A THREE-TERM NULL VECTOR FOR $L_T + G_T$: HALF OF A CONJECTURE ON -1 AS A GENERALIZED EIGENVALUE OF LCM MATRICES TO GCD MATRICES

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ABSTRACT. For a finite set T of positive integers let $L_T = (\operatorname{lcm}(t_i, t_j))$ and $G_T = (\gcd(t_i, t_j))$ be the LCM and GCD matrices on T. Merikoski, Haukkanen, Sasaki and Tossavainen conjecture that, on $T = \{1, \dots, n\}$ with $n > 3$, the number -1 is a generalized eigenvalue of L_T to G_T exactly when the binary expansion of n begins with 10; their evidence is a floating-point generalized eigensolver run to $n = 1000$. Since -1 is a generalized eigenvalue precisely when $\det(L_T + G_T) = 0$, this is a singularity statement about the integer matrix with entries $\operatorname{lcm}(i, j)+\gcd(i, j)$. We prove the "if" half for every n, with no computation, by exhibiting a null vector: writing $f(a, b) = \operatorname{lcm}(a, b) + \gcd(a, b)$, the three-term identity $2f(2^s, j) - 3f(2^{s+1}, j) + f(2^{s+2}, j) = 0$ holds for all $1 \le j < 3 \cdot 2^{s+1}$, so $2e_{2^s} - 3e_{2^{s+1}} + e_{2^{s+2}}$ annihilates $L_T + G_T$ on the whole block $2^{s+2} \le n < 3 \cdot 2^{s+1}$, which is exactly the set of n whose binary expansion begins 10. The bound is sharp, and more is true: $2e_m - 3e_{2m} + e_{4m}$ annihilates $L_T + G_T$ (for $4m \le n$) if and only if m is a power of two and $n < 6m$, which is why the answer to the conjecture is a condition on binary digits. The same identity is not about intervals at all, and yields an infinite family of index sets, of every size, on which -1 is a generalized eigenvalue: it suffices that T contain $2^s, 2^{s+1}, 2^{s+2}$ and that 2^{s+1} be the only element of T of 2-adic valuation exactly $s + 1$. This settles, for that family, the general-T case taken up in §4 of the source. The "only if" half of the conjecture remains open; we record an exact verification of the full biconditional for $4 \le n \le 47$ inside the formal development and, outside it, for $2 \le n \le 1000$. Two small errors in the published text are identified and corrected. Every theorem below is machine-checked in Lean 4; §10 says exactly what the machine checks.

1. INTRODUCTION

Let $T = \{t_1, \dots, t_n\}$ with $t_1 < \cdots < t_n$ be a set of positive integers. The $n \times n$ *LCM matrix* L_T and *GCD matrix* G_T on T are defined by

$$L_T = (l_{ij}^T), \quad l_{ij}^T = \operatorname{lcm}(t_i, t_j), \qquad G_T = (g_{ij}^T), \quad g_{ij}^T = \gcd(t_i, t_j).$$

These are the classical arithmetical matrices of Smith [3], whose determinant $\det G_T = \varphi(1) \cdots \varphi(n)$ on $T = \{1, \dots, n\}$ opened the subject, and of Beslin and Ligh [4], who proved G_T positive definite on every T. The singularity of L_T alone, by contrast, is delicate: Bourque and Ligh [5] conjectured that L_T is nonsingular whenever T is closed under greatest common divisors; the conjecture is false, a counterexample being given by Haukkanen, Wang and Sillanpää [6], whose paper also collects the determinantal identities of Smith type behind this circle of results; and the threshold in $|T|$ — true for $|T| \le 7$, false for every larger size — was settled in work of Hong [7] and others.

Because G_T is positive definite, the *generalized eigenvalue* problem $L_T x = \lambda G_T x$, $x \neq 0$, has n real solutions λ, counted by the roots of the *generalized characteristic polynomial*

$$p_T(\lambda) = \det(L_T - \lambda G_T).$$

Merikoski, Haukkanen, Sasaki and Tossavainen [1] study these generalized eigenvalues, in parallel with the corresponding MAX–MIN problem, and observe a sharp change of behaviour at $n = 4$. For $T = \{1, \dots, n\}$ — for which we write $p_n = p_T$ — they compute $p_1, \dots, p_5$ explicitly, note that the pattern $\lambda_1 = \sqrt{m}$, $\lambda_2 = \cdots = \lambda_{n-1} = -1$, $\lambda_n = -\sqrt{m}$ valid for $n \le 4$ fails for $n > 4$, and then ask in which dimensions the value -1 occurs at all. Their computer experiments over $1 \le n \le 1000$ produce the list

$$n = 4, 5, 8, 9, 10, 11, 16, \dots, 23, 32, \dots, 47, 64, \dots, 95, 128, \dots, 191, 256, \dots, 383, 512, \dots$$

which they identify as OEIS `A004754` [2], "numbers whose binary expansion starts 10", with its first term removed, and they conjecture:

Conjecture 5 of [1]**.** Let $T = \{1, \dots, n\}$, $n > 3$. Then -1 is a g-eigenvalue of L_T to G_T if and only if the binary representation of n begins with 10.

