

FORTUNA'S CONJECTURE ON THE COMPANION LYAPUNOV EQUATION AT THE FIRST OPEN CELL: AN EXACT VERIFICATION AT A DEGREE-FIVE COMPANION MATRIX WITH A COMPLEX-CONJUGATE PAIR

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ABSTRACT. Let $a(s) = s^n + a_{n-1}s^{n-1} + \dots + a_0$ be a Hurwitz polynomial, A its companion matrix and X the solution of the Lyapunov equation $XA + A^\top X = -Q$ with $Q = Q^\top \succeq 0$. Fortuna's conjecture, as stated by Ferrante (arXiv:2606.16492), asserts that X is entrywise nonnegative. Ferrante proves it when A has only real eigenvalues, records that the cases $n \leq 4$ can be settled by brute force and that the diagonal and the two corner entries are always nonnegative, and names the removal of the real-root assumption as the main open question. We give an exact verification in a cell outside those bounds: at the companion matrix of $(s+1)^3(s^2+s+1) = s^5 + 4s^4 + 7s^3 + 7s^2 + 4s + 1$, whose spectrum contains $\exp(\pm 2\pi i/3)$ and which is not diagonalizable, the solution for every rank-one $Q = qq^\top$ is entrywise nonnegative, each of its fifteen distinct entries being written as an explicit sum of squares in q with rational coefficients; the solution is also shown to be unique among symmetric matrices by an explicit inversion of the Lyapunov operator over $\mathbb{Q}$. Two identities valid for every degree and every spectrum are recorded: $2a_0X_{1,n} = Q_{11}$, which turns Ferrante's corner remark into a value, and $X_{k,k-1} = a_{k-1}X_{k,n} - Q_{kk}/2$. Controls show that the companion form and the Hurwitz hypothesis are both needed and that the conclusion is strictly stronger than $X \succeq 0$. Recomputing Ferrante's counterexample to total nonnegativity exactly over $\mathbb{Q}$, we find that the minor he names, the leading 2×2 principal minor, is positive ($+4355916893/28812000$), as it must be when $X \succeq 0$, and that the negative minor is the non-principal one on rows $\{1, 2\}$ and columns $\{2, 3\}$, equal to $-51562093/72030$. Outside the formal development we report a floating-point-free search (57,678 rational instances at $n = 5$, no counterexample), a symbolic verification along the one-parameter family $(s+1)^3(s^2+ps+1)$, $p > 0$, and the identification, for $n \leq 5$, of the inverse of the matrix of the quadratic form X_{nn} with the Hermite matrix of a , whose (n, n) entry is $2a_{n-1}$; this yields $X_{n,n-1} \geq 0$ for every Hurwitz a of degree at most 5. Every pinned statement is verified in Lean 4 against *Mathlib* without compiled evaluation.

1. INTRODUCTION

Fix $n \geq 1$ and a real monic polynomial

$$(1) \quad a(s) = s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \dots + a_1s + a_0.$$

Its *companion matrix* is

$$(2) \quad A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{pmatrix} \in \mathbb{R}^{n \times n},$$

with ones on the superdiagonal and the negated coefficients in the last row. The polynomial a is *Hurwitz* when every root lies in the open left half-plane; then every a_i is positive, and for $Q = Q^\top \succeq 0$ the continuous-time Lyapunov equation

$$(3) \quad XA + A^\top X = -Q$$

has a unique solution X , which is symmetric and positive semidefinite. Ferrante's recent paper [1] is devoted to a different, entrywise, positivity of this X . Its Conjecture 1.1 (page 2 of the arXiv PDF of version 1), attributed in its introduction to L. Fortuna by personal communication [2], reads verbatim:

Let X be the only solution of the Lyapunov equation (1.3), where A has the form (1.2), the polynomial $a(s)$ given by (1.1) is Hurwitz and $Q = Q^\top \succeq 0$. Then X is entrywise nonnegative.

Here (1.1), (1.2), (1.3) of [1] are (1), (2), (3). The motivation given there is the theory of positive systems: for a single-input regulator the optimal gain is expressed through the solution of an algebraic Riccati equation which can be rewritten as a Lyapunov equation, and an entrywise nonnegative solution allows an implementation with single-supply amplifiers.

What the source settles. Ferrante proves the conjecture when A has only real eigenvalues (his Theorem 1, page 4 of the PDF): after reducing to $Q = qq^\top$ and to diagonalizable A (his Section 2), each entry X_{kl} becomes a quadratic form in the coefficients of q in the eigenbasis of $A^\top$, whose matrix is a diagonal congruence of the Cauchy-like matrix $C_{ij} = (r_i(k, l) + r_j(k, l))/(\lambda_i + \lambda_j)$, where $-\lambda_i$ are the (then real, negative and distinct) eigenvalues; the positive semidefiniteness of every such matrix is the subject of the companion paper [3], whose abstract opens with the hypothesis $0 < \lambda_1 < \dots < \lambda_n$. The unnumbered Remark following Theorem 1 (page 4) reads verbatim:

Since X is clearly positive semidefinite, it is obvious that the diagonal elements of X are nonnegative. Moreover, it is not difficult to show by direct computation that the elements $X_{1,n}$ and $X_{n,1}$ namely the entries of X in the top-right and in the bottom-left corners are also nonnegative. Finally, the cases $n = 2, 3, 4$ can be worked out by brute force and, in these cases it is possible to see that the conjecture holds.

Section 5 of [1] shows by a 3×3 example that X need not be totally nonnegative, and proves total nonnegativity (his Theorem 2) when q has a nonnegative expansion in the eigenvectors of $A^\top$; the source calls that property “totally positive”, defining it as nonnegativity of all minors. The Conclusion (Section 6, page 12) states verbatim: “Several questions remain open. The most natural one is whether the real-root assumption can be removed, thereby proving the original conjecture for all Hurwitz companion matrices.”

Indexing the conjecture by the pair (degree, spectrum type), the smallest cell that none of these results covers is therefore $n = 5$ with at least one complex-conjugate pair of eigenvalues.

What this paper does. We treat the conjecture as a finite question at a fixed A . Since X depends linearly on Q and every $Q \succeq 0$ is a sum of rank-one terms $qq^\top$, for each entry (k, l) there is a symmetric matrix $N_{kl}(A)$ with $X_{kl} = q^\top N_{kl}(A) q$ when $Q = qq^\top$ (Section 2); the conjecture at that A is equivalent to $N_{kl}(A) \succeq 0$ for all k, l , and a verification is an exact rational $LDL^\top$ factorisation, that is, a sum-of-squares expression for each entry. Our main result (Theorem 4.1) is such a verification at

$$a(s) = (s+1)^3(s^2+s+1) = s^5 + 4s^4 + 7s^3 + 7s^2 + 4s + 1,$$

whose roots are -1 (triple) and $\exp(\pm 2\pi i/3)$. Both of the source’s standing simplifications fail there at once: the spectrum is not real, so Theorem 1 of [1] does not apply, and A , being a companion matrix with a repeated eigenvalue, is not diagonalizable, a case the source reaches only through a continuity argument. For every $q \in \mathbb{R}^5$ the solution of $XA + A^\top X = -qq^\top$ is entrywise nonnegative; each of its fifteen distinct entries is an explicit sum of squares in q with rational coefficients, the largest integer occurring anywhere in the certificates being 29016. Theorem 4.2 shows, by twenty-five explicit linear combinations of the scalar entries of the matrix equation, that this A admits at most one symmetric solution for each Q , so that the statement is about *the* solution in the sense of the conjecture.

Around this we record what is elementary and what is control. Propositions 3.1 and 3.2 are two identities valid for every n and every spectrum, with no Hurwitz hypothesis and no positivity of Q : the corner entry is exactly $X_{1,n} = Q_{11}/(2a_0)$, which turns the “direct computation” of the Remark into a value, and the first subdiagonal is $X_{k,k-1} = a_{k-1}X_{k,n} - Q_{kk}/2$, so that the whole matrix is determined by its last column and Q . Proposition 5.1 shows that the companion form is essential (a Hurwitz diagonal A with a negative entry in X), that the Hurwitz hypothesis is essential (a companion A with eigenvalue $+1$ and an X that is not even positive semidefinite), that the conclusion is strictly stronger than $X \succeq 0$, and that the main theorem is not vacuous. Proposition 6.1 recomputes the 3×3 example of Section 5 of [1] exactly over $\mathbb{Q}$: it reproduces the printed decimals and is entrywise positive,

as Theorem 1 predicts since its roots are real; but the sentence “the determinant of the 2×2 upper-left corner of X is negative” cannot hold for any positive semidefinite X , and indeed that minor equals $+4355916893/28812000$. The example remains a correct counterexample to total nonnegativity: the negative minor is the non-principal one on rows $\{1, 2\}$ and columns $\{2, 3\}$, equal to $-51562093/72030$.

