KUSNER'S CONJECTURE FAILS AT A SECOND EXPONENT: A CERTIFICATE FOR 58 EQUILATERAL POINTS IN ℓ_6^{56}

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Abstract. An equilateral set in a normed space is a set of points at pairwise equal distance, and $\mathrm{e}(\ell_p^n)$ denotes the largest cardinality of one. Kusner conjectured that $\mathrm{e}(\ell_p^n) = n+1$ for $2 < p < \infty$. Chalmers has recently refuted this at $p = 5$, by exhibiting 58 points in ℓ_5^{56} as the unique zero, inside an explicit box, of a polynomial system with dyadic coefficients; his paper records that it is not known how far that configuration extends in the exponent, and observes that its verification scheme uses the oddness of 5. We address both points. Continued in the exponent inside the square slice that Chalmers uses to make the system square, the solution branch is a closed loop of width 0.0367 around $p = 5$ and reaches no other integer; but the equilateral condition on 58 points in $\mathbb{R}^{56}$ is 1653 equations in 3249 unknowns, so that fold belongs to the slice and not to the problem. Continuing in the full space and refining in exact arithmetic, we obtain a Newton–Kantorovich certificate at the *even* exponent 6, on Chalmers' own frozen set and variable order: a centre with dyadic coordinates, a preconditioner, preconditioned residual $\eta = 4.4320 \times 10^{-16}$ and contraction factor $q = 2.3563 \times 10^{-4}$ at box radius 2^{-33}, so that the Kantorovich test clears by a factor 4176, and still passes at a box radius 4244 times the one used at $p = 5$. The five arithmetic claims of the certificate, the radius window, the constants and the negative controls are checked by the Lean 4 kernel. Because the exponent is even, the sign-constancy hypothesis that the $p = 5$ scheme needs disappears entirely. Subject to the classical reduction of the contraction hypotheses to those arithmetic claims—which is carried out here by hand and is *not* machine-checked, and which we flag at every use—this gives $\mathrm{e}(\ell_6^{56}) \geq 58 > 57$: Kusner's conjecture fails at a second exponent, and at an even one, where it was open and the best bound available was $\mathrm{e}(\ell_6^{56}) \leq 169$.

1. Introduction

1.1. **Equilateral sets.** Write ℓ_p^n for $\mathbb{R}^n$ with the norm $\|x\|_p = \big(\sum_{r<n} |x_r|^p\big)^{1/p}$, $1 \leq p < \infty$. A set $S \subseteq \ell_p^n$ is *equilateral* if $\|x - y\|_p$ takes one and the same positive value for all distinct $x, y \in S$, and

$$\mathrm{e}(\ell_p^n) = \max\{\, |S| : S \subseteq \ell_p^n \text{ equilateral} \,\}.$$

The Euclidean value $\mathrm{e}(\ell_2^n) = n+1$ is classical, and Kusner conjectured, as recorded by Guy [6], that an equilateral set in ℓ_p^n with $2 < p < \infty$ has at most $n+1$ points; the conjecture is Problem 2 of Swanepoel's survey [13]. (Items of [13] are numbered as in `arXiv:math/0406264v1`.) What was known about it, before 2026, is a short list. For $1 < p < 2$ the conjecture is false: Swanepoel constructed equilateral sets of $\lfloor \frac{2^{k+1}}{2^{k+1}-1} n \rfloor$ points, where $k = k(p)$, hence of more than $n+1$ points once $n \geq 2^{k+2} - 2$ [14] (Theorem 4 of [13]). For general p the best upper bound was Alon and Pudlák's $\mathrm{e}(\ell_p^n) \leq c_p n^{(2p+2)/(2p-1)}$, sharpened by them to $\mathrm{e}(\ell_p^n) \leq c_p n \log n$ at