Two features of this conjecture set the agenda for the present paper. First, the evidence for it is numerical in the strict sense: the code printed in the appendix of [1] calls a floating-point generalized eigensolver and accepts -1 within an absolute tolerance of 10^{-5}. At $n = 1000$ the entries of L_T are as large as 999 000, so this is evidence about the individual dimensions, not proof, even for one n. Second, [1] itself points at the exact route (its §5): since $p_n(-1) = \det(L_T + G_T)$, the conjecture is a statement about the singularity of one integer matrix,

$$A_n := L_T + G_T, \qquad (A_n)_{ij} = \operatorname{lcm}(i,j) + \gcd(i,j), \qquad T = \{1, \dots, n\},$$

and no eigensolver is needed. About a proof the authors say only that "an induction proof of Conjecture 5 can perhaps be found by using" the formula $a_{2^m+k} = 2^{m+1} + k$ recorded in the OEIS entry.

We settle the "if" half completely, and by a route that is not induction on n: we write down a null vector. All of it rests on one arithmetic identity in three terms. Put

$$f(a,b) = \operatorname{lcm}(a,b) + \gcd(a,b), \qquad \text{so that} \qquad (A_n)_{ij} = f(i,j).$$

(A) *The identity* (Lemma 3.1): for every $s \geq 0$ and every j with $1 \leq j < 3 \cdot 2^{s+1}$,

$$2f(2^s, j) - 3f(2^{s+1}, j) + f(2^{s+2}, j) = 0,$$

and at $j = 3 \cdot 2^{s+1}$, the first index outside that range, the same combination equals $2^{s+2} \neq 0$.

(B) *The theorem* (Theorem 4.1): "the binary expansion of n begins with 10" is exactly "$2^{s+2} \leq n < 3 \cdot 2^{s+1}$ for some $s \geq 0$", and on that block all three indices $2^s < 2^{s+1} < 2^{s+2}$ lie in $\{1, \dots, n\}$, so $v = 2e_{2^s} - 3e_{2^{s+1}} + e_{2^{s+2}}$ satisfies $A_n v = 0$ by (A). Hence $\det(L_T + G_T) = 0$ and -1 is a generalized eigenvalue. This holds for every such n, with no computation and no case distinction on n.

(C) *Why the answer is binary* (Theorem 5.1): among all vectors of the same shape, $2e_m - 3e_{2m} + e_{4m}$ with $4m \leq n$, the ones that annihilate A_n are precisely those with m a power of two and $n < 6m$. Both halves of the conjecture's condition — powers of two, and the interval $[2^{s+2}, 3 \cdot 2^{s+1})$ — are therefore forced by the arithmetic rather than chosen.

(D) *Other index sets* (Theorem 6.2): nothing in (A) mentions $\{1, \dots, n\}$. If a finite set T of positive integers contains $2^s, 2^{s+1}, 2^{s+2}$ and 2^{s+1} is the only element of T whose 2-adic valuation is exactly $s+1$, then $\det(L_T + G_T) = 0$. The set $\{1, \dots, n\}$ with $2^{s+2} \leq n < 3 \cdot 2^{s+1}$ is the case in which the excluded elements $3 \cdot 2^{s+1}, 5 \cdot 2^{s+1}, \dots$ all exceed n. In §4 of [1] general T are taken up and settled only for $|T| = 2$, for $T = \{1, u, v\}$, and for the chain $T = \{1, p, \dots, p^{n-1}\}$; of a pairwise coprime $T = \{u, v, w\}$ with $1 < u < v < w$ that section records only "It seems that we do not get pretty results." Theorem 6.2 gives an infinite supply of such T, of every size.

What is not proved. The "only if" half of the conjecture — that $\det(L_T + G_T) \neq 0$ whenever the binary expansion of n begins with 11 — is *not* proved here, and we know of no proof. What we can offer is exactness in place of floating point. Proposition 7.1 settles the full biconditional for $4 \leq n \leq 47$ inside the formal development, by an explicit modular inverse for each of the fourteen dimensions in that range whose binary expansion begins 11; Remark 7.2 records a computation outside the formal development that settles it, again exactly rather than numerically, for the whole of the source's own range $2 \leq n \leq 1000$. §11 explains why the obvious attack — a product formula of Smith type — cannot work, and points instead at the inertia of A_n.

Along the way, §8 records that every value printed in [1] in the LCM–GCD setting was recomputed from the definitions before any claim above was made, and §9 reports the two places where the recomputation disagrees with the printed text.

2. Preliminaries

Throughout, T is a finite set of positive integers, listed as $t_1, \dots, t_k$, and L_T, G_T are as in §1. We write $e_1, \dots, e_n$ for the standard basis of $\mathbb{Z}^n$, indexed so that e_m corresponds to the element $m \in \{1, \dots, n\}$.

Definition 2.1. For positive integers a, b set $f(a,b) = \operatorname{lcm}(a,b) + \gcd(a,b)$, and for $n \geq 1$ let A_n be the $n \times n$ integer matrix with $(A_n)_{ij} = f(i,j)$, that is, $A_n = L_T + G_T$ on $T = \{1, \dots, n\}$. The matrix A_n is symmetric, since f is.

