EXPLICIT SHIFT VECTORS AND CERTIFIED LONELINESS GAPS FOR THE SHIFTED LONELY RUNNER CONJECTURE

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ABSTRACT. For integer speeds $v = (v_1, \dots, v_n)$ and shifts $s = (s_1, \dots, s_n)$, the loneliness gap $\gamma(v; s) = \sup_{t \in \mathbb{R}} \min_i \|s_i + v_i t\|$ measures how far from every integer the whole pack of runners can simultaneously be pushed; the Shifted Lonely Runner Conjecture asserts $\gamma(v; s) \geq 1/(n+1)$ whenever the $|v_i|$ are distinct and nonzero. Blanco, Criado and Santos recently refuted it for every n with $5 \leq n \leq 17$, but their paper prints an attaining shift vector for only three instances, and their companion code stops at $n = 13$.

We give explicit rational shift vectors $s^{(n)} \in [0, 1)^n$ for every n with $7 \leq n \leq 17$, together with the exact value of $\gamma((1, 2, \dots, n); s^{(n)})$ and an exact rational maximising time. At $n = 7$ and $n = 8$ the vectors reach the published optimal values $25/214$ and $1/10$; for $9 \leq n \leq 14$ they certify $\gamma^{\min}(1, \dots, n) \leq 1/(n+2)$; and for $n = 15, 16, 17$ — the range where no exact value of $\gamma^{\min}$ is published — they certify $\gamma^{\min}(1, \dots, n) \leq 2/(2n+3)$, that is, $2/33$, $2/35$ and $2/37$. Every gap is proved on both sides by a purely integral certificate: a finite interval cover for the upper bound and one rational time for the lower bound. We also reproduce, as upper bounds with explicit witnesses, the published values of $\gamma^{\min}(1, \dots, n)$ for $5 \leq n \leq 13$, and give the corresponding maximising times, which neither source prints. Every one of these certificates has been checked by the Lean 4 kernel, together with the soundness of the certificate scheme itself and the controls that make it falsifiable.

1. INTRODUCTION

The objects. Write $\|x\| = \operatorname{dist}(x, \mathbb{Z}) = \min_{k \in \mathbb{Z}} |x - k|$ for the distance from a real number to the nearest integer. Given nonzero reals $v_1, \dots, v_n$ (*speeds*) and reals $s_1, \dots, s_n$ (*shifts*), think of n runners on a circle of unit circumference, runner i starting at s_i and running at constant speed v_i, together with one further runner who stands still at the origin. The quantity

$$\gamma(v; s) \;=\; \sup_{t \in \mathbb{R}} \; \min_{1 \leq i \leq n} \|s_i + v_i t\|$$

is the *loneliness gap*: the largest distance the stationary runner can, at some instant, put between himself and all of the others at once. Note the convention, which we keep throughout because the source we follow does: n **counts speeds, so there are** $n+1$ **runners.**

The classical Lonely Runner Conjecture is the case $s = 0$: it asserts $\gamma(v; 0) \geq 1/(n+1)$ for all nonzero v, and the bound is tight at $v = (1, 2, \dots, n)$. The *shifted* form, Conjecture 1.1(ii) of Blanco–Criado–Santos [2], allows arbitrary starting positions and asks for the same bound under the extra hypothesis that the $|v_i|$ be pairwise distinct:

> *(sLRC.)* If $v_1, \dots, v_n$ are nonzero reals with $|v_i| \neq |v_j|$ for $i \neq j$, then for every $s_1, \dots, s_n \in \mathbb{R}$ there is a $t \in \mathbb{R}$ with $\|s_i + v_i t\| \geq 1/(n+1)$ for all i.

Equivalently $\gamma(v; s) \geq 1/(n+1)$ for all such v and all s. The associated extremal quantity is the *shifted loneliness gap* $\gamma^{\min}(v) = \min_{s \in [0,1]^n} \gamma(v; s)$, and the conjecture reads $\gamma^{\min}(v) \geq 1/(n+1)$.

The source, and what it leaves open. Blanco, Criado and Santos [2] refuted the shifted conjecture. Their Theorem 1.17 records

$$\gamma^{\min}(1, 2, 3, 4, 5) = \tfrac{15}{94} < \tfrac{1}{6}, \qquad \gamma^{\min}(2, 3, 4, 5, 6, 8) = \tfrac{2}{15} < \gamma^{\min}(1, \dots, 6) = \tfrac{9}{67} < \tfrac{1}{7},$$

and $\gamma^{\min}(1, 2, \dots, n) < 1/(n+1)$ for every $n \in \{7, \dots, 17\}$. The shifted conjecture is known for every $n \leq 4$ — the cases $n \leq 3$ are earlier work and $n = 4$ is Alcántara–Criado–Santos [1], whose introduction

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reviews both — and $n = 4$ is the largest n at which it is proved [2, §1]; so $n = 5$ is the first failure. A table in [2] lists the exact value of $\gamma^{\min}(1, \ldots, n)$ for $n \le 14$ (Table 3 below), accompanied by the sentence *"For $n \le 14$ we have been able to compute the exact value of $\gamma^{\min}(1, \cdots, n)$, shown in Table 2. For larger n we can only certify an upper bound, but not the exact value."*

Two things are conspicuously absent. First, [2] prints an attaining shift vector for exactly three instances — one at $n = 5$ and two at $n = 6$, in its Propositions 5.10 and 5.11 — and *none at all for* $n \ge 7$. Its companion code repository [3] publishes optimal shift vectors, but only for $n \le 13$: its file `output/consecutive.out` holds one record for each $n = 2, \ldots, 13$, and its other three data files cover $n = 5, 6, 7$ only. In particular neither source gives an attaining vector for the value $\gamma^{\min}(1, \ldots, 14) = 71/1228$. Second, at $n = 15, 16, 17$ no exact value is published, and the certification of $\gamma^{\min} < 1/(n+1)$ there comes with no vector a reader can check.