Section 7 reports, clearly labelled and outside the formal development, three computations: a floating-point-free search in a stated box (57,678 exact rational instances at $n = 5$ and 3,834 optimiser multistarts at $n = 5, 6, 7$, no counterexample), a symbolic verification of the conjecture along the whole one-parameter family $(s + 1)^3(s^2 + ps + 1)$, $p > 0$, and the exact identity $W(a)^{-1} = H(a)$ for $n \leq 5$, where W is the matrix of the quadratic form X_{nn} and H the Hermite matrix of a , which gives $X_{n,n-1} \geq 0$ for every Hurwitz a of degree at most 5 and every $Q \geq 0$.

What is new and what is not. Theorem 4.1 is the new result: the conjecture at a point of the cell the source leaves open. Theorem 4.2 is classical in content (uniqueness of the Lyapunov solution at a Hurwitz A follows from the absence of two eigenvalues summing to zero); what is recorded is the formal instance, proved from the matrix equation with no spectral theory, which makes the phrase “the only solution” literal. Propositions 3.1 and 3.2 are one-line consequences of the matrix equation that the source does not write out; the corner’s nonnegativity is asserted in its Remark, and the subdiagonal reduction is the shape behind the classical recursions for Lyapunov integrals of companion systems (see [7, 8, 9]). Neither is claimed as new. Proposition 5.1 consists of standard controls, and Proposition 6.1 is the source’s own example, new only in its exact arithmetic and in the identification of the correct minor. The three computations of Section 7 are not formalised and are not claimed as theorems of this paper.

Related work. We could not locate a printed statement of the conjecture by Fortuna himself, and quote it here only through [1]; the reference that source gives for it is a personal communication of 2010 [2]. What was searched, on 7 September 2026: arXiv full text, where the queries pairing “companion form” or “entrywise nonnegative” with “Lyapunov equation” each return [1] and nothing else; Crossref and zbMATH author searches for L. Fortuna; OpenAlex full text for “Fortuna conjecture”, “conjecture of Fortuna” and “conjectured by Fortuna” (no hits) together with his own record list; the tables of contents of his books; and the standard open-problem collections in systems and control. Independently, one complete pass over a local mirror of the arXiv full text (892,038 papers in 376 shards, 92 GB) found the phrase “Hurwitz companion” in two papers, one of them [1]; “Fortuna” matched with word boundaries together with “Lyapunov” in 50 papers, of which 49 are bibliography entries for unrelated work and none carries “conjecture” or “communication” near the name; and a Lyapunov *equation* together with a companion *matrix or form* in 184 papers, of which the 51 that use any positivity vocabulary at all were read individually and none says anything about the signs of the entries of the solution. A negative full-text search says nothing about journal-only work or books, and a personal communication cannot be checked this way at all; the citation stands as the source gives it. The classical literature on Lyapunov equations with companion (or Schwarz-form) state matrices is concerned with the Routh–Hurwitz criterion and with evaluating quadratic integrals: Parks [7] proves the Routh–Hurwitz criterion by exhibiting a Lyapunov function for the Schwarz form, Barnett and Storey [8] treat applications of the Lyapunov matrix equation in that setting, and Pták [9] relates Lyapunov equations to Gram matrices; Datta [10] surveys stability and inertia. The positive semidefiniteness of the Cauchy-like matrices behind Theorem 1 of [1] descends from Kwong’s matrices [4], whose modern form is $((p_i^r + p_j^r)/(p_i + p_j))_{ij}$ on distinct positive reals [5]; that literature, like [3], has real positive nodes as its hypothesis and says nothing about a complex-conjugate pair. Total positivity, used in Section 5 of [1], is treated in Pinkus’s monograph [6].

Verification. Every theorem and proposition of Sections 3–6 rests on a declaration verified in Lean 4 against *Mathlib*, with the statements reproduced verbatim in Appendix A; where a printed statement says more than its formal counterpart, the text says so at that point. No proof uses compiled evaluation or floating-point arithmetic. Section 9 gives the details.

2. PRELIMINARIES

Throughout, matrices are indexed from 1 in the text and from 0 in the formal statements of Appendix A; the coefficient vector of (1) is $a = (a_0, \dots, a_{n-1})$ in both. For A as in (2) and any $X \in \mathbb{R}^{n \times n}$, the products in (3) have two-term entries: writing $X_{k,0} := 0$,

$$(4) \quad (XA)_{kl} = X_{k,l-1} - a_{l-1}X_{k,n}, \quad (A^\top X)_{kl} = X_{l,k-1} - a_{k-1}X_{l,n} \quad \text{if } X^\top = X.$$

Everything in Section 3 is read off (4).

The rank-one reduction and the matrices N_{kl} . Suppose A is such that (3) has a unique solution for every Q (this holds whenever no two eigenvalues of A sum to zero, in particular for Hurwitz A). Then $Q \mapsto X$ is linear, and for $Q = qq^\top$ each entry is a quadratic form in q : there are symmetric $N_{kl} = N_{kl}(A) \in \mathbb{R}^{n \times n}$, $N_{kl} = N_{lk}$, with

$$(5) \quad X_{kl} = q^\top N_{kl} q \quad (Q = qq^\top).$$

Since every $Q \succeq 0$ is a finite sum $\sum_i q_i q_i^\top$ (the first simplification of Section 2 of [1], credited there to H. Wimmer), the conjecture at this fixed A is equivalent to: $N_{kl}(A) \succeq 0$ for all k, l . When a is rational the N_{kl} are rational, a positive semidefinite N_{kl} admits an exact rational $LDL^\top$ factorisation with $D \geq 0$, and (5) then becomes a sum of squares

$$(6) \quad X_{kl} = \sum_t d_t (\ell_t \cdot q)^2, \quad d_t \in \mathbb{Q}_{\geq 0}, \ell_t \in \mathbb{Q}^n,$$

whose nonnegativity for every q is evident. This is the form in which Theorem 4.1 is stated and checked: the Lyapunov equation for the matrix of forms (6) is a polynomial identity in q , and nonnegativity is positivity of a sum of squares with nonnegative coefficients.

Hurwitz polynomials. For a monic a of degree n the Routh–Hurwitz criterion [11] says that a is Hurwitz if and only if the n leading principal minors of its Hurwitz matrix are positive. We use it only in the computations of Section 7; in the formal development, the Hurwitz property of the witness is proved directly from the factorisation of a , and “eigenvalue” means: $z \in \mathbb{C}$ such that $Av = zv$ for some $v \in \mathbb{C}^n \setminus \{0\}$.

3. TWO IDENTITIES FOR EVERY DEGREE

The two propositions of this section hold for every $n \geq 1$, every coefficient vector $a \in \mathbb{R}^n$ (no Hurwitz hypothesis), every $Q \in \mathbb{R}^{n \times n}$ (no symmetry or positivity), and every symmetric solution X of (3).

Proposition 3.1 (the corner is a value). *Let A be the companion matrix (2) of $a \in \mathbb{R}^n$, and let $X = X^\top$ satisfy $XA + A^\top X = -Q$. Then*

$$2a_0 X_{1,n} = Q_{11}.$$

Consequently, if $a_0 \neq 0$ then $X_{1,n} = Q_{11}/(2a_0)$, and if $a_0 > 0$ and $Q_{11} \geq 0$ then $X_{1,n} \geq 0$.

Proof. By (4) with $k = l = 1$, $(XA)_{11} = -a_0 X_{1,n}$ and, by symmetry, $(A^\top X)_{11} = -a_0 X_{1,n}$; the (1,1) entry of (3) is therefore $-2a_0 X_{1,n} = -Q_{11}$. The two consequences are division by $2a_0$ and the sign of a quotient. The formal statements are `corner_two_mul`, `corner_eq_div` and `corner_nonneg` of Appendix A. $\square$

For a Hurwitz a and $Q \succeq 0$ this recovers the nonnegativity of the corners $X_{1,n}$ and $X_{n,1}$ asserted in the Remark of [1], with the value: in the rank-one case, $N_{1,n} = e_1 e_1^\top / (2a_0)$.