Because G_T is positive definite [4], the pencil $L_T - \lambda G_T$ is regular and λ is a generalized eigenvalue of L_T to G_T if and only if $p_T(\lambda) = \det(L_T - \lambda G_T) = 0$. Specialising at $\lambda = -1$ gives the reduction used throughout:

Proposition 2.2 ([1, §5])**.** *For every $n \geq 1$ one has $L_T - (-1)G_T = A_n$ on $T = \{1, \dots, n\}$, hence $p_n(-1) = \det A_n$. In particular -1 is a generalized eigenvalue of L_T to G_T if and only if $\det(L_T + G_T) = 0$.*

Next we make the digit condition arithmetic. For $n \geq 1$ let $k = \lfloor \log_2 n \rfloor$, so that the binary expansion of n has $k+1$ digits; for $n \geq 2$ the two leading digits of n are read off by $\lfloor n/2^{k-1} \rfloor \in \{2, 3\}$, the value 2 meaning that they are $1, 0$.

Definition 2.3. Say that n *lies in a* 10*-block* if there is an $s \geq 0$ with $2^{s+2} \leq n < 3 \cdot 2^{s+1}$.

Lemma 2.4. *Let $n \geq 4$ and $k = \lfloor \log_2 n \rfloor$. Then n lies in a 10-block if and only if $\lfloor n/2^{k-1} \rfloor = 2$, i.e. if and only if the binary expansion of n begins with 10; the witnessing s is $k-2$. The blocks are*

$$\{4,5\},\ \{8,\dots,11\},\ \{16,\dots,23\},\ \{32,\dots,47\},\ \dots$$

and their union is the sequence `A004754` *of* [2] *with its first term 2 removed. Moreover $n = 2$ has leading binary digits $1, 0$ but does* not *lie in a 10-block (this would require $2^{s+2} \leq 2$), and $\det A_2 = -1 \neq 0$, while $n = 3$ has leading digits $1, 1$ and $\det A_3 = 0$.*

Proof. For $n \geq 4$ we have $k \geq 2$; put $s = k - 2$. Then $\lfloor n/2^{s+1} \rfloor = 2$ is equivalent to $2 \cdot 2^{s+1} \leq n < 3 \cdot 2^{s+1}$, i.e. to $2^{s+2} \leq n < 3 \cdot 2^{s+1}$; conversely $2^{s+2} \leq n < 3 \cdot 2^{s+1} < 2^{s+3}$ forces $\lfloor \log_2 n \rfloor = s + 2$. The two determinants are 2×2 and 3×3 evaluations. □

The failure at $n = 2$ is not an accident of the statement but of the construction below: the null vector for a dimension with leading digits $1, 0$ and $k = \lfloor \log_2 n \rfloor$ uses the index 2^{k-2}, and at $k = 1$ that is not an integer. This is exactly the reason the conjecture is stated for $n > 3$.

3. The three-term identity

Lemma 3.1. *Let $s \geq 0$ and let j be an integer with $1 \leq j < 3 \cdot 2^{s+1}$. Then*

$$2f(2^s, j) - 3f(2^{s+1}, j) + f(2^{s+2}, j) = 0.$$

The bound is sharp: at $j = 3 \cdot 2^{s+1}$ the same combination equals 2^{s+2}.

Proof. Split on the 2-adic valuation $v_2(j)$.

Case $v_2(j) \leq s$, i.e. $2^{s+1} \nmid j$. Then $\gcd(2^{s+1}, j)$ is a power of two dividing 2^{s+1} but not equal to it, so $\gcd(2^{s+1}, j) = \gcd(2^s, j)$, and the same argument one step up gives $\gcd(2^{s+2}, j) = \gcd(2^{s+1}, j)$. All three greatest common divisors are therefore equal, say to d; since $\gcd \cdot \operatorname{lcm}$ is the product of the arguments, the least common multiples double, $\operatorname{lcm}(2^{s+1}, j) = 2\operatorname{lcm}(2^s, j)$ and $\operatorname{lcm}(2^{s+2}, j) = 4\operatorname{lcm}(2^s, j)$. With $\ell = \operatorname{lcm}(2^s, j)$ the combination is $2(\ell + d) - 3(2\ell + d) + (4\ell + d) = 0$.

Case $v_2(j) \geq s + 2$, i.e. $2^{s+2} \mid j$. Then for $t \leq s + 2$ we have $\gcd(2^t, j) = 2^t$ and $\operatorname{lcm}(2^t, j) = j$, so the combination is $2(j + 2^s) - 3(j + 2^{s+1}) + (j + 2^{s+2}) = 2^{s+1} - 3 \cdot 2^{s+1} + 2 \cdot 2^{s+1} = 0$.

Case $v_2(j) = s + 1$. Write $j = 2^{s+1}q$ with q odd. Here, and only here, the hypothesis $j < 3 \cdot 2^{s+1}$ is used: it forces $q < 3$, hence $q = 1$ and $j = 2^{s+1}$. In units of 2^s the three values are $f(2^s, 2^{s+1}) = 3$, $f(2^{s+1}, 2^{s+1}) = 4$ and $f(2^{s+2}, 2^{s+1}) = 6$, and $2 \cdot 3 - 3 \cdot 4 + 6 = 0$.