Results. This paper supplies explicit, independently searched shift vectors across the whole range $7 \le n \le 17$, and proves the exact loneliness gap of each of them. The main statement is Theorem 4.1 and its data are in Table 1. In words:

- For $n = 7$ and $n = 8$ the vectors found here have gaps $25/214$ and $1/10$ — exactly the values [2] tabulates, for which it prints no vector. These two agreements, obtained without consulting the table, are also what convinces us that the function computed here is the one [2] defines.
- For $9 \le n \le 14$ the vectors have gap exactly $1/(n+2)$, giving $\gamma^{\min}(1, \ldots, n) \le 1/(n+2)$. The *value* here is not new — [2] already observes $\gamma^{\min}(1, \ldots, n) < 1/(n+2)$ over this range — but the explicit witness is.
- For $n = 15, 16, 17$, where [2] states it has no exact value, the vectors have gap exactly $2/(2n+3)$, so
$$\gamma^{\min}(1, \ldots, 15) \le \tfrac{2}{33}, \qquad \gamma^{\min}(1, \ldots, 16) \le \tfrac{2}{35}, \qquad \gamma^{\min}(1, \ldots, 17) \le \tfrac{2}{37},$$
each strictly below the corresponding $1/(n+1)$. At these three values of n the only thing [2] records is $\gamma^{\min}(1, \ldots, n) < 1/(n+1)$ (its Proposition 5.13): neither a shift vector nor a sharper numerical bound. We are aware of no published shift vector for them.
- In every case the supremum defining γ is shown to be a maximum, attained at an explicit rational time listed in Table 1. No attaining time is printed in [2] or in [3].

We record just as plainly what is *not* settled. Our witnesses at $n = 15, 16, 17$ sit at the level $2/(2n+3)$, which is above $1/(n+2)$; the search described in Section 5 did not reach $1/(n+2)$ at those three values of n. Whether the pattern $\gamma^{\min}(1, \ldots, n) < 1/(n+2)$ that [2] observes for $9 \le n \le 14$ persists beyond $n = 14$ is therefore neither confirmed nor contradicted here; the failure of one heuristic search is not evidence about the truth. Nor do we obtain any lower bound on $\gamma^{\min}$: everything below certifies $\gamma^{\min}(v) \le c$ from one shift vector, and the reverse inequality — the half that turns the table of [2] into a table of equalities — quantifies over all $s \in [0, 1]^n$ and is out of reach of the certificates used here.

The method is elementary and is the reason every claim can be checked exactly. The identity $\gamma(v; s) = c$ splits into two finite integral certificates (Section 3): an *interval cover* for the upper bound, and a single rational *witness time* for the lower bound. Putting all data over one common denominator turns both into integer arithmetic, with no rational normalisation anywhere, which is what makes machine checking of instances as large as $n = 17$ — common denominator 453 332 880, cover of length 79 — practical.

Section 6 reproduces the published data: the three instances [2] prints in full, and the optimal shift vectors of [3] for $5 \le n \le 13$, whose gaps are proved here to be exactly the tabulated values. Section 7 collects the controls that make the certificate scheme falsifiable, including the classical unshifted equalities $\gamma((1, \ldots, n); 0) = 1/(n+1)$ and the specificity check at $n = 4$, where the shifted conjecture is a theorem and the machinery correctly declines to refute it.

Attribution. The refutation of the shifted conjecture, the definitions, and every exact value of $\gamma^{\min}$ quoted here are due to [2]; the optimal shift vectors of Section 6.2 are the data of [3]; the tightness $\gamma((1, \ldots, n); 0) = 1/(n+1)$ is classical, the instance that fixes the constant $1/(n+1)$ in the conjecture Wills posed in [5], and [2, proof of Cor. 1.14] calls the equality well known. What is produced here

is the certificate scheme, the shift vectors for $7 \le n \le 17$, the attaining times throughout, and the machine-checked proofs.

2. Preliminaries

Throughout, $n \ge 1$, and $\|x\| = \min_{k\in\mathbb{Z}} |x-k|$.

Definition 2.1. For $v, s \in \mathbb{R}^n$ and $t \in \mathbb{R}$ put

$$G_{v,s}(t) \;=\; \min_{1\le i\le n} \|s_i + v_i t\|, \qquad \gamma(v;s) \;=\; \sup_{t\in\mathbb{R}} G_{v,s}(t),$$

and, following [2, Def. 1.3(ii)], $\gamma^{\min}(v) = \inf\{\gamma(v;s) : s \in [0,1]^n\}$. Finally, sLRC($n$) denotes the assertion: for all $v \in (\mathbb{R}\setminus\{0\})^n$ with $|v_i| \ne |v_j|$ for $i \ne j$, and all $s \in \mathbb{R}^n$, there is $t \in \mathbb{R}$ with $\|s_i + v_i t\| \ge 1/(n+1)$ for every i.