Proposition 3.2 (the subdiagonal reduction). *Under the hypotheses of Proposition 3.1, for every k with $2 \leq k \leq n$,*

$$X_{k,k-1} = a_{k-1} X_{k,n} - \frac{1}{2} Q_{kk}.$$

Proof. The (k,k) entry of (3) is, by (4), $2(X_{k,k-1} - a_{k-1} X_{k,n}) = -Q_{kk}$. The formal statement is `subdiag_eq`, where k is `j.succ` and $k-1$ is `j.castSucc` for `j : Fin m`, $n = m+1$. $\square$

Remark 3.3 (the general recursion; not formalised). The same two-term expansion at a general entry gives, for $1 \leq k, l \leq n$ and with $X_{\cdot,0} := 0$,

$$X_{k,l-1} + X_{l,k-1} = a_{l-1}X_{k,n} + a_{k-1}X_{l,n} - Q_{kl},$$

so a symmetric solution is determined by its last column $(X_{k,n})_k$ and by Q : once column l is known, the equations (k, l) , $1 \leq k \leq n$, give column $l-1$, so the matrix is filled in column by column from the right. Thus the conjecture at a fixed A is a statement about the n quadratic forms $N_{k,n}$, the remaining N_{kl} being linear combinations of them and of Q . Only the diagonal case $k = l$ is part of the formal development.

4. THE CONJECTURE AT A DEGREE-FIVE COMPANION MATRIX WITH A COMPLEX PAIR

Let

$$(7) \quad a(s) = (s+1)^3(s^2+s+1) = s^5 + 4s^4 + 7s^3 + 7s^2 + 4s + 1, \quad a = (1, 4, 7, 7, 4),$$

and let $A \in \mathbb{R}^{5 \times 5}$ be its companion matrix (2), so that the last row of A is $(-1, -4, -7, -7, -4)$. The roots of a are -1 with multiplicity three and $\omega, \bar{\omega} = \exp(\pm 2\pi i/3) = (-1 \pm i\sqrt{3})/2$. Since a companion matrix is non-derogatory, the triple root makes A non-diagonalizable; the source itself records this equivalence at the end of its Section 2 ("since any companion matrix is non-derogatory, then A diagonalizable is equivalent to the fact that all its eigenvalues have algebraic multiplicity equal to 1"), and it is used here only in the discussion, not in the formal development.

For $q \in \mathbb{R}^5$ define fifteen quadratic forms $X_{kl}(q)$, $1 \leq k \leq l \leq 5$, as the sums of squares listed in Appendix A (definitions $\mathbf{x00}, \dots, \mathbf{x44}$, with q_i written $\mathbf{q}(\mathbf{i-1})$), and let $X(q)$ be the symmetric matrix with these entries. Three of them are

$$(8) \quad \begin{aligned} X_{11}(q) &= \frac{1}{1008}(-29q_1 + 7q_3 - 5q_5)^2 + \frac{1}{720}(-7q_2 + 5q_4)^2 + \frac{1}{120}(-15q_1 + 4q_2)^2 \\ &\quad + \frac{1}{42}(-4q_1 + q_5)^2 + \frac{1}{8}q_1^2, \end{aligned}$$

$$(9) \quad \begin{aligned} X_{12}(q) &= \frac{1}{252}(-29q_1 + 7q_3 - 5q_5)^2 + \frac{2}{21}(4q_1 - q_5)^2 + \frac{1}{180}(-7q_2 + 5q_4)^2 \\ &\quad + \frac{1}{30}q_2^2, \end{aligned}$$

$$(10) \quad X_{15}(q) = \frac{1}{2}q_1^2.$$

The last is Proposition 3.1 at $a_0 = 1$. Expanding (8) gives $X_{11}(q) = q^T N_{11} q$ with

$$N_{11} = \frac{1}{144} \begin{pmatrix} 463 & -72 & -29 & 0 & 7 \\ -72 & 29 & 0 & -7 & 0 \\ -29 & 0 & 7 & 0 & -5 \\ 0 & -7 & 0 & 5 & 0 \\ 7 & 0 & -5 & 0 & 7 \end{pmatrix},$$

a matrix with negative off-diagonal entries whose positive semidefiniteness is exactly what (8) certifies.

Theorem 4.1 (the conjecture holds at A for every rank-one Q). *Let A be the companion matrix of (7).*

- (i) (*A is Hurwitz.*) If $z \in \mathbb{C}$ and $v \in \mathbb{C}^5 \setminus \{0\}$ satisfy $Av = zv$, then $\operatorname{Re} z < 0$.
- (ii) (*The spectrum is not real.*) There exist $z \in \mathbb{C}$ with $\operatorname{Im} z \neq 0$ and $v \in \mathbb{C}^5 \setminus \{0\}$ with $Av = zv$; one may take $z = \omega = (-1 + i\sqrt{3})/2$ and $v = (1, \omega, \omega^2, \omega^3, \omega^4)$.
- (iii) (*The forms solve the equation.*) For every $q \in \mathbb{R}^5$, $X(q)A + A^T X(q) = -qq^T$.
- (iv) (*Entrywise nonnegativity.*) For every $q \in \mathbb{R}^5$ and all $1 \leq k, l \leq 5$, $X(q)_{kl} \geq 0$.

Proof. (i) For a companion matrix the eigenvector equation forces $v = (v_1, zv_1, z^2v_1, z^3v_1, z^4v_1)$ from the first four rows, and the last row then reads $v_1 a(z) = 0$; since $v \neq 0$ forces $v_1 \neq 0$, $a(z) = 0$. Now $a(z) = (z+1)^3(z^2+z+1)$, so either $z = -1$ or $z^2+z+1 = 0$; in the latter case the imaginary part of the equation gives $\operatorname{Im} z(2\operatorname{Re} z + 1) = 0$, and $\operatorname{Im} z = 0$ is impossible because $x^2 + x + 1 > 0$ for real x , hence $\operatorname{Re} z = -1/2$. No characteristic polynomial is used. (ii) $\omega^2 + \omega + 1 = 0$ is checked from $(\sqrt{3})^2 = 3$ and $i^2 = -1$, and the Vandermonde vector of any root of a is an eigenvector, the last row producing

$-a(\omega) = 0$. (iii) is a polynomial identity in $q_1, \dots, q_5$, twenty-five entries of degree two, verified entry by entry by normalisation in the polynomial ring. (iv) Each $X_{kl}(q)$ is a sum of squares with positive rational coefficients.

The certificates were found as follows. With Q ranging over the fifteen matrices $\frac{1}{2}(E_{ij} + E_{ji})$, exact rational solutions of (3) give the matrices N_{kl} of (5); here $144N_{kl}$ is integral with entries of absolute value at most 1440. Each N_{kl} was factored as $P^T LDL^T P$ over $\mathbb{Q}$ with $D \geq 0$, the permutation P being chosen among all 120 orderings of the pivots to minimise the largest integer appearing in the resulting expression (6); the largest integer in the whole certificate is 29016 (in X_{22}), and the fifteen forms comprise 65 squares in all. The formal statements are `A_hurwitz`, `A_has_complex_pair`, `Xw_lyap` and `Xw_nonneg` of Appendix A; in the first two the matrix is the complexification Ac of A , in the last two $Q = qq^T$ is `Matrix.vecMulVec q q`. $\square$

Theorem 4.2 (uniqueness at A). *Let A be the companion matrix of (7) and $Q \in \mathbb{R}^{5 \times 5}$ arbitrary. If X and Y are symmetric and $XA + A^T X = -Q = YA + A^T Y$, then $X = Y$.*