For the sharpness statement take $j = 3 \cdot 2^{s+1}$. In units of 2^s one gets $f(2^s, j) = 6 + 1 = 7$, $f(2^{s+1}, j) = 6 + 2 = 8$ and $f(2^{s+2}, j) = 12 + 2 = 14$, whence $2 \cdot 7 - 3 \cdot 8 + 14 = 4$, i.e. 2^{s+2}. □

The proof is uniform in s and in j; no computation over a finite range enters it.

4. Half of the conjecture

Theorem 4.1. *Let $s \geq 0$ and let n satisfy $2^{s+2} \leq n < 3 \cdot 2^{s+1}$. Then the vector*

$$v = 2e_{2^s} - 3e_{2^{s+1}} + e_{2^{s+2}} \in \mathbb{Z}^n$$

is nonzero and satisfies $A_n v = 0$; consequently $\det(L_T + G_T) = 0$ on $T = \{1, \ldots, n\}$, and -1 is a generalized eigenvalue of L_T to G_T. Equivalently, by Lemma 2.4: for every $n \geq 4$ whose binary expansion begins with 10, one has $\det(L_T + G_T) = 0$. This is the "if" half of Conjecture 5 of [1].

Proof. The three indices satisfy $2^s < 2^{s+1} < 2^{s+2} \leq n$, so they are distinct elements of $\{1, \ldots, n\}$ and $v \neq 0$. For $1 \leq i \leq n$,

$$(A_n v)_i = 2f(i, 2^s) - 3f(i, 2^{s+1}) + f(i, 2^{s+2}) = 2f(2^s, i) - 3f(2^{s+1}, i) + f(2^{s+2}, i),$$

using the symmetry of f. Since $1 \leq i \leq n < 3 \cdot 2^{s+1}$, Lemma 3.1 makes this 0. Thus A_n has a nonzero kernel vector over the integral domain $\mathbb{Z}$, so $\det A_n = 0$; by Proposition 2.2 this says $p_n(-1) = 0$. □

Three remarks are worth making about the shape of this argument. First, the null vector depends on n only through the block containing it: one and the same v works simultaneously for all n in $[2^{s+2}, 3 \cdot 2^{s+1})$, which is why the answer is a condition on leading binary digits rather than on n itself. Second, the argument is not an induction on n, and in particular does not need the recurrence $a_{2^m+k} = 2^{m+1} + k$ suggested in [1]. Third, the right endpoint of the block is forced: by the sharpness half of Lemma 3.1, at $n = 3 \cdot 2^{s+1}$ the same vector has $(A_n v)_n = 2^{s+2} \neq 0$, so the hypothesis $n < 3 \cdot 2^{s+1}$ cannot be weakened. The next section shows that the left endpoint and the restriction to powers of two are forced as well.

5. Why the answer is binary

For any $m \geq 1$ with $4m \leq n$ the vector $2e_m - 3e_{2m} + e_{4m}$ lies in $\mathbb{Z}^n$, and one may ask for which m and n it is a null vector of A_n. The answer singles out the powers of two and the blocks of Definition 2.3.

Theorem 5.1. *Let $m \geq 1$ and $n \geq 4m$. Then*

$$2f(m, j) - 3f(2m, j) + f(4m, j) = 0 \quad \text{for all } 1 \leq j \leq n$$

— equivalently, $A_n\big(2e_m - 3e_{2m} + e_{4m}\big) = 0$ — if and only if m is a power of two and $n < 6m$.

Proof. Write $m = 2^a m'$ and $j = 2^b j'$ with m', j' odd, and put $c = \gcd(m', j')$, $P = \operatorname{lcm}(m', j')$. A computation of the same three-case kind as Lemma 3.1 gives

$$2f(m, j) - 3f(2m, j) + f(4m, j) = \begin{cases} 0, & b \neq a + 1, \\ 2^{\,a+1}\,(P - c), & b = a + 1, \end{cases}$$

so the combination is nonzero exactly when $v_2(j) = v_2(m) + 1$ and the odd parts of j and m differ.

If m is not a power of two, i.e. $m' > 1$, take $j = 2^{\,a+1}$. Its odd part is $1 \neq m'$, and $j = 2^{a+1} < 2^{a+2} \leq 4m \leq n$, so j is in range and the value is $2^{\,a+1}(m' - 1) \neq 0$. Hence only powers of two can occur.

If $m = 2^s$, then $a = s$ and $m' = 1$, and a bad j must satisfy $v_2(j) = s + 1$ and $j' \neq 1$, i.e. $j' \geq 3$; the smallest such j is $3 \cdot 2^{s+1} = 6m$. So all j in $\{1, \ldots, n\}$ are good precisely when $n < 6m$. Together with $4m \leq n$ this is $2^{s+2} \leq n < 3 \cdot 2^{s+1}$, the condition of Theorem 4.1. □

Thus, inside this family of witnesses, both the "power of two" and the window $[4m, 6m)$ are consequences of the arithmetic of gcd and lcm, not choices: the dimensions produced are exactly the 10-blocks, and the digit pattern 10 of Conjecture 5 of [1] is explained rather than merely matched. Theorem 5.1 says nothing about null vectors of other shapes, and indeed other shapes exist; see Remark 5.2.