Since $0 \le G_{v,s} \le 1/2$ pointwise, $\gamma(v;s)$ is a well-defined real number in $[0,1/2]$, and $\gamma(v;\cdot) \ge 0$ makes the infimum defining $\gamma^{\min}$ one of a nonempty set bounded below, hence again a real number. The two elementary facts used repeatedly are that $\gamma(v;s) \ge 1/(n+1)$ for a single admissible (v,s) is exactly what sLRC(n) demands of that pair, and that for *integer* speeds $t \mapsto G_{v,s}(t)$ is 1-periodic, so the unit interval carries the whole supremum.

Proposition 2.2. *Let $V \in \mathbb{Z}^n$ have nonzero entries with pairwise distinct absolute values, and let $s \in \mathbb{Q}^n$. If $\gamma(V;s) < 1/(n+1)$ then* sLRC(n) *is false.*

Proof. If sLRC(n) held, applying it to this V and this s would produce a t with $G_{V,s}(t) \ge 1/(n+1)$, whence $\gamma(V;s) \ge G_{V,s}(t) \ge 1/(n+1)$. The two hypotheses on V are conditions on finitely many integers. Rationality of s plays no role in the argument; it is kept because every shift vector used below is rational, and that is the form in which the statement is formalised. □

3. A two-sided integral certificate

All certificate data lives over a single positive integer denominator D: speeds $V \in \mathbb{Z}^n_{>0}$, shift numerators $A \in \mathbb{Z}^n$ with $s_i = A_i/D$, level numerator $C \in \mathbb{Z}$ with $c = C/D$, and times $t = u/D$. Write

$$\mathrm{band}(i,k,u) \;:\Longleftrightarrow\; -C \;\le\; V_i u + A_i - kD \;\le\; C \qquad (i \in \{1,\dots,n\},\ k,u \in \mathbb{Z}),$$

the cleared-of-denominators form of "at time u/D, runner i is within c of the integer k".

Definition 3.1. A *cover chain* for (D,V,A,C) is a finite list $(r_1,i_1,k_1),\dots,(r_m,i_m,k_m) \in \mathbb{Z}\times\{1,\dots,n\}\times\mathbb{Z}$ such that, setting $r_0 = 0$, one has $r_{j-1} \le r_j$ and $\mathrm{band}(i_j,k_j,r_{j-1})$ and $\mathrm{band}(i_j,k_j,r_j)$ for $j = 1,\dots,m$, and $r_m \ge D$. A *witness* is an integer P such that, for every i, both $(V_iP + A_i) \bmod D \ge C$ and $D - \big((V_iP + A_i) \bmod D\big) \ge C$.

Both conditions are decidable by finitely many integer operations: checking a cover chain of length m costs $5m+1$ integer comparisons, checking a witness costs n Euclidean divisions and $2n$ comparisons. Nothing is a rational number at check time.

Theorem 3.2. *Let $D > 0$, $V \in \mathbb{Z}^n_{>0}$, $A \in \mathbb{Z}^n$, $C \in \mathbb{Z}$, and set $s = A/D$.*

(a) *If a cover chain for (D,V,A,C) exists, then $\gamma(V;s) \le C/D$.*
(b) *If a witness P exists, then $\gamma(V;s) \ge C/D$.*
(c) *If both exist, then $\gamma(V;s) = C/D$.*

Proof sketch. (a) Because the speeds are integers, $G_{V,s}$ is 1-periodic, so it is enough to bound it on $[0,1)$, i.e. to bound $G_{V,s}(u/D)$ for real $u \in [0,D)$. For fixed i and k the map $u \mapsto V_iu + A_i - kD$ is affine with *positive* slope, hence monotone; so if it lies in $[-C,C]$ at both endpoints of an interval it lies in $[-C,C]$ throughout. Consequently a cover chain, whose steps march the frontier from 0 to at least D, exhibits for each $u \in [0,D)$ some runner i and some integer k with $|V_iu + A_i - kD| \le C$, whence $\|s_i + V_i(u/D)\| \le |s_i + V_i(u/D) - k| \le C/D$ and therefore $G_{V,s}(u/D) \le C/D$. This monotonicity is the whole trick: no case analysis on the nearest integer is needed anywhere.

(b) At $t = P/D$ one must bound $\|s_i + V_i t\|$ from below, an assertion about *all* integers k. It collapses to one Euclidean division: if $m = V_i P + A_i$ satisfies $m \bmod D \geq C$ and $D - (m \bmod D) \geq C$, then $|m - kD| \geq C$ for every $k \in \mathbb{Z}$, by splitting on the sign of $\lfloor m/D \rfloor - k$. Dividing by D gives $\|s_i + V_i(P/D)\| \geq C/D$ for each i, hence $\gamma(V; s) \geq G_{V,s}(P/D) \geq C/D$.
(c) is (a) and (b) together with antisymmetry. □

The certificates are, of course, verified rather than derived: a cover chain is produced by an external search and then checked. The check is finite and exact, and its correctness — the content of Theorem 3.2 — is proved once and for all, so no claim below depends on the search that produced the data.

Passing from γ to $\gamma^{\min}$ costs only the observation that one admissible shift bounds a minimum over all of them; note that only the *cover* half is needed.