Proof. $Z = X - Y$ is symmetric and satisfies $ZA + A^T Z = 0$, a homogeneous linear system of twenty-five scalar equations h_{ij} in the entries of Z . The Lyapunov operator $Z \mapsto ZA + A^T Z$ on $\mathbb{R}^{5 \times 5}$ is invertible over $\mathbb{Q}$ (its determinant is -1492992 ; this number is a computation outside the formal development), and each entry Z_{uv} is the linear combination $\sum_{ij} c_{ij}^{uv} h_{ij}$ whose coefficients form the corresponding row of the inverse. The twenty-five coefficient vectors were computed exactly and are supplied to the proof assistant literally; every numerator is at most 463 in absolute value and every denominator divides 144. No spectral theory enters: the classical argument (no two eigenvalues of a Hurwitz matrix sum to zero) is replaced by the explicit inverse. The formal statement is `lyap_unique`. $\square$

Corollary 4.3 (the conjecture at A ; the extension to $Q \succeq 0$ is not formalised). *Let A be the companion matrix of (7) and $Q \succeq 0$. Then the unique symmetric solution of $XA + A^T X = -Q$ is entrywise nonnegative.*

Proof. Write $Q = \sum_{i=1}^m q_i q_i^T$. By Theorem 4.1(iii) and linearity, $\sum_i X(q_i)$ is a symmetric solution, by Theorem 4.2 it is the only one, and by Theorem 4.1(iv) it is entrywise nonnegative. The decomposition of Q and the summation are the two steps outside the formal development; the formal statements cover every rank-one Q . $\square$

For orientation, evaluating the certificates at $q = e_1$ gives

$$X(e_1) = \begin{pmatrix} \frac{463}{144} & \frac{175}{36} & \frac{155}{36} & \frac{317}{144} & \frac{1}{2} \\ \frac{175}{36} & \frac{463}{48} & \frac{343}{36} & \frac{31}{6} & \frac{175}{144} \\ \frac{155}{36} & \frac{343}{36} & 10 & \frac{203}{36} & \frac{49}{36} \\ \frac{317}{144} & \frac{31}{6} & \frac{203}{36} & \frac{157}{48} & \frac{29}{36} \\ \frac{1}{2} & \frac{175}{144} & \frac{49}{36} & \frac{29}{36} & \frac{29}{144} \end{pmatrix},$$

an entrywise positive matrix whose $(1, 5)$ entry is $Q_{11}/(2a_0) = 1/2$ (Proposition 3.1); the positivity of this one entry is the non-vacuity control of Proposition 5.1(iv). The Hurwitz determinants of (7) are 4, 21, 87, 216, 216, all positive, in agreement with Theorem 4.1(i); this Routh–Hurwitz check is a computation outside the formal development.

5. CONTROLS

Proposition 5.1 (controls and non-vacuity).

- (i) (*Companion form is essential.*) Let $A_d = \text{diag}(-1, -2)$, $q = (1, -1)$ and $X_d = \begin{pmatrix} 1/2 & -1/3 \\ -1/3 & 1/4 \end{pmatrix}$. Then $X_d A_d + A_d^T X_d = -qq^T$ and $(X_d)_{12} = -1/3 < 0$.

- (ii) (*The Hurwitz hypothesis is essential.*) Let $A_b = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}$, the companion matrix of $s^2 + s - 2 = (s + 2)(s - 1)$, $q = (1, 0)$ and $X_b = \begin{pmatrix} 1/4 & -1/4 \\ -1/4 & -1/4 \end{pmatrix}$. Then $X_b A_b + A_b^\top X_b = -qq^\top$, $(X_b)_{12} < 0$ and $(X_b)_{22} < 0$; in particular X_b is not positive semidefinite.
- (iii) (*Entrywise nonnegativity is strictly stronger than $X \succeq 0$.*) For $q = (1, -1)$ the matrix $qq^\top$ is positive semidefinite and its $(1, 2)$ entry is $-1 < 0$.
- (iv) (*Non-vacuity of Theorem 4.1.*) With A and $X(q)$ as in Section 4, $X(e_1)_{15} = 1/2 > 0$.

Proof. All four are finite rational identities and inequalities, checked by normalisation. The formal statements are `companion_needed`, `hurwitz_needed`, `psd_not_entrywise` and `witness_nonvacuous`. In the same module, but not part of the pinned statements, `Adiag_hurwitz` proves that every eigenvalue of A_d is -1 or -2 , and `Abad_not_hurwitz` exhibits $A_b(1, 1)^\top = (1, 1)^\top$, so that A_b has the eigenvalue $+1$; the reader can verify both by inspection. In (ii) the Lyapunov operator at A_b is still invertible (the eigenvalues $-2, 1$ have no pair summing to zero), so X_b is the unique solution; this remark is outside the formal development. $\square$

Control (i) is the sentence of [1] that entrywise nonnegativity is a property “depending in an essential way on the companion form of the state matrix”, made into a witness. Control (iii) is the observation, also made in [1], that the conclusion of the conjecture does not follow from the classical $X \succeq 0$.

6. THE SOURCE'S EXAMPLE, RECOMPUTED EXACTLY

Section 5 of [1] (page 9 of the PDF) takes

$$A_p = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.03 & -0.4 & -1.3 \end{pmatrix}, \quad q = (4, 36, 29),$$

that is, the companion matrix of $a(s) = s^3 + 1.3s^2 + 0.4s + 0.03 = (s + 0.3)(s^2 + s + 0.1)$, whose roots are -0.3 and $(-1 \pm \sqrt{0.6})/2 \approx -0.887, -0.113$ (the source says “circa $-0.89, -0.11$ and -0.3 ”), prints the solution to one decimal place, and writes: “it is easy to check that the determinant of the 2×2 upper-left corner of X is negative.”

Proposition 6.1 (the example over $\mathbb{Q}$, and the correct minor). *Let A_p be the companion matrix of $a = (3/100, 2/5, 13/10)$ and*

$$X_p = \begin{pmatrix} \frac{981187}{29400} & \frac{216649}{735} & \frac{800}{3} \\ \frac{216649}{735} & \frac{7667197}{2940} & \frac{692929}{294} \\ \frac{800}{3} & \frac{692929}{294} & \frac{314060}{147} \end{pmatrix} \approx \begin{pmatrix} 33.3737 & 294.7605 & 266.6667 \\ 294.7605 & 2607.8901 & 2356.9014 \\ 266.6667 & 2356.9014 & 2136.4626 \end{pmatrix}.$$

- (i) $X_p A_p + A_p^\top X_p = -qq^\top$ for $q = (4, 36, 29)$.
- (ii) Every entry of X_p is strictly positive.
- (iii) The leading principal 2×2 minor is $(X_p)_{11}(X_p)_{22} - (X_p)_{12}(X_p)_{21} = +\frac{4355916893}{28812000} > 0$, while the non-principal minor on rows $\{1, 2\}$ and columns $\{2, 3\}$ is $(X_p)_{12}(X_p)_{23} - (X_p)_{13}(X_p)_{22} = -\frac{51562093}{72030} < 0$.

Proof. Finite rational arithmetic, checked by normalisation; the formal statements are `paper_example_lyap`, `paper_example_entrywise_pos` and `paper_example_minor_correction`. The matrix X_p was obtained by solving (3) exactly over $\mathbb{Q}$; it rounds to the decimals printed in [1], and (ii) is what Theorem 1 of [1] predicts, the roots being real. $\square$

Statement (iii) corrects the source's sentence. A leading principal minor of a positive semidefinite matrix is never negative, so the minor named in [1] cannot be the witness; the example is nevertheless a correct counterexample to total nonnegativity, the failing minor being the non-principal one. For

completeness, the nine 2×2 minors of X_p (a computation outside the formal development) are, by row set and column set:

	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$
$\{1, 2\}$	$\frac{4355916893}{28812000}$	$\frac{160554241}{2881200}$	$-\frac{51562093}{72030}$
$\{1, 3\}$	$\frac{160554241}{2881200}$	$\frac{13726487}{72030}$	$\frac{8916196}{7203}$
$\{2, 3\}$	$-\frac{51562093}{72030}$	$\frac{8916196}{7203}$	$\frac{480459641}{28812}$

(rows label the row set, columns the column set), and $\det X_p = 1479452547/1920800$. Exactly the two non-principal minors on the row set $\{1, 2\}$ with column set $\{2, 3\}$, and its transpose, are negative.

7. COMPUTATIONS OUTSIDE THE FORMAL DEVELOPMENT

Nothing in this section is formalised, and nothing in it is claimed as a theorem of this paper; each item is labelled by what was computed.

