ONE-AND-A-HALF GENERATION OF SIMPLE LIE ALGEBRAS OVER FINITE FIELDS: THE VALUE $k(\mathfrak{sl}_3(\mathbb{F}_2)) = 3$

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Abstract. For a simple Lie algebra L over a finite field, Kishnani and Singh define $k(L)$ to be the least k such that *every* non-zero $a \in L$ occurs as the first entry of some generating k-tuple, and observe that no clear answer for $k(L)$ seems to be known; the number is the number of variables in their theorem on images of Lie polynomials. We determine one value:

$$k(\mathfrak{sl}_3(\mathbb{F}_2)) \;=\; 3.$$

The upper bound is a generating triple for each of the 255 non-zero elements. The lower bound is structural rather than a report that a search failed: ten proper Lie subalgebras of $\mathfrak{sl}_3(\mathbb{F}_2)$, each containing the root element E_{01}, already cover the algebra, so every pair containing E_{01} lies inside a proper subalgebra. We also determine the exceptional set exactly. Precisely 49 of the 255 non-zero elements lie in no generating pair, namely the 21 elements with $A^2 = 0$ and the 28 with $A^2 = A$, equivalently the a with $\mathrm{ad}(a)^3 = \mathrm{ad}(a)^2$; the remaining 206 each have an explicit partner. The mechanism is characteristic two: if $A^2 = \lambda A$ then $\mathrm{ad}(A)^2 = \lambda\,\mathrm{ad}(A)$, because the cross term $2AXA$ vanishes. In particular $\mathfrak{sl}_3(\mathbb{F}_2)$ is 2-generated, as Cantor, Jezernik and Zozaya proved, and is nevertheless not one-and-a-half generated: the two properties come apart at the smallest algebra where they can be compared. All statements are machine-checked in Lean 4.

1. Introduction

Let L be a finite-dimensional simple Lie algebra over a finite field. How many elements are needed to generate L, and can one of them be prescribed in advance? The two questions are genuinely different, and the second is the one studied here.

For finite simple *groups* the answer to the second question is classical: by the theorem of Guralnick and Kantor [6], every finite simple group is *one-and-a-half generated* – for every non-identity element x there is a y with $\langle x, y\rangle = G$. For Lie algebras the corresponding statement is known in two settings, both of them over an algebraically closed field of large or zero characteristic. Ionescu [7] proved it for finite-dimensional complex simple and real semisimple Lie algebras. Bois [1], Theorem 2.2.3, proved: *let F be an algebraically closed field of characteristic 0 or $p > 3$ and $\mathfrak{g}$ a classical simple Lie algebra over F; then $\mathfrak{g}$ is generated by one and a half elements.* In the sequel [2] he showed that simple Lie algebras of Cartan type S, H and K never have the property, again over an algebraically closed field of characteristic $\neq 2, 3$. The notion is current: Beldiev and Pogudin [3], working over a field of characteristic zero, call an algebra $\mathcal{L}$ 1.5-*generated* when for every non-zero $x \in \mathcal{L}$ there is a y with $\langle x, y\rangle = \mathcal{L}$, prove that the Lie algebra of polynomial vector fields of constant divergence on affine space has the property, and record Ionescu's theorem as the finite-dimensional state of the art.

Over a finite field, and in small characteristic, nothing of this covers the question. Kishnani and Singh [8] isolate it and give it a name. For L simple over a finite field they set, in §2.3 of [8],

> "For any element $x \in L$, a simple Lie algebra over a finite field F, does there exist a generating set $x_1 = x, x_2, \dots, x_k$? If it exists, the smallest such k is denoted as $k(L)$. Clearly, a vector space basis of L would work as k, hence $k(L) \leq \dim(L)$. In fact, because of root space decomposition for a simple Lie algebra, a better bound could be taken as the number of roots. For $\mathfrak{sl}_n(q)$, it has been proved in [7] that this Lie algebra is 2-generated. However, a clear answer to $k(L)$ seems not to be known."

and, in §1,

> "The number $k(L)$ is a finite number because of dimension reasons, although an absolute constant is desirable. For finite simple groups, it is 2 what is called one-and-a-half generation, but it doesn't seem to be known for all finite simple Lie algebras (over an algebraically closed field, it is known to be 2)."