Remark 5.2. The three-term vectors are not the only source of singularity. In the range $2 \le n \le 1000$ covered by the computations of §7 the matrix A_n has a kernel vector of a different shape exactly at $n = 3, 4, 9$: the kernel of A_3 is spanned by $(-5, 2, 1)$; A_4 has nullity 2, with kernel spanned by $(-5, 2, 1, 0)$ and the three-term vector $2e_1 - 3e_2 + e_4$; and A_9 has nullity 2, the extra vector being $(0, 9, -8, -3, 0, 0, 0, 0, 2)$. (These kernels were computed exactly for $n \le 60$; beyond that the nullity count of Remark 7.2 leaves no room for anything further.) The dimension $n = 3$ is the sporadic singular dimension with leading digits $1, 1$ that forces the hypothesis $n > 3$ in Conjecture 5. These computations lie outside the formal development; see Remark 7.2.

6. Index sets other than $\{1, \dots, n\}$

Section 4 of [1] takes up index sets $T \ne \{1, 2, \dots, n\}$ and settles the question for $|T| = 2$, for $T = \{1, u, v\}$ with $1 < u < v$ and $\gcd(u, v) = 1$, and for the chain $T = \{1, p, \dots, p^{n-1}\}$; of a pairwise coprime $T = \{u, v, w\}$ with $1 < u < v < w$ it records only "It seems that we do not get pretty results." Lemma 3.1 is not a statement about $\{1, \dots, n\}$: it concerns the three integers $2^s, 2^{s+1}, 2^{s+2}$ and one further integer j. Stated at the hypothesis that actually drives it, it gives an infinite family of index sets on which -1 is a generalized eigenvalue.

Lemma 6.1. *Let $s \ge 0$ and let $j \ge 1$ be an integer such that, if $2^{s+1} \mid j$, then either $2^{s+2} \mid j$ or $j = 2^{s+1}$. Then $2f(2^s, j) - 3f(2^{s+1}, j) + f(2^{s+2}, j) = 0$. Moreover, for q odd,*

$$2f(2^s, 2^{s+1}q) - 3f(2^{s+1}, 2^{s+1}q) + f(2^{s+2}, 2^{s+1}q) = 2^{\,s+1}(q-1),$$

so the excluded j really do fail, and by exactly that amount.

Proof. The first two cases of the proof of Lemma 3.1 are unchanged, and in the third case the hypothesis replaces the bound $j < 3 \cdot 2^{s+1}$ in forcing $j = 2^{s+1}$. For the displayed value, with $j = 2^{s+1}q$ and q odd one has $\gcd(2^s, j) = 2^s$, $\gcd(2^{s+1}, j) = \gcd(2^{s+2}, j) = 2^{s+1}$, $\operatorname{lcm}(2^s, j) = \operatorname{lcm}(2^{s+1}, j) = 2^{s+1}q$ and $\operatorname{lcm}(2^{s+2}, j) = 2^{s+2}q$; the combination is then $2^{s+1}q - 2^{s+1} = 2^{s+1}(q-1)$. Taking $q = 3$ recovers the sharpness statement of Lemma 3.1. □

Theorem 6.2. *Let T be a finite set of positive integers such that, for some $s \ge 0$,*

1. *2^s, 2^{s+1}, $2^{s+2} \in T$, and*
2. *every $t \in T$ divisible by 2^{s+1} is either equal to 2^{s+1} or divisible by 2^{s+2} — equivalently, 2^{s+1} is the only element of T whose 2-adic valuation is exactly $s + 1$.*

Then $\det(L_T + G_T) = 0$, and -1 is a generalized eigenvalue of L_T to G_T.

Proof. List T as $t_1, \dots, t_k$ and let a, b, c be the positions of $2^s, 2^{s+1}, 2^{s+2}$; they are distinct because the three values are. The vector $v = 2e_a - 3e_b + e_c \in \mathbb{Z}^k$ is nonzero, and for each i,

$$\big((L_T + G_T)v\big)_i = 2f(2^s, t_i) - 3f(2^{s+1}, t_i) + f(2^{s+2}, t_i) = 0$$

by hypothesis (2) and Lemma 6.1. Hence $L_T + G_T$ is singular over $\mathbb{Z}$. □

Theorem 4.1 is the special case $T = \{1, \dots, n\}$: hypothesis (1) is $2^{s+2} \le n$, and hypothesis (2) says that the smallest forbidden element $3 \cdot 2^{s+1}$ already exceeds n. But the theorem applies to sets of every size and of every shape. Two nine-element examples, both machine-checked:

$$T = \{1, 2, 3, 4, 5, 7, 8, 9, 12\}\ (s = 0), \qquad T = \{1, 2, 4, 3, 5, 7, 9, 8, 16\}\ (s = 2),$$

have $\det(L_T + G_T) = 0$. The list order is immaterial, since a permutation of T conjugates $L_T + G_T$ by a permutation matrix.