7.1. A search for a counterexample. Every instance below is a polynomial $a(s) = \prod_i (s + r_i) \prod_j (s^2 + p_j s + q_j)$ with $r_i, p_j, q_j > 0$ and $p_j^2 < 4q_j$, so that the Hurwitz property is structural and every sample is a legitimate instance of the conjecture; a counterexample would have been fully finite (rational a , q and X).

Exact arm, $n = 5$. The parameters r_i, p_j, q_j were drawn uniformly from the rationals u/v with $1 \leq u \leq 60$, $1 \leq v \leq 12$ (quadratic factors resampled until $p_j^2 < 4q_j$), the number of complex pairs uniformly from $\{1, 2\}$. For each of 57,678 instances (28,810 with one complex pair, 28,868 with two) all fifteen matrices N_{kl} were tested for positive semidefiniteness by an exact rational LDL^T with symmetric pivoting. No counterexample was found; no floating-point arithmetic was used in this arm.

Guided arm, $n = 5, 6, 7$. 3,834 Nelder–Mead multistarts (1,963 at $n = 5$, 1,132 at $n = 6$, 739 at $n = 7$) in logarithmic parameters minimised the diagonally normalised margin $\min_{k,l} \lambda_{\min}(DN_{kl}D)$, $D = \text{diag}((N_{kl})_{ii}^{-1/2})$; every start whose float margin fell below -10^{-8} was re-evaluated in exact rational arithmetic. The best float margins were $-6.7 \cdot 10^{-10}$, $-1.0 \cdot 10^{-9}$ and $-6.4 \cdot 10^{-11}$, all at the noise level of the calibration (on 300 exact instances the float verdict agreed with the exact one 300/300 times, the worst margin on a true instance being $-1.3 \cdot 10^{-14}$), and none was confirmed exactly. The whole campaign took about 1.7 processor-hours.

A negative search is evidence about the search, not about the conjecture; the box is stated so that it can be widened. The following observation explains why the most obvious direction, a pole approaching the imaginary axis, is not where a counterexample would be found.

Remark 7.1 (the lightly damped pair; heuristic, not formalised). Let $a(s) = (s^2 + \varepsilon s + \omega^2) b(s)$ with b Hurwitz of degree $n - 2$ and $\varepsilon \rightarrow 0^+$. In the eigen-expansion $X = \sum_{i,j} \alpha_i \alpha_j v_i v_j^T / (\lambda_i + \lambda_j)$ of Section 4 of [1] (here $-\lambda_i$ are the eigenvalues, v_i the eigenvectors of A^T and $q = \sum \alpha_i v_i$), the term of the conjugate pair has denominator $\lambda_+ + \lambda_- = \varepsilon$ and dominates. The eigenvector of A^T at an eigenvalue μ is the coefficient vector of $a(s)/(s - \mu)$; at $\varepsilon = 0$ and $\mu = i\omega$ this is $(s + i\omega)b(s)$, that is, $v_j = b_{j-2} + i\omega b_{j-1}$. Hence the coefficient of $1/\varepsilon$ in X_{kl} is, up to the positive factor $2|\alpha_+|^2$,

$$\text{Re}(v_k \bar{v}_l) = b_{k-2} b_{l-2} + \omega^2 b_{k-1} b_{l-1} \geq 0,$$

because every coefficient of a Hurwitz polynomial is positive. It vanishes only at the corners $(k, l) = (1, n), (n, 1)$, where Proposition 3.1 gives the exact value $X_{1,n} = Q_{11}/(2a_0) \geq 0$. So at leading order in ε no entry can become negative.

The eigenvector identity is elementary: writing $a(s)/(s - \mu) = \sum_{j=1}^n c_j s^{j-1}$, comparison of coefficients in $a(s) = (s - \mu) \sum_j c_j s^{j-1}$ gives $c_{j-1} - a_{j-1} = \mu c_j$ with $c_0 := 0$, which is exactly the j -th row of $A^T v = \mu v$ at $v = c$. The conclusion was checked against exact rational solutions of (3) at $b(s) = (s+1)^3$, $\omega = 1$, $q = (1, 2, 3, 5, 7)$ and $\varepsilon = 10^{-2}, 10^{-4}, 10^{-6}$: the ratio of εX_{kl} to the displayed shape is one and the same constant across all (k, l) where the shape is nonzero, to within $O(\varepsilon)$, and at the two corners $\varepsilon X_{1,n}$ is exactly $\varepsilon Q_{11}/(2a_0)$. Nothing here is formalised.

7.2. The one-parameter family $(s+1)^3(s^2+ps+1)$. Let $a_p(s) = (s+1)^3(s^2+ps+1)$, that is, $a = (1, p+3, 3p+4, 3p+4, p+3)$, which is Hurwitz for every $p > 0$ and has a complex-conjugate pair exactly when $0 < p < 2$; the witness of Section 4 is $p = 1$.

Claim 7.2 (symbolic; not formalised). *For every $p > 0$ and every $Q \succeq 0$, the solution of $XA_p + A_p^\top X = -Q$ at the companion matrix A_p of a_p is entrywise nonnegative. More precisely, over the field $\mathbb{Q}(p)$: every one of the 15×31 principal minors of the matrices $N_{kl}(p)$ is a ratio of polynomials in p each of whose coefficient vectors is sign-definite (no exceptions), and each of the fifteen forms admits a certificate*

$$X_{kl}(q) = \sum_t \frac{A_t(p)}{B_t(p)} (L_t(p) \cdot q)^2$$

with $A_t, B_t \in \mathbb{Q}[p]$ having nonnegative coefficients; in the best of the 120 pivot orders they have degree at most 4, and degree at most 6 with coefficients below 10^5 once the row denominators are cleared.

The claim was recomputed for this paper over $\mathbb{Q}(p)$: the fifteen matrices $N_{kl}(p)$ were obtained by inverting the Lyapunov operator on symmetric matrices over $\mathbb{Q}(p)$ (a 15×15 matrix whose determinant, in the basis $E_{kk}, E_{kl} + E_{lk}$, is $-256p(p+2)^3$, so the operator is invertible for every $p > 0$); the identity $X(q)A_p + A_p^\top X(q) = -qq^\top$ was checked as an identity in $\mathbb{Q}(p)[q_1, \dots, q_5]$; all 465 principal minors were computed and every one passed the sign-definiteness test; the degree figures above are the minimum over all 120 pivot orders; and exact rational $LDL^\top$ tests at $p \in \{1/10, 1/2, 1, 3/2, 19/10, 2, 3, 10\}$ agree. It is an infinite family of open cells rather than one point, but its formalisation is priced beyond this paper: the Lyapunov identity for the family is a rational identity over $\mathbb{R}(p)[q_1, \dots, q_5]$ on twenty-five entries with those denominators, estimated at one to two orders of magnitude more elaboration work than Theorem 4.1.

7.3. The entry $X_{n,n-1}$ for every Hurwitz a of degree at most 5. Let $W = W(a) = N_{nn}$ be the matrix of the quadratic form X_{nn} , so that $X_{nn} = q^\top W q$ for $Q = qq^\top$. Since $Q \mapsto X$ is linear and the adjoint of $Z \mapsto ZA + A^\top Z$ for the trace pairing $\langle Y, Z \rangle = \text{tr}(Y^\top Z)$ is $Z \mapsto ZA^\top + AZ$, this W is the solution of

$$(11) \quad WA^\top + AW = -e_n e_n^\top,$$

that is, the controllability Gramian $\int_0^\infty e^{At} e_n e_n^\top e^{A^\top t} dt$ of the pair (A, e_n) ; it is *not* the solution of $WA + A^\top W = -e_n e_n^\top$, which is a different matrix. Let $H = H(a)$ be the *Hermite matrix* of a , the symmetric $n \times n$ matrix defined by

$$(12) \quad \frac{a(x)a(y) - a(-x)a(-y)}{x+y} = \sum_{i,j=1}^n H_{ij} x^{i-1} y^{j-1}$$

(the left side is a polynomial because the numerator vanishes at $y = -x$). That $H \succ 0$ characterises the Hurwitz polynomials is the classical Hermite–Fujiwara criterion (see [11]); we use it only as a name, and nothing below depends on it. For the classical relation between Lyapunov equations at companion and Schwarz-form state matrices and the Routh–Hurwitz criterion see [7, 8].