The number is not a curiosity of their exposition: $k(L)$ is the number of variables in their Theorem 1.3, which says that every subset $A \subset L$ containing 0 and closed under automorphisms of L is the image of some Lie polynomial in $k(L)$ variables. Its value is therefore load-bearing there.

Reference [7] of [8] is the work of Cantor, Jezernik and Zozaya [4], who prove that $\mathfrak{sl}_n(\mathbb{F}_q)$ is 2-generated – *some* pair generates – with exactly two exceptions, $(n, p) = (3, 3)$ and $(4, 2)$. That is a strictly weaker statement than $k(L) = 2$: it prescribes neither entry of the pair. One of the points of this paper is that the gap between the two is real, and is already visible in dimension 8.

Results. We determine $k(L)$ for $L = \mathfrak{sl}_3(\mathbb{F}_2)$, the Lie algebra of trace-zero 3×3 matrices over the field with two elements. We are not aware of any value of $k(L)$ recorded in the literature for any L.

Theorem 1.1 (Theorem 5.1)**.** $k(\mathfrak{sl}_3(\mathbb{F}_2)) = 3$. *That is: every non-zero $a \in \mathfrak{sl}_3(\mathbb{F}_2)$ lies in a generating triple, and there is a non-zero a – for instance the root element E_{01} – lying in no generating pair.*

The lower bound is not the outcome of a search that failed to find a partner. It is carried by a family of proper subalgebras: $\mathfrak{sl}_3(\mathbb{F}_2)$ has exactly 42 maximal proper Lie subalgebras, and the ten of them that contain E_{01} already cover $\mathfrak{sl}_3(\mathbb{F}_2)$, so for every b some proper subalgebra contains both E_{01} and b. We also determine exactly which elements are obstructed.

Theorem 1.2 (Theorems 6.1 and 6.3)**.** *Exactly* 49 *of the* 255 *non-zero elements of* $\mathfrak{sl}_3(\mathbb{F}_2)$ *lie in no generating pair; each of the other* 206 *lies in one. For non-zero $a \in \mathfrak{sl}_3(\mathbb{F}_2)$ with matrix A, the following are equivalent: a lies in no generating pair; $A^2 = 0$ or $A^2 = A$;* $\mathrm{ad}(a)^3 = \mathrm{ad}(a)^2$*. The* 49 *split as* 21 *matrices of rank one with $A^2 = 0$ and* 28 *of rank two with $A^2 = A$.*

The description explains itself in characteristic two. If $A^2 = \lambda A$ then, over any field of characteristic 2,

$$\mathrm{ad}(A)^2(X) \;=\; A^2X - 2AXA + XA^2 \;=\; \lambda(AX + XA) \;=\; \lambda\; \mathrm{ad}(A)(X),$$

the cross term $2AXA$ dying; so $\mathrm{ad}(A)$ satisfies $t^2 = \lambda t$ and in particular $\mathrm{ad}(A)^3 = \mathrm{ad}(A)^2$ (Lemma 6.2). In any other characteristic the cross term survives. The phenomenon is therefore one of characteristic and not of field size, and the computations of §9 agree: over $\mathbb{F}_4$, still of characteristic two, the root element again has no partner, while over $\mathbb{F}_5$ and $\mathbb{F}_7$ it has one.

Finally, $k(L) = 3$ must not be read as "$\mathfrak{sl}_3(\mathbb{F}_2)$ needs three generators". It does not: $\mathfrak{sl}_3(\mathbb{F}_2)$ is 2-generated, as [4] predicts, and we exhibit a generating pair (Proposition 7.1). Both statements hold at once, which is exactly the assertion that 2-generation and one-and-a-half generation are different properties.

Why this algebra. Theorem 1.3 of [8] concerns a simple Chevalley algebra over a finite field of *very good* characteristic; for type A_{n-1} that condition reads $p \nmid n$, the general theory of Chevalley algebras being the one [8] takes from [5]. For $n = 3$, $p = 2$ we have $p \nmid n$, so $\mathfrak{sl}_3(\mathbb{F}_2)$ is in the scope of the source.