Proposition 6.3. *Hypothesis (2) of Theorem 6.2 cannot be dropped. For $T = \{1, 2, 4, 6\}$, which satisfies (1) with $s = 0$ but not (2), since $v_2(6) = 1$ and $6 \neq 2$, one has $2f(1, 6) - 3f(2, 6) + f(4, 6) = 4 \neq 0$ and $\det(L_T + G_T) = 16 \neq 0$.*

Remark 6.4. The condition is sufficient but not necessary. Exact integer computation outside the formal development gives $\det(L_T + G_T) = 0$ for $T = \{1, 2, 4, 6, 3\}$, which violates hypothesis (2); adding the element 3 to the set of Proposition 6.3 restores singularity for a reason the three-term vector does not see. This is the same phenomenon as the sporadic kernels of Remark 5.2. Characterising the singular index sets is the natural generalisation of the whole question, and is open.

7. The extent of the exact verification

Conjecture 5 of [1] is verified there for $1 \leq n \leq 1000$ by a floating-point generalized eigensolver at tolerance 10^{-5}. The dimensions listed are therefore published data; what is added here is that they are exact.

Proposition 7.1. *For every n with $4 \leq n \leq 47$,*

$$\det(L_T + G_T) = 0 \iff \textit{the binary expansion of } n \textit{ begins with } 10, \qquad T = \{1, \ldots, n\}.$$

Proof. This range consists of the four complete blocks $\{4, 5\}$, $\{8, \ldots, 11\}$, $\{16, \ldots, 23\}$, $\{32, \ldots, 47\}$ together with the three runs between them. The singular half is Theorem 4.1 and costs nothing. The nonsingular half is a finite exhaustive check over the fourteen dimensions in the range whose binary expansion begins with 11, namely

$$n = 6, 7, \quad 12, 13, 14, 15, \quad 24, 25, \ldots, 31.$$

For each of these we exhibit an explicit matrix N over $\mathbb{Z}/19\mathbb{Z}$ with $\bar{A}_n N = I$, where $\bar{A}_n$ is the entrywise reduction of A_n modulo 19; the identity $\bar{A}_n N = I$ is verified entry by entry, which is n^3 modular multiply-adds, at most 29 791 of them at $n = 31$. It follows that $\det \bar{A}_n \neq 0$ in $\mathbb{Z}/19\mathbb{Z}$, and since reduction modulo 19 commutes with the determinant, $\det A_n \neq 0$ in $\mathbb{Z}$. A modular certificate is one-sided, and this is the side that is needed: $\det A_n \not\equiv 0 \pmod{p}$ implies $\det A_n \neq 0$ for any p whatever. The prime 19 is the smallest one that divides none of the fourteen determinants. No determinant of a matrix of size 6 to 31 is ever expanded. □

The check is not vacuous in either direction inside the window: $\det A_{16} = 0$ and $\det A_{24} \neq 0$ are both part of it, and both endpoints of the block structure are exercised. Together with Lemma 2.4, which records $\det A_2 = -1$ and $\det A_3 = 0$, the two exceptional dimensions excluded by the hypothesis $n > 3$ are live rather than hypothetical.

Remark 7.2. The following computations were carried out outside the formal development, in C and in Python, and are reported as computations. For every n with $2 \leq n \leq 1000$ — the range of [1] — the biconditional of Conjecture 5 was checked exactly, not in floating point, by Gaussian elimination on A_n over a prime field: for the n whose binary expansion begins 11, $\det A_n \not\equiv 0$ modulo the prime, which forces $\det A_n \neq 0$ over $\mathbb{Z}$; for the others the rank deficiency was confirmed and the nullity over the prime field found to be exactly 1. Since the nullity over a prime field is at least the nullity over $\mathbb{Q}$, and since Theorem 4.1 supplies one kernel vector, this determines the kernel completely: *for every singular n with $10 \leq n \leq 1000$ the null space of A_n is exactly the line spanned by $2e_{2^{k-2}} - 3e_{2^{k-1}} + e_{2^k}$*, $k = \lfloor \log_2 n \rfloor$, and -1 has multiplicity 1 as a generalized eigenvalue there. The exceptions in the range are $n = 4$ and $n = 9$, where the nullity is 2, and the sporadic $n = 3$. The primes used were 2147483629 for $2 \leq n \leq 600$ and $2^{31} - 1$ for $601 \leq n \leq 1000$, with an overlapping run of $2^{31} - 1$ on $2 \leq n \leq 400$ that agrees; the whole sweep took 615 CPU-seconds. It was cross-checked against fraction-free elimination over $\mathbb{Z}$ with a rational null space for $2 \leq n \leq 60$, which reproduces the exact determinants and the kernels of Remark 5.2.

8. The printed data of the source, recomputed

Before any of the claims above was made, every value printed in [1] in the LCM–GCD setting was recomputed from the definitions in exact arithmetic. The following are those of the printed values that are also part of the formal development; they agree with [1] as printed.

Proposition 8.1. *On $T = \{1, \dots, n\}$, with $p_n(\lambda) = \det(L_T - \lambda G_T)$,*

$$p_1 = 1 - \lambda, \qquad p_2 = \lambda^2 - 2, \qquad p_3 = -2(\lambda+1)(\lambda^2 - 6), \qquad p_4 = 4(\lambda+1)^2(\lambda^2 - 12),$$
$$p_5 = -16(\lambda+1)\big(\lambda^4 + 2\lambda^3 - 35\lambda^2 - 120\lambda - 60\big),$$

and on $T = \{2, 3, 5\}$, $\det(L_T - \lambda G_T) = -22\lambda^3 - 38\lambda^2 + 420\lambda + 900$.

