1 12:2 13:1 14:1 15:1 16:1 18:1 19:1	–
$\{2,7,7\}$	19	1:16 2:11 3:7 4:2 5:6 6:5 9:1 10:4 11:2 12:2 15:2 18:1 19:1	–
$\{2,7,8\}$	20	1:36 2:18 3:5 4:8 5:4 6:3 7:1 8:3 10:3 11:4 12:1 13:2 14:1 16:2 19:1 20:1	–
$\{2,8,8\}$	21	1:32 2:9 3:7 4:7 5:7 6:4 7:7 10:1 11:6 12:4 14:2 15:2 17:2 20:1 21:1	–
$\{3,3,3\}^*$	12	1:1 2:1 3:1 4:1 6:1 12:1	yes
$\{3,3,4\}$	13	1:2 11:1 13:1	–
$\{3,3,5\}$	14	1:1 2:1 7:1 14:1	yes
$\{3,3,6\}$	15	1:2 2:1 3:2 4:1 5:1 6:1 12:1 15:1	–
$\{3,3,7\}$	16	1:2 2:2 3:1 4:2 6:1 8:1 12:1 16:1	–
$\{3,3,8\}$	17	1:4 7:2 13:1 17:1	–
$\{3,4,4\}$	14	1:3 2:3 3:2 4:2 6:2 7:1 12:2 14:1	–
$\{3,4,5\}$	15	1:3 3:2 5:1 9:1 13:1 15:1	–
$\{3,4,6\}$	16	1:4 2:3 4:2 7:1 8:2 13:1 14:1 16:1	–
$\{3,4,7\}$	17	1:3 3:1 5:1 13:1 15:1 17:1	–
$\{3,4,8\}$	18	1:6 2:6 3:2 4:2 5:1 6:2 7:1 8:2 9:1 10:1 14:1 16:1 18:1	–
$\{3,5,5\}$	16	1:3 2:3 4:3 8:3 16:1	yes
$\{3,5,6\}$	17	1:4 2:2 5:1 7:2 10:1 14:1 17:1	–
$\{3,5,7\}$	18	1:2 2:2 3:1 6:1 7:1 9:1 14:1 18:1	–
$\{3,5,8\}$	19	1:5 3:3 5:1 9:2 11:1 15:1 19:1	–
$\{3,6,6\}$	18	1:3 2:1 3:3 5:2 6:1 9:1 15:2 18:1	–
$\{3,6,7\}$	19	1:5 2:2 3:2 4:2 5:1 8:2 9:1 15:1 16:1 19:1	–
$\{3,6,8\}$	20	1:6 2:4 4:3 5:2 8:2 10:2 11:1 16:1 17:1 20:1	–
$\{3,7,7\}$	20	1:3 2:3 4:3 5:1 8:2 10:1 16:2 20:1	–
$\{3,7,8\}$	21	1:5 3:1 7:2 11:1 17:2 21:1	–
$\{3,8,8\}$	22	1:11 2:11 3:6 4:6 6:6 8:2 9:2 11:1 12:4 18:2 22:1	–
$\{4,4,4\}$	15	1:4 3:1 5:1 13:3 15:1	–
$\{4,4,5\}$	16	1:5 2:5 4:1 5:2 7:2 8:1 10:2 14:2 16:1	–
$\{4,4,6\}$	17	1:6 2:1 3:4 5:2 7:1 9:2 14:1 15:2 17:1	–
$\{4,4,7\}$	18	1:6 2:6 3:3 4:2 6:3 7:1 8:2 9:1 14:1 16:2 18:1	–
$\{4,4,8\}$	19	1:8 3:1 5:1 7:2 11:2 15:1 17:2 19:1	–
$\{4,5,5\}$	17	1:6 3:3 5:1 9:2 11:2 15:1 17:1	–
$\{4,5,6\}$	18	1:8 2:6 3:4 4:3 5:2 6:3 8:2 9:1 10:1 11:1 12:1 15:1 16:1 18:1	–
$\{4,5,7\}$	19	1:5 3:1 5:2 13:1 15:1 17:1 19:1	–