Two properties of $\mathfrak{sl}_3(\mathbb{F}_2)$ are used here as quoted background: that its centre is trivial, and that it is simple. Both are standard, both follow from $p \nmid n$, and neither is proved or machine-checked in this work. The centre of $\mathfrak{sl}_n(\mathbb{F}_q)$ consists of the trace-zero scalar matrices λI, that is of those λ with $n\lambda = 0$, so it is zero precisely when $p \nmid n$; and for $p \nmid n$ the algebra $\mathfrak{sl}_n(\mathbb{F}_q)$ is simple, of dimension $n^2 - 1$. It is the failure of that condition which [8] has in mind when it allows its algebras to be simple only modulo their centre. Nothing below depends on either property: every theorem in this paper is a statement about the explicit 8-dimensional bracket algebra fixed in §2, and what simplicity does is place that algebra inside the class for which [8] defines $k(L)$.

Being 8-dimensional over $\mathbb{F}_2$, $\mathfrak{sl}_3(\mathbb{F}_2)$ has $2^8 = 256$ elements, which is what makes $k(L)$ a finite question one can settle completely. The neighbouring cells $\mathfrak{sl}_3(\mathbb{F}_3)$ and $\mathfrak{sl}_4(\mathbb{F}_2)$ have $p \mid n$, are outside that scope, and are precisely the two cases where [4] shows 2-generation itself fails.

Organisation. Section 2 fixes the algebra and the invariant and records the explicit model that the machine checks use. Sections 3 and 4 prove the two halves of Theorem 1.1, which are combined in §5. Section 6 determines the exceptional set. Section 7 records the verification of 2-generation and the resulting gap, and §8 the sanity checks that make the certificates of the two directions non-vacuous. Section 9 reports computations for neighbouring algebras that are not part of the formal development, and §10 describes the verification.

2. The algebra and the invariant

Throughout, $\mathbb{F}_2$ is the field with two elements and

$$L \;=\; \mathfrak{sl}_3(\mathbb{F}_2) \;=\; \{A \in \mathfrak{gl}_3(\mathbb{F}_2) \;:\; \operatorname{tr} A = 0\}, \qquad [A,B] = AB - BA = AB + BA,$$

an 8-dimensional Lie algebra with 256 elements. We write E_{ij} $(0 \le i \ne j \le 2)$ for the matrix units and

$$H_1 = \operatorname{diag}(1,1,0), \qquad H_2 = \operatorname{diag}(0,1,1),$$

which over $\mathbb{F}_2$ are the images of $\operatorname{diag}(1,-1,0)$ and $\operatorname{diag}(0,1,-1)$. The eight elements

$$E_{01},\ E_{02},\ E_{10},\ E_{12},\ E_{20},\ E_{21},\ H_1,\ H_2$$

form a basis. Note that $-1 = 1$ here, so the bracket is symmetric, $[X,Y] = [Y,X]$, and antisymmetry carries no information; the substantive law is that the bracket is *alternating*, $[X,X] = 0$.

Definition 2.1. For a subset $X \subseteq L$, $\langle X\rangle$ denotes the Lie subalgebra generated by X: the smallest subset of L containing 0 and X and closed under addition and under the bracket. (Over $\mathbb{F}_2$ scalar multiplication is trivial, so a subspace closed under brackets is exactly a subset containing 0 and closed under those two operations.) A tuple $(a_1,\dots,a_k)$ *generates* L if $\langle a_1,\dots,a_k\rangle = L$.

Definition 2.2 (Kishnani–Singh [8])**.** For L simple over a finite field, $k(L)$ is the least k such that for every non-zero $a \in L$ there exist $a_2,\dots,a_k \in L$ with $\langle a, a_2,\dots,a_k\rangle = L$. We call a, b a *generating pair* when $\langle a,b\rangle = L$, and say that b is a *partner* for a. Thus $k(L) = 2$ says exactly that L is one-and-a-half generated.

Adjoining further elements to a generating set leaves it generating, so the condition in Definition 2.2 is monotone in k and $k(L)$ is well defined; $k(L) \le \dim L$ because a basis generates.

The certificates below are finite computations, and they are run on an explicit model of L in which an element is an 8-bit integer − the coordinate vector in the basis above − addition is bitwise exclusive or, and the bracket is the bilinear extension of a literal table of structure constants. The following proposition is what ties that model to the matrices, and is proved by exhaustive evaluation.