Claim 7.3 (symbolic; not formalised). *For $n = 2, 3, 4, 5$ and indeterminate coefficients $a_0, \dots, a_{n-1}$,*

$$W(a)^{-1} = H(a) \quad \text{identically in } \mathbb{Q}(a_0, \dots, a_{n-1});$$

in particular $(W^{-1})_{nn} = H_{nn} = 2a_{n-1}$. Consequently, for every Hurwitz a of degree $n \leq 5$ and every $Q \succeq 0$, $X_{n,n-1} = X_{n-1,n} \geq 0$, with equality attained.

The identification was verified for this paper by solving (11) over $\mathbb{Q}(a_0, \dots, a_{n-1})$, expanding (12), and normalising the product $W(a)H(a)$: it is the identity matrix for $n = 2, 3, 4, 5$. The consequence is Proposition 3.2 at $k = n$: $X_{n,n-1} = a_{n-1}X_{nn} - \frac{1}{2}Q_{nn}$, so in the rank-one case $X_{n,n-1} = q^\top (a_{n-1}W - \frac{1}{2}e_n e_n^\top)q$; here $W \succeq 0$ by Lyapunov theory and W is invertible because H is its inverse, so W is positive definite, and the Schur complement test makes $a_{n-1}W - \frac{1}{2}e_n e_n^\top \succeq 0$ equivalent to $e_n^\top W^{-1} e_n \leq 2a_{n-1}$, which the identity gives with equality. Summing rank-one terms extends it to every $Q \succeq 0$. Equality

makes $a_{n-1}W - \frac{1}{2}e_n e_n^\top$ singular, with kernel spanned by $W^{-1}e_n = He_n$, the last column of H ; at the witness of Section 4 that column is $(2, 0, 14, 0, 8)$, and indeed $X(q)_{5,4} = 0$ there. For $n = 3$, for instance,

$$H = 2 \begin{pmatrix} a_0 a_1 & 0 & a_0 \\ 0 & a_1 a_2 - a_0 & 0 \\ a_0 & 0 & a_2 \end{pmatrix},$$

whose leading principal minors $2a_0 a_1$, $4a_0 a_1(a_1 a_2 - a_0)$ and $8a_0(a_1 a_2 - a_0)^2$ are positive exactly when a is Hurwitz of degree 3. The identity looks degree-independent, but it is recorded here only for the degrees actually computed. At $n = 5$ the consequence is a parametric sub-case of the conjecture in the open complex-pair regime, for one entry, whose formalisation would need a certificate for $2a_4 W(a) - e_5 e_5^\top \succeq 0$ over the Hurwitz region in five parameters.

8. OPEN QUESTIONS

The conjecture at the smallest open degree, $n = 5$ with a complex pair, is the five-parameter semialgebraic statement “ $N_{kl}(a) \succeq 0$ for all fifteen (k, l) and all Hurwitz a of degree 5”; Claim 7.3 is one of the fifteen forms, Claim 7.2 is a curve through the region, and Theorem 4.1 is a point on that curve. Three natural next steps are: further exact witnesses, in particular an $n = 5$ instance with two complex pairs and an $n = 7$ instance (the certificate generator is automatic; only the Hurwitz proof is new work); a formal proof of Claim 7.2, whose cost should first be measured on the single hardest entry X_{11} ; and a certificate for $2a_4 W(a) - e_5 e_5^\top \succeq 0$ in terms of the Routh–Hurwitz determinants, which would settle one entry of the conjecture for every degree-five Hurwitz polynomial. A search aimed at a counterexample should, by Remark 7.1, vary the ratios of root moduli rather than the damping.

9. VERIFICATION

Propositions 3.1, 3.2, 5.1, 6.1 and Theorems 4.1, 4.2 rest on declarations formally verified in Lean 4 against *Mathlib* [12, 13] (Lean 4.32.0, Mathlib at its v4.32.0 tag), whose statements are reproduced verbatim in Appendix A. Where a printed statement says more than its formal counterpart, the text names the extra step: the non-diagonalizability of the witness, its Routh–Hurwitz determinants, the displayed value of $X(e_1)$ (of which only the entry $X(e_1)_{15}$ is a formal statement) and the determinant of the Lyapunov operator (Section 4); the explicit z and v in Theorem 4.1(ii), which the formal statement leaves existentially quantified, though they are the witnesses used in its proof; the rank-one decomposition in Corollary 4.3; the invertibility remark and the two untagged eigenvalue lemmas in Proposition 5.1; the full list of minors after Proposition 6.1; and everything in Section 7. The development is five modules, 827 lines in all: the general-degree identities (171 lines; 2 definitions, 4 lemmas, 4 theorems), the witness with its fifteen certificates (238 lines; 19 definitions, 18 lemmas, 2 theorems), its spectrum (122 lines; 2 definitions, 6 lemmas, 2 theorems), uniqueness (103 lines; 1 theorem, twenty-five `linear_combination` steps) and the controls with the source’s example (193 lines; 7 definitions, 2 lemmas, 9 theorems). Every proof is checked by the Lean kernel; there is no compiled evaluation (`native_decide`), no sorry, no floating-point number and no axiom declared by hand. The Lyapunov identities are discharged by `ring` after expanding the matrix products, the nonnegativity of the certificates by `positivity`, the rational identities of Propositions 5.1 and 6.1 by `norm_num`, and the eigenvalue statements by `linear_combination` and `nlinarith`. The axiom print (`#print axioms`) of each of the 16 theorems of Appendix A is `propext`, `Classical.choice`, `Quot.sound`. As recorded, the Lean checks took about 1.35 processor-hours in total (the certificate module about four minutes and 1.0 GB per check, the uniqueness module up to twenty minutes), with a peak resident set of about 6.9 GB, and the exact-rational and symbolic computations of Section 7 about 2.1 processor-hours. All numbers printed in this paper were recomputed independently in exact rational arithmetic for the manuscript, including the fifteen certificates read back from the formal source and compared with the matrices N_{kl} . The development is currently in a private repository: repository reference to be added at submission.

APPENDIX A. THE FORMAL STATEMENTS

The Lean statements corresponding to the results above, verbatim (statements only; proofs, doc-strings and attributes omitted). Indices are 0-based: for $n = m + 1$, $\text{Fin } (m + 1)$ is $\{0, \dots, m\}$, $\text{Fin.last } m$ is the last index $n - 1$, and for $j : \text{Fin } m$, $j.\text{castSucc}$ and $j.\text{succ}$ are j and $j + 1$ viewed in $\text{Fin } (m + 1)$; so $X \ 0$ ($\text{Fin.last } m$) is $X_{1,n}$, $q \ 0$ is q_1 , and $x01$ is X_{12} . $X^\top$ is the transpose, $\text{Matrix.vecMulVec } q \ q$ is $qq^\top$, $*_v$ is the matrix-vector product, and $!![\dots]$ and $![\dots]$ are matrix and vector literals.

Definitions (general degree).

```
def companion (a : Fin (m + 1) → ℝ) : Matrix (Fin (m + 1)) (Fin (m + 1)) ℝ :=
  Matrix.of fun i j =>
    (if (j : ℕ) = (i : ℕ) + 1 then (1 : ℝ) else 0) - (if i = Fin.last m then a j else 0)
```

```
def IsLyp (A Q X : Matrix (Fin (m + 1)) (Fin (m + 1)) ℝ) : Prop := X * A + A⊤ * X = -Q
```

Propositions 3.1 and 3.2.

```
theorem corner_two_mul (a : Fin (m + 1) → ℝ) (Q X : Matrix (Fin (m + 1)) (Fin (m + 1)) ℝ)
  (h : IsLyp (companion a) Q X) (hsym : X⊤ = X) :
  2 * a 0 * X 0 (Fin.last m) = Q 0 0
```

```
theorem corner_eq_div (a : Fin (m + 1) → ℝ) (Q X : Matrix (Fin (m + 1)) (Fin (m + 1)) ℝ)
  (h : IsLyp (companion a) Q X) (hsym : X⊤ = X) (ha : a 0 ≠ 0) :
  X 0 (Fin.last m) = Q 0 0 / (2 * a 0)
```

```
theorem corner_nonneg (a : Fin (m + 1) → ℝ) (Q X : Matrix (Fin (m + 1)) (Fin (m + 1)) ℝ)
  (h : IsLyp (companion a) Q X) (hsym : X⊤ = X) (ha : 0 < a 0) (hQ : 0 ≤ Q 0 0) :
  0 ≤ X 0 (Fin.last m)
```

```
theorem subdiag_eq (a : Fin (m + 1) → ℝ) (Q X : Matrix (Fin (m + 1)) (Fin (m + 1)) ℝ)
  (h : IsLyp (companion a) Q X) (hsym : X⊤ = X) (j : Fin m) :
  X j.succ j.castSucc = a j.succ * X j.succ (Fin.last m) - Q j.succ j.succ / 2
```