Proposition 3.3. *$\gamma^{\min}$ is well defined, $\gamma^{\min}(v) \geq 0$, and $\gamma^{\min}(v) \leq \gamma(v; s)$ for every $s \in [0,1]^n$. In particular, if $D > 0$, $V \in \mathbb{Z}^n_{>0}$, $0 \leq A_i \leq D$ for all i, and a cover chain for (D, V, A, C) exists, then $\gamma^{\min}(V) \leq C/D$.*

Proof. The set $\{\gamma(v; s) : s \in [0,1]^n\}$ is nonempty and bounded below by 0, so its infimum is a real number and is a lower bound for each of its elements; that gives the first three claims. For the last, $0 \leq A_i \leq D$ places $s = A/D$ in $[0,1]^n$, and Theorem 3.2(a) bounds $\gamma(V; s)$. □

Finally, the supremum in the definition of γ is attained, and when a certificate pair passes the certificate itself names a maximiser.

Proposition 3.4. *Let $v, s \in \mathbb{R}^n$.*

(a) *$G_{v,s}$ is Lipschitz with constant $\sum_i |v_i|$, hence continuous.*
(b) *If $v \in \mathbb{Z}^n$, there is $t \in [0,1]$ with $G_{v,s}(t) = \gamma(v; s)$: the supremum is a maximum.*
(c) *For every $N \geq 1$ and every t there is $u \in \mathbb{Z}$ with $|t - u/N| \leq 1/(2N)$ and $G_{v,s}(t) \leq G_{v,s}(u/N) + M/(2N)$, where $M \geq \max_i |v_i|$.*
(d) *If $D > 0$, $V \in \mathbb{Z}^n_{>0}$ and both a cover chain and a witness P pass for (D, V, A, C), then $G_{V,A/D}(P/D) = \gamma(V; A/D)$. In particular the maximum is attained at a rational time whose denominator divides the certificate's own D.*

Proof sketch. (a) $\|\cdot\|$ is 1-Lipschitz, since $\|x\| \leq |x - \mathrm{round}(y)| \leq |x - y| + \|y\|$; a minimum of finitely many $|v_i|$-Lipschitz functions is $\max_i |v_i|$-Lipschitz, and $\max_i |v_i| \leq \sum_i |v_i|$. (b) For integer speeds $G_{v,s}$ is 1-periodic, so its supremum over $\mathbb{R}$ equals its supremum over the compact $[0,1]$, where by (a) it is attained. (c) Take $u = \mathrm{round}(tN)$ and apply the Lipschitz bound. (d) The cover bounds G by C/D at *every* real time, and the witness computation of Theorem 3.2(b) gives $G_{V,A/D}(P/D) \geq C/D$; so the value C/D is both the supremum and the value at P/D. □

Remark 3.5. Part (d) is a statement about instances that carry a certificate, and it is proved without any enumeration of critical points. The general statement — that for $V \in \mathbb{Z}^n$ and $s \in (1/D)\mathbb{Z}^n$ some maximiser has denominator dividing $D \cdot \mathrm{lcm}_{i<j}(V_i + V_j)$, because a local maximum of a minimum of V-shaped functions occurs where a rising branch meets a falling one — is what the search of Section 5 uses to compute γ exactly, but it is not needed for, and is not part of, any claim proved here.

4. Explicit shift vectors for $7 \leq n \leq 17$

We now state the main result. For n in the stated range let $v = (1, 2, \ldots, n)$ and let $s^{(n)}$ be the shift vector recorded in Table 1, written there as $(1/D)(a_1, \ldots, a_n)$ with $0 \leq a_i < D$.

Theorem 4.1. *For each n with $7 \leq n \leq 17$,*

$$\gamma\big((1, 2, \ldots, n); s^{(n)}\big) \;=\; c_n, \qquad \text{where} \qquad c_n = \begin{cases} 25/214, & n = 7, \\ 1/10, & n = 8, \\ 1/(n+2), & 9 \leq n \leq 14, \\ 2/(2n+3), & 15 \leq n \leq 17, \end{cases}$$

and in each case $c_n < 1/(n+1)$.

n	D shift numerators $(a_1, \dots, a_n)$	$\gamma = c_n$	$1/(n+1)$	t_n
7	214 $(0, 117, 145, 106, 62, 1, 198)$	25/214	1/8	67/214
8	10 $(0, 4, 3, 3, 4, 2, 3, 1)$	1/10	1/9	3/20
9	22 $(0, 8, 9, 9, 15, 9, 0, 2, 17)$	1/11	1/10	1/8
10	24 $(0, 16, 15, 14, 12, 13, 2, 22, 3, 18)$	1/12	1/11	1/3
11	26 $(0, 16, 16, 16, 13, 15, 3, 24, 4, 20, 24)$	1/13	1/12	2/13
12	14 $(0, 9, 8, 8, 6, 7, 0, 12, 0, 9, 8, 6)$	1/14	1/13	1/14
13	30 $(0, 19, 19, 18, 9, 17, 28, 28, 17, 23, 27, 7, 1)$	1/15	1/14	1/12
14	32 $(0, 13, 13, 18, 18, 18, 30, 30, 0, 14, 28, 30, 28, 2)$	1/16	1/15	1/16
15	33 $(0, 14, 14, 15, 22, 15, 5, 5, 32, 28, 7, 3, 5, 29, 16)$	2/33	1/16	17/99
16	35 $(0, 24, 23, 23, 20, 22, 4, 33, 10, 29, 0, 23, 31, 2, 33, 2)$	2/35	1/17	1/5
17	37 $(0, 12, 15, 13, 22, 15, 36, 1, 34, 33, 33, 13, 33, 31, 2, 33, 36)$	2/37	1/18	8/37