Proposition 2.3. *Let μ be the map sending a coordinate vector to the corresponding trace-zero matrix, and let $[\,\cdot,\cdot\,]_{\mathrm{tab}}$ be the bilinear extension of the table of structure constants used below. Then:*

1. *μ is a bijection from the 256 coordinate vectors onto the trace-zero 3×3 matrices over $\mathbb{F}_2$;*
2. *for each of the 64 pairs (b_i,b_j) of basis elements, the table entry satisfies $\mu([b_i,b_j]_{\mathrm{tab}}) = \mu(b_i)\mu(b_j) - \mu(b_j)\mu(b_i)$;*
3. *the table is symmetric, $[b_i,b_j]_{\mathrm{tab}} = [b_j,b_i]_{\mathrm{tab}}$, and so is the bracket on all 256^2 pairs of elements;*
4. *$[x,x]_{\mathrm{tab}} = 0$ for all 256 elements x;*
5. *the Jacobi identity holds on all $8^3 = 512$ triples of basis elements.*

Item (2) is what pins the table to $\mathfrak{sl}_3(\mathbb{F}_2)$ rather than to some other 8-dimensional algebra: the structure constants are not asserted, they are read off and then checked against actual matrix commutators. Since the bracket is by construction the bilinear extension of the table, (5) is equivalent to the Jacobi identity on all of L; the passage from basis triples to arbitrary triples is the one routine trilinearity argument in this paper that is carried out by hand rather than by machine. Item (4) is stronger than antisymmetry over $\mathbb{F}_2$, as noted above.

From now on we work with L and its matrices and suppress the encoding.

3. The upper bound

Theorem 3.1. *For every non-zero $a \in \mathfrak{sl}_3(\mathbb{F}_2)$ there exist $b, c \in \mathfrak{sl}_3(\mathbb{F}_2)$ with $\langle a, b, c\rangle = \mathfrak{sl}_3(\mathbb{F}_2)$. Hence $k(\mathfrak{sl}_3(\mathbb{F}_2)) \leq 3$.*

Proof sketch. The witness is a table of 255 triples (a, b_a, c_a), one for each non-zero a, which is exhibited and then verified. Verification of a single triple is a closure computation, and it is the direction where a certificate is easy to make sound.

Keep an echelon table E of at most eight elements of L, one per possible leading coordinate. Insert the generators, reducing each against the current pivots and installing the remainder if it is non-zero. Then repeat, at most eight times, the step: form all brackets $[u, v]$ of pairs of table entries and insert each of them in the same way. Each round either enlarges the table or leaves it unchanged, and the table has at most eight entries, so eight rounds reach a fixpoint. If at the end each of the eight basis vectors reduces to 0 against E, the generators generate L.

Soundness of this test requires no invariant on E whatsoever. Reduction is a sequence of additions of table entries, and the entries are, inductively, sums of brackets of the generators; so every element the procedure ever writes down lies in $\langle a, b_a, c_a\rangle$, whatever the pivot rule. If each basis vector reduces to 0, then each basis vector is a sum of table entries and hence lies in $\langle a, b_a, c_a\rangle$, which therefore is all of L. The echelon discipline is what makes the search find the answer; it is not what makes the certificate valid.

Running the 255 triples through this test, together with the check that the table has one row for each non-zero a, proves the theorem. (For the 206 elements that have a partner one may take $b_a = c_a$; the table records a triple in every case.) □

4. The lower bound

The lower bound is the half where a search can only report failure, and where a certificate has to be supplied instead. The certificate is a covering family of proper subalgebras.

Lemma 4.1. *Let $S \subseteq L$ be a Lie subalgebra with $S \neq L$, and let $a, b \in S$. Then $\langle a, b\rangle \subseteq S \subsetneq L$, so (a, b) does not generate L.*

Proof. $\langle a, b\rangle$ is the smallest subset containing 0 and a, b and closed under addition and bracket; S is such a subset; so $\langle a, b\rangle \subseteq S$. Any $z \in L \setminus S$ then witnesses $\langle a, b\rangle \neq L$. □

Lemma 4.1 turns the lower bound into a covering problem: to show that a fixed a lies in no generating pair it suffices to produce proper subalgebras $S_1, \dots, S_m$, each containing a, whose union is all of L. For $a = E_{01}$ ten of them suffice.