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odd integers p [1]. At even integers p there is a linear bound, $\mathrm{e}(\ell_p^n) \le (p/2)n + 1$ when $p \equiv 2 \pmod 4$ and $\mathrm{e}(\ell_p^n) \le (p/2-1)n + 1$ when $p \equiv 0 \pmod 4$, due to Swanepoel [14] (Theorem 10 of the survey [13]). Recently Ge, Xu and Zhou [5] proved the conjecture in the range $2 \le p \le 4$ for every n: this is the case $k = 0$ of their Theorem 1.1, which gives $\mathrm{e}(\ell_p^n) \le (2k+1)n + 1$ for every integer $k \ge 0$ and every $p \in [4k+2, 4k+4]$, while their Theorem 1.2 gives $\mathrm{e}(\ell_p^n) \le C_p n \log(2n)$ on the complementary intervals $p \in (4k, 4k+2)$. At $p = 6$, $n = 56$ both their bound (at $k = 1$) and the even-exponent bound above read $\mathrm{e}(\ell_6^{56}) \le 169$.

1.2. **The counterexample at $p = 5$, and the questions it leaves.** Chalmers [3] refuted the conjecture at $p = 5$: there are 58 points in ℓ_5^{56} at pairwise equal distance, so $\mathrm{e}(\ell_5^{56}) \ge 58 > 57 = n + 1$. The configuration is not written down as a list of coordinates. It is obtained as the unique zero, inside an explicit box, of the system

$$F_{ij}(z) = \sum_{r<56} \left| x_r^{(i)} - x_r^{(j)} \right|^5 - D = 0, \qquad 1 \le i < j \le 58, \tag{1}$$

made square by freezing 1596 of the 3248 coordinates, and existence and uniqueness are reduced by a Newton–Kantorovich contraction argument to six claims about explicitly given integers, archived as a data set [4]. The paper's abstract calls this "the first equilateral set of more than $n+1$ points in ℓ_p^n for any finite $p \ge 2$"; before it, no exponent was known at which the conjecture fails above $p = 2$.

Two of its own remarks bound what it can say about *other* exponents. Its Corollary 2 asserts, non-constructively, that there is an $\varepsilon > 0$ with $\mathrm{e}(\ell_p^{56}) \ge 58$ for all p with $|p-5| < \varepsilon$, and Remark 6, which follows the proof of that corollary, says, verbatim,

> The argument gives no quantitative control of ε_0, and we do not know how far the branch $p \mapsto z(p)$ extends: it may leave X, or cease to exist, well inside $(4, 6)$. Nor does the certificate of Section 4 transfer directly to a given non-integer p: it is the oddness of the exponent, together with the sign constancy of Section 2, that makes (2) polynomial with dyadic coefficients and the verification a finite integer computation.

Throughout, references to numbered items of [3] (Theorem 1, Corollary 2, Lemma 3, Proposition 4, Remark 6, and Sections 2 and 4) are to the numbering of the preprint's version 1, the only version at the time of writing.

1.3. **Results.** This paper answers the first sentence of that remark and, in the course of doing so, produces a second counterexample — conditional, as (v) below and Section 6 make explicit, on an unformalised reduction.

(i) *The branch does not extend, inside the slice.* Continued in p with a pseudo-arclength corrector, the solution branch of Chalmers' square system through his own configuration closes into a loop of arclength 3.28 which never leaves $4.999974 \le p \le 5.036695$; it carries exactly two solutions at $p = 5$, and both turning points are simple folds, confirmed at 40 decimal digits. This is floating-point computation and is reported as such (Section 3); it is the search that motivates the rest, and nothing later depends on it.

(ii) *But that fold belongs to the slice.* The equilateral condition itself is 1653 equations in 3249 unknowns, so its solution set near the configuration is

about 1596-dimensional. Continued in the full space with a minimum-norm Gauss–Newton corrector, the branch runs from $p = 5$ to $p = 7$ with no fold at all. This too is floating-point computation (Section 3).

(iii) *A certificate at* $p = 6$ (Section 4). Rounding the $p = 6$ point of that continuation to the grid 2^{-50}, refining the free coordinates in exact integer arithmetic and inverting the Jacobian gives a new centre c_6, a new common sixth-power distance $D_c = 687876319827206/2^{50}$ and a new preconditioner B_6, on Chalmers' own frozen set and variable order. Theorem 4: the exponent-6 analogues of the six integer claims—five of them; the sign-constancy claim is not needed—hold, with $\eta = 4.4320 \times 10^{-16}$ (a factor 121 *smaller* than the residual of the published certificate) and $q = 2.3563 \times 10^{-4}$ at $\rho = 2^{-33}$.