TABLE 1. The shift vectors of Theorem 4.1. Speeds are $v = (1, 2, \dots, n)$ and $s^{(n)} = (1/D)(a_1, \dots, a_n)$. The last column is the time at which the supremum defining γ is attained.

n	D_n^*	cover steps	n	D_n^*	cover steps
7	17 976	18	13	1 801 800	51
8	600	22	14	2 882 880	56
9	27 720	27	15	3 963 960	63
10	15 120	32	16	25 225 200	71
11	360 360	37	17	453 332 880	79
12	388 080	43			

TABLE 2. Size of the certificates behind Theorem 4.1: the common denominator D_n^* over which all data is cleared, and the length of the cover chain.

Corollary 4.2. *For each n with $7 \le n \le 17$:*

(a) $\gamma^{\min}(1, 2, \dots, n) \le c_n < 1/(n+1)$*, and in particular* sLRC(n) *fails;*

(b) *the supremum defining $\gamma\big((1, \dots, n); s^{(n)}\big)$ is attained at the rational time t_n listed in Table 1.*

Proof. (a) Each $s^{(n)}$ lies in $[0, 1)^n \subseteq [0, 1]^n$, so Proposition 3.3 gives the first inequality; the speeds $1, 2, \dots, n$ are nonzero with distinct absolute values, so Proposition 2.2 gives the failure of sLRC(n). That failure is of course not new — it is Theorem 1.17 of [2] — and is restated here only because it now rests on a witness the reader can check. (b) The number t_n is precisely the witness time P/D_n^* of the certificate proving Theorem 4.1 at that n, so Proposition 3.4(d) applies. These times are printed in neither [2] nor [3]. □

Proof of Theorem 4.1. Each of the eleven cases is an instance of Theorem 3.2(c) and Proposition 3.4(d). For each n one fixes a common denominator D_n^* (Table 2), rescales the speeds, the shift $s^{(n)}$ and the target level c_n over it, and exhibits a cover chain and a witness time; the entire verification is then a finite list of integer inequalities and n Euclidean divisions, and the numbers involved — D_n^* runs from 17 976 at $n = 7$ to 453 332 880 at $n = 17$, cover chains from 18 to 79 steps — stay well inside machine-checkable range. The strict inequalities $c_n < 1/(n+1)$ are comparisons of explicit rationals.

Two points deserve emphasis. First, the cover chains and witness times are the entire content of the proof: they were produced by the search of Section 5, but the correctness of a chain does not depend on how it was found. Second, no step of the verification enumerates candidate times. The cover bounds $G_{V,s}$ at every real t simultaneously, by the monotonicity argument of Theorem 3.2(a); the only "exhaustive" ingredient is the finite check of the chain itself. □

Remark 4.3 (what the range $15 \le n \le 17$ does and does not say). For $n = 15, 16, 17$, Theorem 4.1 gives the first explicit shift vectors we know of, and the bounds $\gamma^{\min} \le 2/33,\ 2/35,\ 2/37$. These are strictly better than the bound $1/(n+1)$ that [2] certifies, but strictly worse than the level $1/(n+2)$ that [2] reports for $9 \le n \le 14$: indeed $1/17 < 2/33 < 1/16$, and similarly at $n = 16, 17$. So the natural guess

— that $\gamma^{\min}(1,\dots,n) < 1/(n+2)$ continues past $n = 14$ — is untouched by this paper. Our search reached the level $1/(n+2)$ at $n = 9,\dots,14$ and at no $n \geq 15$; see Section 5 for the numbers, which record a limitation of the search and nothing more.

5. The search

Theorem 4.1 does not depend on this section: a certificate is checked, not trusted. We describe the search anyway, because it is what produced the data and because its negative outcomes are the honest boundary of the results.

The obvious approach — simulated annealing on $s \mapsto \gamma((1,\dots,n);s)$ directly, with γ computed exactly by enumerating the local maxima of a minimum of V-shaped functions — works for small n and produced the vectors at $n = 7$ and $n = 8$. It stalls from about $n = 9$ because γ is badly plateaued: almost every single-coordinate move leaves the value unchanged, so there is no gradient to follow.

The search that scales anneals instead on a surrogate with a real gradient. Fix a target level c; the *uncovered measure* of a shift s at level c is the Lebesgue measure of $\{t \in [0,1] : G_{v,s}(t) > c\}$, and it vanishes exactly when $\gamma(v;s) \leq c$. Annealing this quantity to zero therefore produces a certificate-ready witness at level c, and every success is a refutation by construction as soon as $c < 1/(n+1)$. Levels were taken along the ladder $c = k/\big(k(n+1)+1\big)$, $k = 1, 2, \dots$, which increases from $1/(n+2)$ towards $1/(n+1)$, with s restricted to grids of denominator $m \cdot \big(k(n+1)+1\big)$.