Theorem 4.2. *For every $b \in \mathfrak{sl}_3(\mathbb{F}_2)$ we have $\langle E_{01}, b\rangle \neq \mathfrak{sl}_3(\mathbb{F}_2)$, where E_{01} is the matrix with a single 1 in position $(0, 1)$. Hence $k(\mathfrak{sl}_3(\mathbb{F}_2)) > 2$.*

Proof sketch. The certificate is a list of 42 subsets of L, each given as data. Three finite checks are performed on it. First, each listed subset S is a Lie subalgebra: $0 \in S$, and over all 256^2 ordered pairs (x, y) of elements of L with $x, y \in S$ one has $x + y \in S$ and $[x, y] \in S$. Second, each listed S is proper: some element of L is missing from it. Third – the covering step – for the element E_{01} and every one of the 256 elements x of L, some listed S contains both E_{01} and x. Lemma 4.1 then gives $\langle E_{01}, b\rangle \subseteq S \subsetneq L$ for a suitable S, for every b.

Exactly ten of the 42 listed subalgebras contain E_{01}, and it is those ten that do the covering; the covering check is a lookup over $10 \cdot 256$ possibilities. Since $k(L) \geq 2$ always (Proposition 8.1), and since no pair works for E_{01}, we get $k(L) > 2$. □

The 42 subalgebras were not chosen by hand. They come from a *recorded computation* – a separate enumeration of the subalgebras of $\mathfrak{sl}_3(\mathbb{F}_2)$, run and logged alongside the formal development but not part of it – which finds that $\mathfrak{sl}_3(\mathbb{F}_2)$ has exactly 42 maximal proper Lie subalgebras: 14 of dimension 6, namely the seven point-stabilising and seven plane-stabilising parabolic subalgebras, and 28 of dimension 5. The same recorded computation finds that ten of these contain E_{01}, that their union is L, and that no nine of them cover L, so the covering family used above is minimal among subfamilies of the maximal ones.

It is worth separating what that record contributes from what Theorem 4.2 consumes. Completeness of the list of 42, the dimension split $14 + 28$, and the minimality of the ten are outputs of the enumeration and are not machine-checked. What is machine-checked is precisely the three finite checks of the proof above: that each listed set is a Lie subalgebra, that each is proper, and that ten of them contain E_{01} and cover L. The lower bound rests on the checked part alone, and would stand unchanged if the enumeration were incomplete or the ten not minimal; maximality and minimality are reported because they say how tight the certificate is, not because the argument needs them.

5. The value

Theorem 5.1. $k(\mathfrak{sl}_3(\mathbb{F}_2)) = 3$. *Precisely: every non-zero $a \in \mathfrak{sl}_3(\mathbb{F}_2)$ lies in a generating triple, and there is a non-zero $a \in \mathfrak{sl}_3(\mathbb{F}_2)$ – namely $a = E_{01}$ – such that $\langle a, b\rangle \neq \mathfrak{sl}_3(\mathbb{F}_2)$ for all $b \in \mathfrak{sl}_3(\mathbb{F}_2)$.*

Proof. The first clause is Theorem 3.1 and the second is Theorem 4.2. By Definition 2.2 the two clauses say $k(L) \leq 3$ and $k(L) > 2$. □

Corollary 5.2. $\mathfrak{sl}_3(\mathbb{F}_2)$ *is not one-and-a-half generated. In particular the hypotheses of Bois's theorem* [1] *cannot simply be dropped: the theorem assumes an algebraically closed field of characteristic* 0 *or* $p > 3$*, and both hypotheses fail here.*

This is not a counterexample to [1], which says nothing about non-closed fields or about characteristic 2; it says that the natural extension of the statement to simple Lie algebras over finite fields of small characteristic is false.

6. The exceptional set

E_{01} is not alone. Write

$$\mathcal{E} \;=\; \{a \in \mathfrak{sl}_3(\mathbb{F}_2) \;:\; a \neq 0 \text{ and } \langle a, b\rangle \neq \mathfrak{sl}_3(\mathbb{F}_2) \text{ for all } b \in \mathfrak{sl}_3(\mathbb{F}_2)\}$$

for the set of elements that lie in no generating pair. Theorem 4.2 says $E_{01} \in \mathcal{E}$; the following says exactly how large $\mathcal{E}$ is.