$\{4,5,8\}$	20	1:9 2:9 3:3 4:6 5:2 6:3 7:1 8:2 9:1 10:2 12:2 14:1 16:1 18:1 20:1	–
$\{4,6,6\}$	19	1:6 2:2 4:2 8:2 11:2 16:2 17:1 19:1	–
$\{4,6,7\}^*$	20	1:9 2:6 3:3 4:4 5:2 6:2 7:1 8:1 9:2 10:2 12:1 16:1 17:1 18:1 20:1	–
$\{4,6,8\}$	21	1:11 2:2 3:6 4:1 6:2 7:2 9:3 12:1 13:2 17:1 18:1 19:1 21:1	–
$\{4,7,7\}$	21	1:8 3:3 7:3 17:2 19:1 21:1	–
$\{4,7,8\}$	22	1:11 2:11 3:2 4:3 5:2 6:2 7:1 8:2 9:2 10:2 11:1 14:1 18:2 20:1 22:1	–
$\{4,8,8\}$	23	1:18 3:5 5:4 7:1 9:2 11:2 13:2 15:2 19:2 21:1 23:1	–
$\{5,5,5\}$	18	1:7 2:7 3:1 5:6 6:1 9:1 10:6 18:1	–
$\{5,5,6\}$	19	1:10 2:3 3:4 4:3 5:2 6:2 8:1 9:2 11:2 12:2 16:1 19:1	–
$\{5,5,7\}$	20	1:4 2:4 3:2 4:4 5:1 6:2 8:1 10:1 12:2 16:1 20:1	–
$\{5,5,8\}$	21	1:8 3:1 7:1 11:2 13:4 17:1 21:1	–
$\{5,6,6\}$	20	1:13 2:7 4:5 5:3 7:2 10:3 13:2 17:2 20:1	–
$\{5,6,7\}$	21	1:12 2:4 3:7 5:1 6:2 7:2 9:1 10:1 11:2 14:1 17:1 18:1 21:1	–
$\{5,6,8\}$	22	1:14 2:10 3:5 4:3 5:3 6:4 7:1 8:1 9:1 10:1 11:1 12:2 13:1 14:1 15:1 18:1 19:1 22:1	–
$\{5,7,7\}$	22	1:5 2:5 3:2 6:2 9:2 11:1 18:2 22:1	–
$\{5,7,8\}$	23	1:11 3:2 5:3 7:1 9:1 13:1 15:1 19:2 23:1	–
$\{5,8,8\}$	24	1:13 2:13 3:1 4:7 5:4 6:1 7:4 8:3 10:4 12:1 14:4 16:2 20:2 24:1	–
$\{6,6,6\}$	21	1:4 2:3 3:4 6:3 7:1 9:3 18:3 21:1	–
$\{6,6,7\}$	22	1:10 2:6 3:3 4:2 6:3 9:1 11:3 12:2 18:1 19:2 22:1	–
$\{6,6,8\}$	23	1:14 2:6 4:4 5:4 7:2 8:2 10:2 11:2 14:2 19:1 20:2 23:1	–
$\{6,7,7\}$	23	1:10 2:5 3:2 4:5 5:1 6:2 8:2 10:1 12:2 13:2 19:2 20:1 23:1	–
$\{6,7,8\}$	24	1:11 2:7 3:5 4:5 5:4 6:2 7:2 8:1 10:3 12:2 13:1 14:1 15:1 20:2 21:1 24:1	–
$\{6,8,8\}$	25	1:20 2:7 3:12 4:2 5:3 6:4 7:2 8:2 9:2 11:3 13:2 15:2 16:2 21:2 22:1 25:1	–
$\{7,7,7\}^*$	24	1:4 2:4 3:1 4:4 5:3 6:1 8:1 10:3 12:1 20:3 24:1	–
$\{7,7,8\}$	25	1:8 3:5 5:1 7:3 9:2 11:2 21:3 25:1	–
$\{7,8,8\}$	26	1:16 2:16 3:6 4:6 6:6 8:2 11:3 12:4 13:1 16:2 22:3 26:1	–
$\{8,8,8\}$	27	1:34 3:1 7:6 9:1 13:6 17:6 23:3 27:1	–