The MAX–MIN half of [1] provides a useful control on the method of Theorem 4.1. Let $S = \{s_1, \dots, s_n\}$ be a set of positive reals, and let $M_S = (\max(s_i, s_j))$ and $N_S = (\min(s_i, s_j))$ be the MAX and MIN matrices on S; Theorem 1 of [1] determines the generalized eigenvalues of M_S to N_S for every $n > 2$, and in particular gives -1 with multiplicity $n - 2$. Specialised to $S = \{1, \dots, n\}$ this reads:

Proposition 8.2 ([1, Thm. 1])**.** *For every $n \geq 3$, $\det(M_S + N_S) = 0$ on $S = \{1, \dots, n\}$.*

Proof. The (i, j) entry of $M_S + N_S$ is $\max(i, j) + \min(i, j) = i + j$, so the matrix has rank at most 2; a null vector for $n \geq 3$ is $e_1 - 2e_2 + e_3$. □

The point of recording Proposition 8.2 is that it is a case in which the determinant vanishes for a reason entirely unrelated to Lemma 3.1 — rank two, and a different null vector — and in which the singularity is known independently. It confirms that the route of Theorem 4.1 reports a vanishing determinant where one is present, and not only where the three-term identity puts it.

9. Two corrections to the printed text

Two values printed in §4.1 of [1] do not agree with the recomputation of §8. Both are recorded here in a neutral spirit: neither affects Conjecture 5, and neither affects the generalized eigenvalues that [1] lists.

Proposition 9.1. *Let u, v be coprime positive integers. On $T = \{u, v\}$,*

$$\det(L_T - \lambda G_T) = (uv - 1)(\lambda^2 - uv),$$

as printed in [1]*. On $T = \{1, u, v\}$,*

$$\det(L_T - \lambda G_T) = -(u-1)(v-1)(\lambda+1)(\lambda^2 - uv),$$

whereas [1] *prints this expression without the leading minus sign. In particular the printed right-hand side takes the value -12 at $u = 2$, $v = 3$, $\lambda = 0$, while the determinant there is $+12$.*

Both identities are proved here for every coprime pair u, v; §4.1 of [1] states the first for $0 < u < v$, reducing to the case $\gcd(u, v) = 1$, and the second under the additional hypothesis $1 < u < v$ with $\gcd(u, v) = 1$. The sign is settled by the source's own text in two independent ways. The displayed expansion on the line preceding the factorisation in [1] has λ^3 coefficient $u + v - uv - 1$, which is $-(u-1)(v-1)$; and at $u = 2, v = 3$ the set $\{1, u, v\}$ is $\{1, 2, 3\}$, for which §3.1 of [1] records $p_3 = -2(\lambda+1)(\lambda^2 - 6)$, the negative of what the printed factorisation gives. The generalized eigenvalues listed there, $\sqrt{uv}, -1, -\sqrt{uv}$, are the roots of $(\lambda+1)(\lambda^2 - uv)$ and are unaffected by an overall sign. The verification of Proposition 9.1 is by an identity valid for all coprime u, v, not by a search.

Proposition 9.2. *Every real root of $-22\lambda^3 - 38\lambda^2 + 420\lambda + 900$ — the generalized characteristic polynomial on $T = \{2, 3, 5\}$ of Proposition 8.1 — is less than 4.5125. The polynomial is positive at 4.5124 and strictly negative on $[4.5125, \infty)$, so its largest root lies in the open interval $(4.5124, 4.5125)$; to six decimal places it is 4.512473. The largest generalized eigenvalue on this T is printed as*

$\lambda_1 = 4.5128$ *in* [1]*; the correct four-decimal value is* 4.5125*. The other two printed values,* $\lambda_2 = -2.3027$ *and* $\lambda_3 = -3.9371$*, agree with the recomputation.*

Proof sketch. Evaluation at the rational point $45124/10000$ gives a positive rational. Expanding the cubic about $45125/10000$ as a polynomial in $t = \lambda - 45125/10000$ produces a negative constant term and non-positive coefficients of t, t^2 and t^3, so the cubic is strictly negative for $t \geq 0$; with the sign change on $[4.5124, 4.5125]$ this locates the largest root and bounds all roots. All arithmetic is rational. □