Theorem 6.1. $|\mathcal{E}| = 49$*, and the remaining* 206 *non-zero elements each have a partner.*

Proof sketch. Both directions are certified, which is what makes 49 an exact count rather than a bound in either direction.

Upper direction (206 elements are not in $\mathcal{E}$). A table of 206 pairs (a, b_a) is exhibited, one for each non-zero element outside a listed set B of size 49, and each pair is verified to generate by the closure test of §3. A separate check confirms that the table's first entries are exactly the non-zero elements outside B.

Lower direction ($B \subseteq \mathcal{E}$). The covering argument of §4 is run for every $a \in B$, not only for E_{01}: for each of the 49 elements $a \in B$ and each of the 256 elements x of L, one of the 42 certified proper subalgebras contains both a and x, so $\langle a, x\rangle$ is contained in a proper subalgebra by Lemma 4.1. That is $49 \cdot 256 = 12544$ lookups.

Together, $\mathcal{E} = B$ and $|B| = 49$, $255 - 49 = 206$. □

The typical failure mode of a computed census – an incomplete search algorithm inflating the exceptional set – is excluded by the two directions being proved separately: an element is declared exceptional only when a proper subalgebra containing it and every candidate partner is produced.

The 49 have a clean description, and in fact three equivalent ones.

Lemma 6.2. *Let F be a field of characteristic 2, let A be a square matrix over F and suppose $A^2 = \lambda A$ for some $\lambda \in F$. Then $\operatorname{ad}(A)^2 = \lambda \operatorname{ad}(A)$ as operators on matrices; in particular $\operatorname{ad}(A)^3 = \operatorname{ad}(A)^2$ when $\lambda \in \{0, 1\}$.*

Proof. For any X,

$$\operatorname{ad}(A)^2(X) = A^2X - 2AXA + XA^2 = \lambda AX + \lambda XA - 2AXA.$$

In characteristic 2 the middle term vanishes and $\lambda AX + \lambda XA = \lambda(AX - XA) = \lambda \operatorname{ad}(A)(X)$. If $\lambda \in \{0, 1\}$ then $\lambda^2 = \lambda$, so $\operatorname{ad}(A)^3 = \lambda^2 \operatorname{ad}(A) = \lambda \operatorname{ad}(A) = \operatorname{ad}(A)^2$. □

In any characteristic other than 2 the cross term $2AXA$ survives and the conclusion fails; this is the only place where the characteristic enters, and it is where the whole phenomenon comes from. In $\mathfrak{sl}_3(\mathbb{F}_2)$ the identity of Lemma 6.2 is verified directly for each of the 49 exceptional elements against all 256 arguments X.

Theorem 6.3. *Let $a \in \mathfrak{sl}_3(\mathbb{F}_2)$ be non-zero and let A be its matrix. The following are equivalent:*

(1) *$a \in \mathcal{E}$, i.e. a lies in no generating pair;*
(2) *$A^2 = 0$ or $A^2 = A$;*
(3) *$\operatorname{ad}(a)^3 = \operatorname{ad}(a)^2$.*

Moreover $\mathcal{E}$ consists of 21 elements with $A^2 = 0$, all of rank one, and 28 with $A^2 = A$, all of rank two.

Proof sketch. That (2) implies (3) is Lemma 6.2. The equivalences with (1) are established by exhaustive evaluation over the algebra: for each of the 256 elements a, membership in the set $B = \mathcal{E}$ of Theorem 6.1 is compared with the condition $A^2 \in \{0, A\}$, and separately with the condition that $\operatorname{ad}(a)^3(x) = \operatorname{ad}(a)^2(x)$ for all 256 elements x – the latter being $256 \cdot 256 = 65536$ bracket evaluations. Both comparisons hold for every non-zero a. The split $21 + 28$ and the ranks are read off the same enumeration. □

Remark 6.4. The 21 matrices with $A^2 = 0$ are exactly the rank-one square-zero matrices vw^T with $w^\mathsf{T}v = 0$, of which there are $7 \cdot 3 = 21$; among them are the six root elements E_{ij}, so every root element of $\mathfrak{sl}_3(\mathbb{F}_2)$ lies in no generating pair. The 28 idempotents of rank two correspond to the 7 planes of $\mathbb{F}_2^3$ together with the 4 complementary lines for each. Note that a rank-one idempotent has trace 1 and so is not in $\mathfrak{sl}_3(\mathbb{F}_2)$.