The outcome, cell by cell: for every n with $9 \leq n \leq 14$ the first rung $k = 1$, that is $c = 1/(n+2)$, succeeded on a grid of denominator $n+2$ or $2(n+2)$ — which is exactly the range over which [2] observes $\gamma^{\min}(1,\dots,n) < 1/(n+2)$. At $n = 15, 16, 17$ the first rung failed on all three grids tried, the uncovered measure bottoming out at 6.5×10^{-4}, 1.1×10^{-3} and 2.9×10^{-3} respectively; success came at $k = 2$, that is $c = 2/(2n+3)$, which is the source of the values 2/33, 2/35, 2/37 in Table 1. (The two smaller cells came from the direct search instead, and the ladder says nothing about them: at $n = 7$ the rung $k = 1$ is not available at all, since $1/9 < 25/214 = \gamma^{\min}(1,\dots,7)$, so no shift attains it; and at $n = 8$ that rung sits exactly at the optimum, $1/10 = \gamma^{\min}(1,\dots,8)$.)

Two further runs are worth recording, both negative.

- At $n = 9$ we tried to reach the published optimum 5/58 on the grid of denominator 58. Both the annealer and a deterministic exhaustive coordinate sweep — every value of every coordinate, iterated to convergence — stalled at uncovered measure 6.16×10^{-4}. The explanation appeared later, from the data of [3]: the optimal shift at $n = 9$ needs denominator 29 232, so no search on the 1/58 grid can reach it. The same is true further along — at $n = 11$ the optimum needs denominator 1 422 960 while $\gamma^{\min} = 1/14$.
- At $n = 4$, where the shifted conjecture is a theorem [1], the same annealer was run on five grids (denominators 5, 20, 60, 210, 420) and never returned a value below $1/5 = 1/(n+1)$. A search that manufactured counterexamples wherever pointed would have produced one here.

Neither of these is a theorem, and neither is quoted anywhere above as evidence about $\gamma^{\min}$.

6. The published data, reproduced

6.1. **The three printed instances.** [2] prints an attaining shift vector in exactly three places, its Propositions 5.10 and 5.11. All three are reproduced exactly, together with their maximising times, which [2] does not print.

Theorem 6.1. *With $\|\cdot\|$ and γ as above,*

$$\gamma\big((1,2,3,4,5);\tfrac{1}{94}(0,46,38,47,72)\big) = \tfrac{15}{94},$$
$$\gamma\big((2,3,4,5,6,8);\tfrac{1}{30}(0,29,17,0,16,22)\big) = \tfrac{2}{15},$$
$$\gamma\big((1,2,3,4,5,6);\tfrac{1}{67}(0,39,37,51,62,54)\big) = \tfrac{9}{67}.$$

Proof sketch. Three applications of Theorem 3.2(c), over the common denominators 2820, 1800 and 4020, with cover chains of 11, 19 and 13 steps. The speeds and shifts are quoted verbatim from [2]; the values 15/94, 2/15, 9/67 agree with the ones it states. The three witness times used are 63/188,

n	$g_n = \gamma^{\min}(1,\dots,n)$ from [2]	common denom.	cover steps	maximising time
5	15/94	11 280	11	1413/3760
6	9/67	4 020	13	69/268
7	25/214	629 160	18	4397/12840
8	1/10	67 200	23	1/10
9	5/58	29 232	27	101/348
10	27/349	4 397 400	35	18961/209400
11	1/14	4 268 880	41	11/48
12	6/91	7 567 560	46	42241/540540
13	1/16	74 954 880	55	3001/40560

TABLE 3. The values of $\gamma^{\min}(1,\dots,n)$ tabulated in [2], certified here as upper bounds with the explicit witnesses of [3]. Maximising times are computed here and are printed in neither source. The table of [2] also lists $1/3, 1/4, 1/5$ at $n = 2, 3, 4$ and $71/1228$ at $n = 14$; no attaining vector for $n = 14$ is published in either source.

17/120 and 37/134, so by Proposition 3.4(d) these are the maximisers; they are computed here and are not printed in [2]. Since each shift lies in $[0,1)^n$, Proposition 3.3 turns the three equalities into $\gamma^{\min}(1,2,3,4,5) \le 15/94$, $\gamma^{\min}(2,3,4,5,6,8) \le 2/15$ and $\gamma^{\min}(1,\dots,6) \le 9/67$; [2] asserts equality in all three, which we do not prove. □

6.2. **The tabulated values, as upper bounds with witnesses.** The optimal shift vectors behind the table of [2] are published, for $n \le 13$, in its companion code repository [3], as records giving the velocities, the starting positions in exact rational arithmetic, and the resulting gap. Each record was recomputed here from the definition of γ, in exact rational arithmetic, before any formal proof was written: all twelve records $n = 2,\dots,13$ reproduce the gap they state, and those with $n \ge 5$ land on the tabulated value.

Theorem 6.2. *Let $s^{*(n)}$ denote the shift vector recorded in* [3] *for $v = (1,2,\dots,n)$. Then for $5 \le n \le 13$,*

$$\gamma\big((1,\dots,n); s^{*(n)}\big) \;=\; g_n, \qquad (g_5,\dots,g_{13}) = \Big(\tfrac{15}{94}, \tfrac{9}{67}, \tfrac{25}{214}, \tfrac{1}{10}, \tfrac{5}{58}, \tfrac{27}{349}, \tfrac{1}{14}, \tfrac{6}{91}, \tfrac{1}{16}\Big),$$

each $s^{*(n)}$ lies in $[0,1]^n$, and consequently $\gamma^{\min}(1,\dots,n) \le g_n$ for $5 \le n \le 13$. In each case the supremum is attained at the explicit rational time listed in Table 3.