7. Statements already in the literature, recovered

The classification of §6 is produced by one procedure applied uniformly to every coefficient triple, so the previously known answers are not corollaries of it but the gate it passes through: they are produced by the same procedure that produces the new rows, and they must come out right.

Theorem 7.1 (The two settled coefficient triples)**.** *For $k \geq 1$:*

$$\mathcal{S}(0,0,0;k) \neq \emptyset \iff k \in \{1,3\}, \qquad \mathcal{S}(2,0,0;k) \neq \emptyset \iff k \in \{1,2,3,4,5\}.$$

The first is the Markov–Hurwitz classification of $x^2+y^2+z^2 = kxyz$ [4, 5]*; the second is* [1, Theorem 4.8]*: "There exist positive integer solutions to the Diophantine equations $x^2+y^2+z^2+2yz = kxyz$ if and only if $k = 1,2,3,4,5$."*

Theorem 7.2 (One orbit at the top value, for every coefficient triple)**.** *For all $k_1, k_2, k_3 \in \mathbb{N}$, the only fundamental solution of $\mathcal{E}(k_1,k_2,k_3;3+K)$ is $(1,1,1)$. Hence $\mathcal{N}(k_1,k_2,k_3;3+K) = 1$: every positive integer solution of* (1) *at $k = 3+K$ lies in the orbit $\hat{\Gamma}[(1,1,1)]$. Moreover the orbit data of* [1] *are reproduced representative by representative:*

$$\mathcal{F}(2,0,0;1) = \{(9,6,3),(9,3,6),(8,4,4),(5,5,5)\}, \quad \mathcal{F}(2,0,0;2) = \{(4,2,2)\},$$
$$\mathcal{F}(2,0,0;3) = \{(3,2,1),(3,1,2)\}, \quad \mathcal{F}(2,0,0;4) = \{(2,1,1)\},$$

and $\mathcal{F}(2,0,0;5) = \{(1,1,1)\}$, $\mathcal{F}(0,0,0;1) = \{(3,3,3)\}$, $\mathcal{F}(0,0,0;3) = \{(1,1,1)\}$, $\mathcal{F}(0,2,2;7) = \{(1,1,1)\}$.

Proof sketch. The first assertion is proved with no search, uniformly in (k_1,k_2,k_3), by three cases on a sorted fundamental solution $u \geq M \geq m \geq 1$ at $k = 3+K$ using Theorem 3.3, with (A,B,C) the coefficients transported into the sorted frame. If $m \geq 2$ then $2k \leq k\,m \leq 4+2A+B+C$ forces $2+B+C \leq 0$, so $m = 1$. If $M \geq 2$ then $(\star)$ reads $M + A\,M + B\,M \leq 2+2A+B$, which with $M \geq 2$ forces $B = 0$ and then $M = 2$; but then the equation reads $u^2+5+2A = (6+2A)u$ while fundamentality reads $u^2 \leq 5+2A$, and with $u \geq 2$ the two sides differ. So $M = 1$, and at $m = M = 1$ the equation reads $u^2+2+A = (3+A)u$ against $u^2 \leq 2+A$, which forces $u = 1$. A duplicate-free list all of whose members equal $(1,1,1)$ and which contains it has length 1, so $\mathcal{N}(k_1,k_2,k_3;3+K) = 1$ by Theorem 4.6. The displayed sets of fundamental solutions are the enumeration of §6 at those parameters, and agree with the statements of [1] including the listing order. ☐

The first assertion of Theorem 7.2 is the statement [1, Remark 7.2(1)] attributes to [2, Theorem 1.1]; the proof above establishes it for every coefficient triple at once, without a search. The displayed data reproduce [1, Theorem 4.6] — "For $k = 1$, there are exactly four orbits of solutions: $\hat{\Gamma}[(9,6,3)]$, $\hat{\Gamma}[(9,3,6)]$, $\hat{\Gamma}[(8,4,4)]$ and $\hat{\Gamma}[(5,5,5)]$; For $k = 2$, there is only one orbit of solutions: $\hat{\Gamma}[(4,2,2)]$; For $k = 3$, there are exactly two orbits of solutions: $\hat{\Gamma}[(3,2,1)]$ and $\hat{\Gamma}[(3,1,2)]$; For $k = 4$, there is only one orbit of solutions: $\hat{\Gamma}[(2,1,1)]$; For $k = 5$, there is only one orbit of solutions: $\hat{\Gamma}[(1,1,1)]$" — together with the other two rows of its Table 2, the classical Markov equation and Lampe's equation $x^2+y^2+z^2+2xy+2xz = 7xyz$ [3].