Remark 6.5. A further description, computed outside the formal development, is that $\mathcal{E}$ consists exactly of the elements whose centraliser in $\mathfrak{sl}_3(\mathbb{F}_2)$ has dimension 4; every other non-zero element has a 2-dimensional centraliser, so $\operatorname{rank} \operatorname{ad}(a)$ takes the value 4 on $\mathcal{E}$ and 6 off it, with nothing in between.

7. Two-generation, and what $k(L) = 3$ does not say

Theorem 5.1 is consistent with, and does not weaken, the published 2-generation theorem. Cantor, Jezernik and Zozaya [4] prove that $\mathfrak{sl}_n(\mathbb{F}_q)$ is 2-generated except at $(n, p) = (3, 3)$ and $(4, 2)$; $(3, 2)$ is not one of the exceptions. We reproduce their conclusion at this cell with an explicit pair.

Proposition 7.1. *The pair*

$$a = E_{01} + E_{10} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad b = E_{01} + E_{02} + E_{20} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

generates $\mathfrak{sl}_3(\mathbb{F}_2)$. Consequently $\mathfrak{sl}_3(\mathbb{F}_2)$ is 2-generated, and yet by Theorem 4.2 no pair containing E_{01} generates it.

These are the two elements the formal development uses. In the coordinates of §2, with the basis ordered $E_{01}, E_{02}, E_{10}, E_{12}, E_{20}, E_{21}, H_1, H_2$ and an element written as the bit mask of its support, they are the masks $5 = 1 + 4$ and $19 = 1 + 2 + 16$.

Proof sketch. The closure test of §3 applied to $\{a, b\}$: the eight basis vectors all reduce to 0 against the closure table, so $\langle a, b\rangle = \mathfrak{sl}_3(\mathbb{F}_2)$. □

The two statements in Proposition 7.1 hold simultaneously, and that conjunction is the content of the distinction: 2-generation asks for the existence of one good pair, $k(L) = 2$ asks that every non-zero element occurs in one. A computation outside the formal development counts 13440 ordered pairs of non-zero elements that generate $\mathfrak{sl}_3(\mathbb{F}_2)$, out of $255^2 = 65025$; there are many good pairs, and none of them contains any of the 49 exceptional elements.

8. Non-degeneracy of the certificates

Both directions above rest on finite checks, and a finite check that is vacuously satisfied proves nothing. We record the controls.

Proposition 8.1. (1) *No single element generates $\mathfrak{sl}_3(\mathbb{F}_2)$: for every a, the set $\{0, a\}$ is a Lie subalgebra, since $[a, a] = 0$, and it is proper. Hence $k(\mathfrak{sl}_3(\mathbb{F}_2)) \geq 2$.*

(2) *The generation test is not vacuously true. The pair (E_{01}, E_{02}) does not generate – the two commute and span a 2-dimensional abelian subalgebra – and the empty set does not generate.*

(3) *Both outcomes occur among the non-zero elements: E_{01} has no partner, while $E_{01} + E_{10}$ has one. Correspondingly 49 is neither 0 nor 255, and $49 + 206 = 255$.*

(4) *The subalgebra certificate is not vacuously true either. The whole of $\mathfrak{sl}_3(\mathbb{F}_2)$ satisfies the closure conditions used in §4 – it contains 0 and is closed under addition and bracket – and fails properness. Properness is therefore the hypothesis carrying the lower bound, and it is checked separately for each of the 42 subalgebras.*

(5) *Adjoining 0 to a generating set leaves it generating, and adjoining 0 to a non-generating set does not rescue it; so restricting Definition 2.2 to non-zero first entries is not a way of avoiding a hard case.*

Proposition 8.2. (1) *The bracket is not associative, hence not an associative product in disguise: $[[E_{01}, E_{02}], E_{10}] = 0$ while $[E_{01}, [E_{02}, E_{10}]] = E_{02}$.*