Proof sketch. As before: one application of Theorem 3.2(c) per cell, with the common denominators and cover lengths of Table 3, followed by Proposition 3.3. The largest instance, $n = 13$, clears everything over $D = 74\,954\,880$ and uses a cover of 55 steps. □

Remark 6.3. Theorem 6.2 certifies the table of [2] in one direction only. The reverse inequality $\gamma^{\min}(1,\dots,n) \ge g_n$, which is what makes those entries equalities, quantifies over all $s \in [0,1]^n$ and is exactly what the polytrope computation of [2] establishes; no certificate of the shape used here can reach it. The one tabulated value we cannot even bound this way is $n = 14$: [3] stops at $n = 13$, so no attaining vector for 71/1228 is published, and Theorem 4.1 leaves $\gamma^{\min}(1,\dots,14) \le 1/16$ as the best explicit bound here.

6.3. **Cross-checks.** Four of the cells of Theorem 6.2 coincide with cells for which a different shift vector was already available, which gives independent confirmation that the function computed here is the one [2] defines.

Proposition 6.4. *At $n = 5$ and $n = 6$ the shift vector of* [3] *differs from the one printed in* [2]*, and at $n = 7$ and $n = 8$ it differs from the one found here (Table 1); in all four cases the two vectors have the same loneliness gap. Moreover, for each n with $9 \le n \le 13$,*

$$\gamma\big((1,\dots,n); s^{*(n)}\big) \;<\; \gamma\big((1,\dots,n); s^{(n)}\big),$$

so the cells of Theorem 6.2 strictly improve on those of Theorem 4.1.

Proof. Each equality of gaps is a comparison of two values already computed exactly; each inequality compares g_n with $1/(n+2)$; and the vectors are seen to differ by comparing one coordinate, e.g. at $n = 8$ the fourth coordinate is $19180/67200$ against $180/600$. □

6.4. **The full range of failures.** Collecting Corollary 4.2 with the instances at $n = 5, 6$ of Theorem 6.1 gives the following, which is Theorem 1.17 of [2] and is stated here in the form in which it has been machine-checked.

Theorem 6.5. sLRC(n) *is false for every* n *with* $5 \le n \le 17$. *Each of the thirteen failures is witnessed by an explicit shift vector whose loneliness gap is computed exactly.*

7. Controls

A certificate scheme that was too generous in either direction would prove all of the above and prove nothing. The controls below are of three kinds, and each is a theorem.

7.1. **The classical anchors.** The first control is that the machinery returns the right answer on the instances whose value is classical and not in dispute.

Theorem 7.1. *For every* n *with* $2 \le n \le 14$, $\gamma\big((1, 2, \dots, n); 0\big) = 1/(n+1)$.

Proof sketch. Thirteen applications of Theorem 3.2(c), one per n; the witness time used is $t = 1/(n+1)$ in every case, so that is the maximiser by Proposition 3.4(d). Mathematically this is classical: at $t = 1/(n+1)$ runner i sits at $i/(n+1)$, at distance at least $1/(n+1)$ from $\mathbb{Z}$ with equality at $i = 1$ and $i = n$, and no time does better; the statement is the tightness of the unshifted Lonely Runner Conjecture at $v = (1, \dots, n)$, the instance that fixes the constant $1/(n+1)$ in the conjecture of Wills [5]. It is reproved here purely as validation, and its force is comparative: the *same* certificate scheme on the *same* speed vectors, with a nonzero shift, lands strictly below $1/(n+1)$ from $n = 5$ on. Anything that made the scheme systematically optimistic would have to break these equalities first. The values at $n = 2, 3, 4$ agree with the entries $1/3, 1/4, 1/5$ tabulated in [2]. □

We stress that nothing in this paper bears on the unshifted Lonely Runner Conjecture, which is a different and much harder statement; it is known for up to 13 runners [4, Thm. 1.3].

7.2. **Neither half of the certificate can be satisfied past the truth.** The next control refutes each half of the scheme one integer unit beyond the true value. Note that these are genuine universally quantified statements — over *all* cover chains of *all* lengths, and over *all* witness times — rather than finite searches: they follow from the exact value, by Theorem 3.2 run backwards.

Theorem 7.2. *Let* $V = (1, 2, 3, 4, 5)$, $A = (0, 1380, 1140, 1410, 2160)$, $D = 2820$, *so that* A/D *is the shift vector of* [2] *at* $n = 5$ *and* $\gamma(V; A/D) = 450/2820 = 15/94$. *Then:*

(a) *no cover chain for* $(D, V, A, 449)$ *exists, of any length, using any runners and any integers* k*; in particular the chain that certifies level* 450 *fails at level* 449*, and the chain obtained from it by moving a single breakpoint by one unit (from* 710 *to* 711*) fails already at level* 450*;*
(b) *no integer* P *is a witness for* $(D, V, A, 451)$*; in particular the true maximiser* $t = 63/188$ *is not, and the maximiser moved by one unit of* $1/2820$ *is not a witness at level* 450 *either;*
(c) $G_{V,A/D}(0) = 0$*;*
(d) $\gamma(V; A/D) < \gamma(V; 0) = 1/6$*: the refutation is a property of the shift, not of the speeds;*
(e) *moving one coordinate of the shift by one unit of* $1/94$*, from* $46/94$ *to* $47/94$*, changes the gap from* $15/94$ *to exactly* $1/6$*, so the perturbed vector is not a counterexample.*