Remark 7.3 (How to read Table 2 of [1]). That table has one row for each of the three fundamental Markov-type equations, and its third column is headed "Number of $\hat{\Gamma}$-orbits of solutions $\mathcal{N}$"; so it tabulates orbit counts, not admissible sets, and its three rows should not be read the same way. Row 1 is the classical Markov equation, with entry "$k = 1,3:\ \mathcal{N} = 1$", and row 3 is $x^2+y^2+z^2+2yz = kxyz$, with entries "$k = 1:\ \mathcal{N} = 4$", "$k = 3:\ \mathcal{N} = 2$" and "$k = 2,4,5:\ \mathcal{N} = 1$"; in both the listed k happen to be exactly the admissible ones, by Theorem 7.1. Row 2 is Lampe's equation $x^2+y^2+z^2+2xy+2xz = kxyz$, and its entry is "$k = 7:\ \mathcal{N} = 1$" alone — whereas the admissible set of that coefficient triple is $\mathcal{K}(0,2,2) = \{1,2,3,4,5,6,7\}$ by Theorem 6.1; for instance $(3,12,15)$ solves that equation at $k = 1$. So row 2 lists the one value of k whose orbits the cited literature classifies, not the values of k for which the equation is solvable, and reading it as a solvability classification would be a mistake. Nothing in [1] claims otherwise.

8. Negative controls

Exhaustive computation over a proved box is only as good as the box and the hypotheses, so we record the checks that the statements above are not vacuous and that the naive alternatives fail.

Proposition 8.1. (1) $(0,0,0)$ *solves* $\mathcal{E}(a,b,c;k)$ *for every* a,b,c,k*, so restricting to positive solutions is essential.*

(2) $\mathcal{S}(a,b,c;0)=\emptyset$ *for all* a,b,c*, so* $k\geq 1$ *is a real hypothesis.*

(3) $\mathcal{K}(0,1,1)=\{1,5\}$*: solvable at* $k=1$ *(witness* $(5,5,5)$*) and at* $k=5$ *(witness* $(1,1,1)$*) but not at* $k=2$*. So an admissible set need not be an interval.*

(4) $(1,4,5)\in\mathcal{S}(2,2,2;5)$ *and* $5\nmid 9=3+K$*. So an admissible set need not be* $\mathrm{D}(3+K)$*, although it is always divisor-closed by Proposition 3.8.*

(5) $(36,6,30)$ *is a* fundamental *solution of* $\mathcal{E}(2,3,3;1)$*. So the descent does not push solutions into a small box, and the bound of Corollary 3.4 cannot be replaced by a constant.*

(6) $\mathcal{E}(3,3,3;1)$ *has no solution with all coordinates* ≤ 11*, yet* $(12,12,12)$ *is a solution. A search over* $[1,11]^3$ *would have reported "no solution" for a solvable instance — exactly the failure mode Theorem 3.3 rules out.*

Proposition 8.2. (1) $\mathcal{N}(a,b,c;0)=0$ *for every* a,b,c*: at* $k=0$ *the box of Corollary 3.4 is empty, so the hypothesis* $k\geq 1$ *in Theorem 4.6 is not a formality. More generally* $\mathcal{N}(a,b,c;k)>0$ *if and only if* $k\in\mathcal{K}(a,b,c)$*, for* $k\geq 1$*.*

(2) $\mathcal{N}(2,0,0;1)=4$ *and* $\mathcal{N}(2,2,8;1)=42$*: the orbit count is neither always* 1 *nor bounded by any of the small constants the examples in* [1] *suggest.*

(3) $\mathcal{F}(2,3,3;1)=\{(36,30,6),(36,6,30),(20,10,10),(11,11,11)\}$*: orbit representatives need not be small.*

(4) $(5,5,5)\not\approx(8,4,4)$ *in* $\mathcal{S}(2,0,0;1)$*. This is what separates "four fundamental solutions" from "four orbits", and it is deduced from Theorem 4.5, not from an inspection of orbits.*

(5) $(20,5,5)=\hat{\mu}_1(5,5,5)$ *lies in the orbit of* $(5,5,5)$ *and is not fundamental, so* $\approx$ *is not equality and Definition 2.2 is a real restriction.*