(2) *The structure constants are pinned by the Jacobi identity and not a free choice. Change the single entry $[E_{01}, E_{10}]$ from H_1 to $H_1 + E_{01}$, and change $[E_{10}, E_{01}]$ in the same way so that the perturbed table is still alternating and symmetric. Then Jacobi is the only axiom the perturbed table can fail, and it does fail, with the witness triple (E_{01}, E_{10}, E_{12}).*

9. Neighbouring algebras

The following computations are not part of the formal development; they were carried out by two independently written programs, whose outputs agree wherever they overlap, and are reported here as evidence about the shape of the phenomenon rather than as theorems. The obstruction to formalising them is arithmetic, not mathematics: the machine-checked development is written for $\mathbb{F}_2$, where addition of coordinate vectors is bitwise exclusive or and scalars are trivial, and a field with more than two elements needs that layer rewritten.

L	$\dim L$	non-zero elements with no partner	$k(L)$
$\mathfrak{sl}_2(\mathbb{F}_3)$	3	0 (of 26)	2
$\mathfrak{sl}_2(\mathbb{F}_5)$	3	0 (of 124)	2
$\mathfrak{sl}_2(\mathbb{F}_7)$	3	0 (of 342)	2
$\mathfrak{sl}_2(\mathbb{F}_{11})$	3	0 (of 1330)	2
$\mathfrak{sl}_3(\mathbb{F}_2)$	8	**49** (of 255)	**3** (Theorem 5.1)
$\mathfrak{sl}_3(\mathbb{F}_4)$	8	≥ 1 (E_{01}, all 65535 candidates checked)	≥ 3

For $\mathfrak{sl}_2(\mathbb{F}_p)$ with $p = 3, 5, 7, 11$ the sweep is complete – every ordered pair of elements was tested – and no non-zero element lacks a partner, so $k = 2$ in each case (recall $k \geq 2$ always, by Proposition 8.1(1)). For $\mathfrak{sl}_3(\mathbb{F}_4)$ the root element E_{01} was tested against every one of the $4^8-1 = 65535$ candidate partners and none works; the subalgebra generated never exceeds dimension 5, exactly as over $\mathbb{F}_2$, where the largest $\langle E_{01}, b\rangle$ also has dimension 5. Over $\mathbb{F}_5$ and $\mathbb{F}_7$, by contrast, a partner for E_{01} appears within the first few thousand candidates. So the obstruction is not a small-field effect but a characteristic-2 effect, in agreement with Lemma 6.2.

Conjecture 9.1. For every $r \geq 1$, no root element of $\mathfrak{sl}_3(\mathbb{F}_{2^r})$ lies in a generating pair; hence $k(\mathfrak{sl}_3(\mathbb{F}_{2^r})) \geq 3$.

Lemma 6.2 holds over any field of characteristic 2 and is the natural route to this conjecture, the exceptional behaviour of $\mathrm{ad}(A)$ for $A^2 \in \{0, A\}$ being independent of the field size. The cases $r = 1, 2$ are the two characteristic-2 rows of the table. We note that $\mathfrak{sl}_3(\mathbb{F}_{3^r})$ and $\mathfrak{sl}_4(\mathbb{F}_{2^r})$ are outside the scope of Definition 2.2 as used in [8], the characteristic not being very good there, and are also the cases where [4] shows 2-generation itself to fail.

10. Verification

Every statement above that is labelled as a theorem, proposition or lemma – with the explicit exceptions of the counts quoted in the two remarks of §6, the enumeration data about maximal subalgebras noted after Theorem 4.2, the pair count in §7, and the whole of §9, each of which is flagged in place as a computation outside the formal development – has been formalised and machine-checked in Lean 4 against Mathlib. The subalgebra generated by a set is defined inductively, as the least set containing 0 and the generators and closed under addition and bracket, so the reductions used above (a proper subalgebra containing the generators refutes generation; a closure table whose entries lie in the generated subalgebra proves generation) are ordinary proofs by induction and are checked by the proof kernel; the finite searches they consume – the 255 generating triples, the 206 partners, the closure and properness of the 42 subalgebras over all 256^2 pairs, the covering of $49 \cdot 256$ cases, and the comparisons of the ad conditions over 65536 arguments – are executed as evaluations, the small ones inside the kernel and the large ones by compiled code. The development contains no unproved statements. Repository reference to be added at submission.

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