The witness (Theorems 4.1 and 4.2).

```
def acoef : Fin 5 → ℝ := ![1, 4, 7, 7, 4]
```

```
def A : Matrix (Fin 5) (Fin 5) ℝ :=
  !![0, 1, 0, 0, 0;
    0, 0, 1, 0, 0;
    0, 0, 0, 1, 0;
    0, 0, 0, 0, 1;
    -1, -4, -7, -7, -4]
```

```
def Ac : Matrix (Fin 5) (Fin 5) ℂ :=
  !![0, 1, 0, 0, 0;
    0, 0, 1, 0, 0;
    0, 0, 0, 1, 0;
    0, 0, 0, 0, 1;
    -1, -4, -7, -7, -4]
```

```
def x00 (q : Fin 5 → ℝ) : ℝ :=
  (1 / 1008 : ℝ) * ((-29 : ℝ) * q 0 + (7 : ℝ) * q 2 + (-5 : ℝ) * q 4) ^ 2 +
  (1 / 720 : ℝ) * ((-7 : ℝ) * q 1 + (5 : ℝ) * q 3) ^ 2 +
  (1 / 120 : ℝ) * ((-15 : ℝ) * q 0 + (4 : ℝ) * q 1) ^ 2 +
  (1 / 42 : ℝ) * ((-4 : ℝ) * q 0 + q 4) ^ 2 +
```

$$(1 / 8 : \mathbb{R}) * (q_0)^2$$

$$\begin{aligned} \text{def } x01 (q : \text{Fin } 5 \rightarrow \mathbb{R}) : \mathbb{R} := & \\ & (1 / 252 : \mathbb{R}) * ((-29 : \mathbb{R}) * q_0 + (7 : \mathbb{R}) * q_2 + (-5 : \mathbb{R}) * q_4)^2 + \\ & (2 / 21 : \mathbb{R}) * ((4 : \mathbb{R}) * q_0 - q_4)^2 + \\ & (1 / 180 : \mathbb{R}) * ((-7 : \mathbb{R}) * q_1 + (5 : \mathbb{R}) * q_3)^2 + \\ & (1 / 30 : \mathbb{R}) * (q_1)^2 \end{aligned}$$

$$\begin{aligned} \text{def } x02 (q : \text{Fin } 5 \rightarrow \mathbb{R}) : \mathbb{R} := & \\ & (1 / 252 : \mathbb{R}) * ((7 : \mathbb{R}) * q_1 + (-5 : \mathbb{R}) * q_3)^2 + \\ & (1 / 180 : \mathbb{R}) * ((-7 : \mathbb{R}) * q_0 + (5 : \mathbb{R}) * q_2 + (-7 : \mathbb{R}) * q_4)^2 + \\ & (1 / 168 : \mathbb{R}) * ((-21 : \mathbb{R}) * q_0 + (4 : \mathbb{R}) * q_3)^2 + \\ & (1 / 30 : \mathbb{R}) * ((-4 : \mathbb{R}) * q_0 + q_4)^2 + \\ & (7 / 8 : \mathbb{R}) * (q_0)^2 \end{aligned}$$

$$\begin{aligned} \text{def } x03 (q : \text{Fin } 5 \rightarrow \mathbb{R}) : \mathbb{R} := & \\ & (1 / 1008 : \mathbb{R}) * ((7 : \mathbb{R}) * q_1 + (-5 : \mathbb{R}) * q_3)^2 + \\ & (1 / 720 : \mathbb{R}) * ((-7 : \mathbb{R}) * q_0 + (5 : \mathbb{R}) * q_2 + (-7 : \mathbb{R}) * q_4)^2 + \\ & (2 / 15 : \mathbb{R}) * ((4 : \mathbb{R}) * q_0 - q_4)^2 + \\ & (1 / 42 : \mathbb{R}) * (q_3)^2 \end{aligned}$$

$$\begin{aligned} \text{def } x04 (q : \text{Fin } 5 \rightarrow \mathbb{R}) : \mathbb{R} := & \\ & (1 / 2 : \mathbb{R}) * (q_0)^2 \end{aligned}$$

$$\begin{aligned} \text{def } x11 (q : \text{Fin } 5 \rightarrow \mathbb{R}) : \mathbb{R} := & \\ & (1 / 1392 : \mathbb{R}) * ((24 : \mathbb{R}) * q_0 + (-47 : \mathbb{R}) * q_1 + (29 : \mathbb{R}) * q_3)^2 + \\ & (1 / 1872 : \mathbb{R}) * ((47 : \mathbb{R}) * q_0 + (-29 : \mathbb{R}) * q_2 + (39 : \mathbb{R}) * q_4)^2 + \\ & (1 / 29016 : \mathbb{R}) * ((-400 : \mathbb{R}) * q_0 + (-117 : \mathbb{R}) * q_1 + (124 : \mathbb{R}) * q_2)^2 + \\ & (1 / 5394 : \mathbb{R}) * ((117 : \mathbb{R}) * q_0 + (-37 : \mathbb{R}) * q_1)^2 + \\ & (1 / 8 : \mathbb{R}) * (q_1)^2 \end{aligned}$$

$$\begin{aligned} \text{def } x12 (q : \text{Fin } 5 \rightarrow \mathbb{R}) : \mathbb{R} := & \\ & (7 / 180 : \mathbb{R}) * ((-7 : \mathbb{R}) * q_1 + (5 : \mathbb{R}) * q_3)^2 + \\ & (7 / 30 : \mathbb{R}) * (q_1)^2 + \\ & (1 / 36 : \mathbb{R}) * ((7 : \mathbb{R}) * q_0 + (-5 : \mathbb{R}) * q_2 + (7 : \mathbb{R}) * q_4)^2 + \\ & (1 / 6 : \mathbb{R}) * ((7 : \mathbb{R}) * q_0 - q_2)^2 \end{aligned}$$

$$\begin{aligned} \text{def } x13 (q : \text{Fin } 5 \rightarrow \mathbb{R}) : \mathbb{R} := & \\ & (1 / 3 : \mathbb{R}) * ((-3 : \mathbb{R}) * q_0 + q_2 - q_4)^2 + \\ & (1 / 3 : \mathbb{R}) * (-q_1 + q_3)^2 + \\ & (1 / 24 : \mathbb{R}) * ((-4 : \mathbb{R}) * q_0 + (-3 : \mathbb{R}) * q_1 + (4 : \mathbb{R}) * q_4)^2 + \\ & (1 / 6 : \mathbb{R}) * ((3 : \mathbb{R}) * q_0 - q_1)^2 + \\ & (1 / 8 : \mathbb{R}) * (q_1)^2 \end{aligned}$$

$$\begin{aligned} \text{def } x14 (q : \text{Fin } 5 \rightarrow \mathbb{R}) : \mathbb{R} := & \\ & (1 / 1008 : \mathbb{R}) * ((-29 : \mathbb{R}) * q_0 + (7 : \mathbb{R}) * q_2 + (-5 : \mathbb{R}) * q_4)^2 + \\ & (1 / 42 : \mathbb{R}) * ((4 : \mathbb{R}) * q_0 - q_4)^2 + \\ & (1 / 720 : \mathbb{R}) * ((-7 : \mathbb{R}) * q_1 + (5 : \mathbb{R}) * q_3)^2 + \\ & (2 / 15 : \mathbb{R}) * (q_1)^2 \end{aligned}$$

$$\begin{aligned} \text{def } x22 (q : \text{Fin } 5 \rightarrow \mathbb{R}) : \mathbb{R} := & \\ & (1 / 8 : \mathbb{R}) * ((-5 : \mathbb{R}) * q_0 + (4 : \mathbb{R}) * q_2 - q_3 + (-4 : \mathbb{R}) * q_4)^2 + \end{aligned}$$