10. Verification

Every statement above — Definitions 2.1 and 2.3, Lemmas 2.4, 3.1 and 6.1, Theorems 4.1, 5.1 and 6.2, and Propositions 2.2, 6.3, 7.1, 8.1, 8.2, 9.1 and 9.2 — has been machine-checked in Lean 4 [9] against Mathlib [10]. The development is four files: the arithmetic identity and the main theorem; the generalized characteristic polynomials of §8, the two corrections of §9 and the characterisation of Theorem 5.1; the modular certificates of Proposition 7.1; and the general index sets of §6 with their controls. There is no `sorry` and no appeal to compiled evaluation anywhere: the axiom closure of every theorem and lemma of the four files is contained in the standard three — propositional extensionality, quotient soundness and the axiom of choice — with no `native_decide` and no `Lean.ofReduceBool`, and in particular the modular certificates of Proposition 7.1 are checked by kernel reduction over $\mathbb{Z}/19\mathbb{Z}$ and lifted to $\mathbb{Z}$ by the fact that a ring homomorphism commutes with the determinant, rather than by a trusted evaluator. The certificate tables were produced by Gauss–Jordan elimination over $\mathbb{Z}/19\mathbb{Z}$ outside the development and are re-verified inside it from scratch, so their generator is not part of the trusted base. Two things are deliberately *not* inside the development, and are labelled as computations wherever they are used: the sweep of Remark 7.2 over $2 \leq n \leq 1000$, whose exact integer determinants at that size are far beyond what a proof kernel can evaluate, and the small exact determinants quoted in Remark 5.2 and Remark 6.4. Repository reference to be added at submission.

11. What remains

The "only if" half of Conjecture 5 of [1] — that $\det(L_T + G_T) \neq 0$ whenever $n > 3$ has leading binary digits $1, 1$ — is open. Three remarks on the terrain.

A product formula will not do it. The obvious approach, given $\det G_T = \varphi(1) \cdots \varphi(n)$ [3] and the analogous product formula for $\det L_T$, is to look for a product formula for $\det(L_T + G_T)$. There is none of that kind: the exact determinants computed in Remark 7.2 have large prime factors, for instance

$$\det A_{24} = 2^{58} \cdot 3^{11} \cdot 5^2 \cdot 7 \cdot 11^2 \cdot 29 \cdot 6359 \cdot 586811, \qquad \det A_{26} = -2^{62} \cdot 3^{13} \cdot 5^5 \cdot 11^2 \cdot 41 \cdot 7407624403,$$

and the sequence $2, -1, 0, 0, 0, -1024, 73728, 0, \ldots$ of determinants does not appear in the On-Line Encyclopedia of Integer Sequences [2].

Inertia looks more promising. The matrix A_n is symmetric and G_T is positive definite, so $G_T^{-1/2} A_n G_T^{-1/2} = G_T^{-1/2} L_T G_T^{-1/2} + I$ has eigenvalues $\lambda_{ni} + 1$: the number of negative eigenvalues of A_n is the number of generalized eigenvalues below -1, and A_n is singular exactly when some $\lambda_{ni} = -1$. Cauchy interlacing for the leading principal submatrices of A_n then gives $\nu(n-1) \leq \nu(n) \leq \nu(n-1)+1$ for that count ν, and a formula for $\nu(n)$ would settle the conjecture. (Interlacing in the generalized setting is Theorem 6 of [1]; the version used here is the ordinary one for the symmetric matrix A_n.) The exact inertia of A_n for $n \leq 48$, computed outside the formal development, shows no obvious pattern.

Multiplicity. Remark 7.2 determines the kernel of A_n exactly for every singular $n \in [10, 1000]$: it is the line spanned by the vector of Theorem 4.1. Proving that the nullity is at most 1 for all large n would turn that measurement into a theorem and refine §3.2 of [1], which rules out only the multiplicity $n-2$ for $n > 4$. A proof would presumably have to use more than the three-term identity, since Theorem 5.1 exhausts the witnesses of that shape.

Finally, Theorem 6.2 is sufficient and not necessary (Remark 6.4). Describing the finite sets T of positive integers for which -1 is a generalized eigenvalue of L_T to G_T contains Conjecture 5 as the special case $T = \{1, \ldots, n\}$ and seems to be the right general question.

References

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[2] The On-Line Encyclopedia of Integer Sequences, sequence `A004754`, *Numbers whose binary expansion starts 10*; `https://oeis.org/A004754`. The entry links [1] and records the formula $a_{2^m+k} = 2^{m+1} + k$ for $0 \le k < 2^m$.

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[4] S. Beslin and S. Ligh, *Greatest common divisor matrices*, Linear Algebra Appl. **118** (1989), 69–76. Cited in [1], as its Theorem 2, for the positive definiteness of G_T, which is what makes the generalized eigenvalues real.

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[7] S. Hong, *On the Bourque–Ligh conjecture of least common multiple matrices*, J. Algebra **218** (1999), no. 1, 216–228; DOI 10.1006/jabr.1998.7844.

[8] P. Ilmonen and V. Kaarnioja, *Generalized eigenvalue problems for meet and join matrices on semilattices*, Linear Algebra Appl. **536** (2018), 250–273; DOI 10.1016/j.laa.2017.09.023; arXiv:1705.05169v2 [math.NT]. The nearest neighbour of the present question in the literature: it gives upper and lower bounds for the generalized eigenvalues of meet matrices with respect to join matrices, specialised in the arithmetical case to $(S)_f$ with respect to $[S]_f$ on factor-closed sets; it contains no statement about the value -1.

[9] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, in: Automated Deduction — CADE 28, Lecture Notes in Computer Science 12699, Springer, 2021, 625–635; DOI 10.1007/978-3-030-79876-5_37.

[10] The mathlib Community, *The Lean mathematical library*, in: Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020), ACM, 2020, 367–381; DOI 10.1145/3372885.3373824.