Proof. (a) A cover chain at level 449 would give $\gamma(V; A/D) \le 449/2820$ by Theorem 3.2(a), contradicting $\gamma(V; A/D) = 450/2820$; (b) symmetrically with Theorem 3.2(b). The concrete instances in (a), (b) and the assertion (c) are finite integer computations. For (d) and (e), $\gamma(V; 0) = 1/6$ is Theorem 7.1 at $n = 5$ and the perturbed value $1/6$ is a further application of Theorem 3.2(c), over $D = 8460$ with an 8-step chain. □

Theorem 7.3. *The analogues of Theorem 7.2(a),(b) hold at the cells $n = 9$ and $n = 13$ of Theorem 6.2, with units $1/29\,232$ and $1/74\,954\,880$ respectively; concretely, the $n = 9$ chain with one breakpoint moved by a single unit no longer covers, and the maximising time moved by one unit is no longer a witness. Moreover:*

(a) (specificity) *at $n = 4$, where the shifted conjecture is a theorem* [1] *and* [2] *tabulates $\gamma^{\min}(1, 2, 3, 4) = 1/5$, the same machinery returns $\gamma^{\min}(1, 2, 3, 4) \leq 1/5$; since $1/5 < 1/5$ is false, no refutation follows;*
(b) (non-vacuity) $\gamma^{\min}(1, \ldots, 9) < 1/10$ *and* $\gamma^{\min}(1, \ldots, 9) \geq 0$*;*
(c) *the hypothesis $0 \leq A_i \leq D$ of Proposition 3.3 is a genuine restriction: it fails for an explicit A at $n = 9$.*

Proof. As in Theorem 7.2, from the exact values of Theorem 6.2 and Theorem 7.1, together with Proposition 3.3. Item (b) records that the infimum defining $\gamma^{\min}$ is taken over a nonempty set, so that the bounds proved above are bounds on a real number and are not vacuous. □

8. Formal verification

Every theorem and proposition in this paper — the certificate scheme of Section 3, the eleven instances of Theorem 4.1, the reproductions of Section 6 and all of the controls of Section 7 — has been formalised and checked in Lean 4 [6] against Mathlib [7]. Corollary 4.2 is the one place where a composition is left to the reader: its two halves are Proposition 3.3 and Proposition 3.4(d), both formalised, applied to the certificate data of Theorem 4.1, which is formalised too; the composite itself is not a separate Lean theorem. The definitions of $\|\cdot\|$, $G_{v,s}$, γ, $\gamma^{\min}$ and $\mathrm{sLRC}(n)$ are transcribed directly from [2, Conj. 1.1 and Def. 1.3], with γ a supremum over all of $\mathbb{R}$ and $\gamma^{\min}$ an infimum over the closed box $[0, 1]^n$, so that no finiteness is built into the statements being proved. The development contains 91 theorems in the modules carrying the definitions, the anchors, the witnesses of Theorem 4.1 and the controls, and a further 89 in the module carrying the cells of Theorem 6.2; it is free of `sorry`, and the axioms used are the three standard ones (propositional extensionality, choice, and quotient soundness). Every arithmetic claim is discharged by the Lean kernel's own evaluator rather than by compiled code: clearing all data over a single denominator keeps every check in $\mathbb{Z}$, so no rational normalisation — and in particular no greatest-common-divisor computation — ever has to be unfolded, which is what makes kernel checking of the largest instance ($n = 17$, denominator $453\,332\,880$, cover of 79 steps) inexpensive. The whole development compiles in a few minutes, with a peak resident set of about 2.6 GB, nearly all of which is the library import rather than the checking: the module carrying the cells of Theorem 6.2 adds about 190 MB and five seconds on top of the modules it is built on. Repository reference to be added at submission.

9. Open questions

(1) *The lower bound.* Everything here certifies $\gamma^{\min}(v) \leq c$ from a single shift vector. A certificate for $\gamma^{\min}(v) \geq c$ must quantify over all $s \in [0, 1]^n$; finding a finite certificate shape for that is the natural next step, and would turn Theorem 6.2 into a verification of the table of [2] as stated.
(2) *The missing cell $n = 14$.* The value $\gamma^{\min}(1, \ldots, 14) = 71/1228$ is tabulated in [2], but no attaining shift vector for it appears in [2] or [3]. An explicit witness would complete Theorem 6.2 over the whole tabulated range.
(3) *Beyond $n = 14$.* Is $\gamma^{\min}(1, \ldots, n) < 1/(n + 2)$ for $n = 15, 16, 17$? Theorem 4.1 gives only $2/(2n + 3)$, and the search of Section 5 says nothing about the truth.
(4) Two matters [2] leaves open and that are untouched here: whether its list of counterexamples at $n = 7$ is complete (it believes the $n = 6$ list is), and how few vectors a configuration failing the Lonely Vector Property can have — [2] exhibits failures at $|S| = 12$ and $|S| = 16$, and claims no lower bound.

References

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