(6) $\mathcal{K}(0,0,5)=\mathcal{K}(1,1,3)=\{1,2,4,8\}$ *while the orbit counts are* $3,3,3,1$ *and* $1,1,1,1$ *respectively. So* $\mathcal{N}$ *is not a function of* $\mathcal{K}$*, and the orbit half of Question 7.1 is not a corollary of the existence half. Likewise* $\mathcal{N}$ *is not monotone in* k*: at* $\{2,2,8\}$ *the profile is* $42,18,15,6,5,6,2,2,2,3,2,2,1$ *over* $k=1,2,3,4,5,6,7,8,9,11,12,14,15$*, rising at* $k=6$ *and again at* $k=11$*.*

9. Formal verification

Every numbered statement in this paper has been formalized and machine-checked in Lean 4 against Mathlib, with one exception: the identification of $\hat{\Gamma}$-orbits with $\approx$-classes is argued only in prose, in Remark 4.3, and the formal counterparts of Lemma 4.2, Theorem 4.5 and Theorem 4.6 are stated for $\approx$. The reasoning layer — Propositions 3.1, 3.8, 5.1, Lemmas 4.2, 4.4, Theorems 3.2, 3.3, 3.5, 3.6, 4.5, 4.6, 5.2, 7.2 and Corollaries 3.4, 5.3 — is checked by the Lean kernel with no computational reflection and no additional axioms beyond the three that Mathlib itself uses (propositional extensionality, quotient soundness and choice). The classification statements that evaluate a concrete list of numbers — Theorems 6.1, 6.2, 6.3, 7.1, the displayed enumerations in Theorem 7.2, and the numerical items of Propositions 8.1 and 8.2 — are discharged by compiled evaluation of the decision procedure of Theorem 3.5 and the enumeration of Theorem 4.6, one such evaluation per coefficient triple, which introduces one additional axiom per evaluated statement asserting that the compiler's evaluation agrees with the kernel's. The whole development contains no incomplete proof. As an independent check against a mistake in the enumeration rather than in the proof, the 165 admissible sets and the 1430 orbit counts were recomputed by a reimplementation in C sharing no code with the Lean development, with no disagreement, and the orbit claim itself was falsification-tested on ten instances by a bound-free enumeration of all positive solutions with coordinates at most 60 together with a breadth-first search of the mutation graph from each fundamental solution, verifying that the balls are pairwise disjoint and that no solution in that range escapes them. Repository reference to be added at submission.

10. Questions

(1) *Beyond* $k_i\leq 8$*.* Nothing in §3–§4 depends on a bound for the coefficients; only the size of the enumeration does, and it grows by a factor of roughly 2.2 per unit increase of the bound. The stopping point above was chosen for the size of Table 1.

(2) *A closed form for $\mathcal{N}$.* The 1430 orbit counts of Table 1 exhibit no rule beyond the value 1 at the top. Proposition 3.8 relates solutions at k to solutions at multiples of k by scaling. Whether the counts decompose accordingly over *primitive* orbits — equivalently, whether the greatest common divisor of a solution is a $\hat{\Gamma}$-orbit invariant — is not settled here, and would be the natural next step. For its own coefficient triple $(2, 0, 0)$, [1, Proposition 4.7] establishes exactly such a divisibility correspondence between the orbits at $k = 1$ and the solutions at $k = 2, 3, 4, 5$.

(3) *The uniqueness conjectures.* Conjectures 7.3 and 7.4 of [1] are ordered uniqueness statements in the style of the Markov uniqueness conjecture: Conjecture 7.3, for the symmetric case $k_1 = k_2 = k_3$ at $k = 3 + K$, and Conjecture 7.4, for $x^2 + y^2 + z^2 + 2yz = kxyz$ with $k = 2, 4, 5$. Each quantifies over all positive solutions of its instance, of which there are infinitely many, so neither is a finite check and nothing above makes it one. What the machinery does supply is a finite generating description: Theorem 4.6 exhibits, for any (k_1, k_2, k_3, k), an explicit finite list of fundamental solutions from which every positive solution is reached by mutations, so such a conjecture becomes a statement about finitely many mutation trees with explicitly known roots rather than about an unstructured solution set.

References

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