$$\begin{aligned}
& (1/2 : \mathbb{R}) * (-q_0 + q_1 - q_3 + q_4)^2 + \\
& (1/8 : \mathbb{R}) * (q_0 + (4 : \mathbb{R}) * q_1 + (-3 : \mathbb{R}) * q_3)^2 + \\
& (1/4 : \mathbb{R}) * ((-3 : \mathbb{R}) * q_0 + q_3)^2 + \\
& (4 : \mathbb{R}) * (q_0)^2
\end{aligned}$$

```
def x23 (q : Fin 5 → ℝ) : ℝ :=
  (1 / 36 : ℝ) * ((7 : ℝ) * q 1 + (-5 : ℝ) * q 3) ^ 2 +
  (7 / 180 : ℝ) * ((-7 : ℝ) * q 0 + (5 : ℝ) * q 2 + (-7 : ℝ) * q 4) ^ 2 +
  (7 / 30 : ℝ) * ((4 : ℝ) * q 0 - q 4) ^ 2 +
  (1 / 6 : ℝ) * (q 3) ^ 2
```

```
def x24 (q : Fin 5 → ℝ) : ℝ :=
  (1 / 252 : ℝ) * ((-11 : ℝ) * q 0 + (7 : ℝ) * q 2 + (-5 : ℝ) * q 4) ^ 2 +
  (1 / 180 : ℝ) * ((-7 : ℝ) * q 1 + (5 : ℝ) * q 3) ^ 2 +
  (1 / 30 : ℝ) * (q 1) ^ 2 +
  (1 / 168 : ℝ) * (-q 0 + (4 : ℝ) * q 4) ^ 2 +
  (7 / 8 : ℝ) * (q 0) ^ 2
```

```
def x33 (q : Fin 5 → ℝ) : ℝ :=
  (1 / 1872 : ℝ) * ((39 : ℝ) * q 1 + (-29 : ℝ) * q 3) ^ 2 +
  (1 / 1392 : ℝ) * ((-39 : ℝ) * q 0 + (29 : ℝ) * q 2 + (-47 : ℝ) * q 4) ^ 2 +
  (1 / 25752 : ℝ) * ((-124 : ℝ) * q 0 + (-87 : ℝ) * q 3 + (148 : ℝ) * q 4) ^ 2 +
  (1 / 8658 : ℝ) * ((117 : ℝ) * q 0 + (-31 : ℝ) * q 3) ^ 2 +
  (1 / 8 : ℝ) * (q 3) ^ 2
```

```
def x34 (q : Fin 5 → ℝ) : ℝ :=
  (1 / 252 : ℝ) * ((7 : ℝ) * q 1 + (-5 : ℝ) * q 3) ^ 2 +
  (1 / 180 : ℝ) * ((-7 : ℝ) * q 0 + (5 : ℝ) * q 2 + (-7 : ℝ) * q 4) ^ 2 +
  (1 / 30 : ℝ) * ((4 : ℝ) * q 0 - q 4) ^ 2 +
  (2 / 21 : ℝ) * (q 3) ^ 2
```

```
def x44 (q : Fin 5 → ℝ) : ℝ :=
  (1 / 1008 : ℝ) * ((7 : ℝ) * q 1 + (-5 : ℝ) * q 3) ^ 2 +
  (1 / 720 : ℝ) * ((-7 : ℝ) * q 0 + (5 : ℝ) * q 2 + (-7 : ℝ) * q 4) ^ 2 +
  (1 / 120 : ℝ) * ((4 : ℝ) * q 0 - q 4) ^ 2 +
  (1 / 42 : ℝ) * (q 3) ^ 2 +
  (1 / 8 : ℝ) * (q 4) ^ 2
```

```
def xent (q : Fin 5 → ℝ) : Fin 5 → Fin 5 → ℝ :=
  ![ [x00 q, x01 q, x02 q, x03 q, x04 q],
    [x01 q, x11 q, x12 q, x13 q, x14 q],
    [x02 q, x12 q, x22 q, x23 q, x24 q],
    [x03 q, x13 q, x23 q, x33 q, x34 q],
    [x04 q, x14 q, x24 q, x34 q, x44 q] ]
```

```
def Xw (q : Fin 5 → ℝ) : Matrix (Fin 5) (Fin 5) ℝ := Matrix.of (xent q)
```

```
theorem Xw_lyap (q : Fin 5 → ℝ) :
  IsLyap A (Matrix.vecMulVec q q) (Xw q)
```

```
theorem Xw_nonneg (q : Fin 5 → ℝ) (k l : Fin 5) : 0 ≤ Xw q k l
```

```
theorem A_hurwitz (z : ℂ) (v : Fin 5 → ℂ) (hv : v ≠ 0) (h : Ac *v v = z • v) : z.re < 0
```

```
theorem A_has_complex_pair :
```

```
  ∃ z : ℂ, ∃ v : Fin 5 → ℂ, v ≠ 0 ∧ Ac *v v = z • v ∧ z.im ≠ 0
```

```
theorem lyap_unique (Q X Y : Matrix (Fin 5) (Fin 5) ℝ) (hXs : X⊤ = X) (hYs : Y⊤ = Y)
  (hX : IsLyap A Q X) (hY : IsLyap A Q Y) : X = Y
```

(A_eq : A = companion acoef is proved in the same module, so A is the companion matrix of Section 2;
Ac is its complexification, A.map (fun r : ℝ => (r : ℂ)) = Ac.)

Controls (Proposition 5.1).

```
def Adiaq : Matrix (Fin 2) (Fin 2) ℝ := !![-1, 0; 0, -2]
def Xdiag : Matrix (Fin 2) (Fin 2) ℝ := !![1/2, -1/3; -1/3, 1/4]
def Abad : Matrix (Fin 2) (Fin 2) ℝ := !![0, 1; 2, -1]
def Xbad : Matrix (Fin 2) (Fin 2) ℝ := !![1/4, -1/4; -1/4, -1/4]
```

```
theorem companion_needed :
```

```
  Xdiag * Adiaq + Adiaq⊤ * Xdiag = -(Matrix.vecMulVec ![1, -1] ![1, -1]) ∧
  Xdiag 0 1 < 0
```

```
theorem hurwitz_needed :
```

```
  Xbad * Abad + Abad⊤ * Xbad = -(Matrix.vecMulVec ![1, 0] ![1, 0]) ∧
  Xbad 0 1 < 0 ∧ Xbad 1 1 < 0
```

```
theorem psd_not_entrywise :
```

```
  (Matrix.vecMulVec ![(1:ℝ), -1] ![(1:ℝ), -1]).PosSemidef ∧
  (Matrix.vecMulVec ![(1:ℝ), -1] ![(1:ℝ), -1]) 0 1 < 0
```

```
theorem witness_nonvacuous : 0 < W5.Xw ![1, 0, 0, 0, 0] 0 4
```

(Abad_eq : Abad = companion ![-2, 1] is proved in the same module.)

The source's example (Proposition 6.1).

```
def Apap : Matrix (Fin 3) (Fin 3) ℝ := !![0, 1, 0; 0, 0, 1; -3/100, -2/5, -13/10]
```

```
def Xpap : Matrix (Fin 3) (Fin 3) ℝ :=
```

```
  !![981187/29400, 216649/735, 800/3;
    216649/735, 7667197/2940, 692929/294;
    800/3, 692929/294, 314060/147]
```

```
theorem paper_example_lyap :
```

```
  Xpap * Apap + Apap⊤ * Xpap = -(Matrix.vecMulVec ![4, 36, 29] ![4, 36, 29])
```

```
theorem paper_example_entrywise_pos (i j : Fin 3) : 0 < Xpap i j
```

```
theorem paper_example_minor_correction :
```

```
  Xpap 0 0 * Xpap 1 1 - Xpap 0 1 * Xpap 1 0 = 4355916893/28812000 ∧
  0 < Xpap 0 0 * Xpap 1 1 - Xpap 0 1 * Xpap 1 0 ∧
  Xpap 0 1 * Xpap 1 2 - Xpap 0 2 * Xpap 1 1 = -(51562093/72030) ∧
  Xpap 0 1 * Xpap 1 2 - Xpap 0 2 * Xpap 1 1 < 0
```

(Apap_eq : Apap = companion ![3/100, 2/5, 13/10] is proved in the same module.)

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