(iv) *Sharpness* (Section 5). Theorem 6: the Kantorovich test for this certificate still passes at box radius $556278737/2^{50} = 4.9407 \times 10^{-7}$, which is 4244 times the radius used at $p = 5$, and fails one unit of the grid above; there $q = 0.999999997$, so the window closes exactly where the contraction does. Negative controls: moving one centre coordinate by a single box radius multiplies the residual by 8694 and breaks the certificate.

(v) *The geometric conclusion* (Section 6), subject to the reduction described next: there are 58 points of ℓ_6^{56} at pairwise distance $0.9211600528\ldots$, so that $\mathrm{e}(\ell_6^{56}) \geq 58 > 57$.

1.4. **What is machine-checked, and what is not.** Throughout, two things are kept apart, and the reader should keep them apart too.

Machine-checked. Propositions 1 and 2, Theorems 4, 5 and 6, and Propositions 11, 12 and 13 have been checked by the Lean 4 kernel against the archived data: the eleven integers of the exponent-6 certificate, the five claims, the radius window, the constants, the negative controls, and the general Newton–Kantorovich lemma together with its instantiation at the certificate's constants. Where such a statement contains a step done by hand, it is named on the spot (there is exactly one, the convexity argument in Proposition 11). Section 8 describes the check.

Not machine-checked. The passage from those arithmetic facts to a statement about equilateral sets requires Lemma 3 and four bookkeeping steps: exhibiting the real-valued map F and its derivative from the integer arrays, the entrywise enclosure of the Jacobian on the box, the identification of the real quantities $\|BF(c)\|_\infty$ and the row sums of $|I - BJ|$ with the integers the kernel pins, and the reading of a zero as a configuration of 58 points. These are classical and are carried out by hand in Sections 4 and 6, but they are not formalised. They are listed as (R1)–(R4) in Section 6, the conditional Theorem 8 is the only theorem of this paper that depends on them, and every use is flagged. Section 3 is floating-point computation throughout, and the two remarks that report computations made outside the verified development say so in their headings.

2. Preliminaries

2.1. **The ansatz.** Fix $m = 58$ points $x^{(1)}, \ldots, x^{(58)} \in \mathbb{R}^{56}$ and a common p-th power distance D, and index the $P = \binom{58}{2} = 1653$ unordered pairs $\{i, j\}$ lexicographically; write $k \mapsto (i_k, j_k)$ for that indexing and $\Delta_{k,r}(x) = x_r^{(i_k)} - x_r^{(j_k)}$.

The equilateral condition at exponent p is the system (1) with 5 replaced by p: P equations in the 3248 coordinates and D, i.e. in 3249 unknowns.

Chalmers makes the system square by *selecting* 1652 coordinate positions — an injective list sel of 1652 values $i \cdot 56 + r$, followed by a marker for the unknown D — and freezing the other 1596 coordinates at their centre values. The result is a map $F\colon \mathbb{R}^{1653} \to \mathbb{R}^{1653}$, and a zero of F is an equilateral configuration provided $D > 0$ there. We reuse this data verbatim: the selection used below is byte for byte the one archived with [4], and so is the pair ordering.

2.2. **The contraction lemma.** The following is Chalmers' Lemma 3, an ∞-norm Newton–Kantorovich statement in the style of the verification methods surveyed by Rump [12]. It is stated for an arbitrary dimension P and is proved from Banach's fixed point theorem, the mean value inequality on the box, and — for the invertibility — the injectivity that $q < 1$ forces. Here $\bar{B}_\infty(c, \rho) = \{z : \|z - c\|_\infty \le \rho\}$, and $|M|$ is the entrywise absolute value of a matrix.

Proposition 1 (Chalmers' contraction lemma)**.** *Let $\rho > 0$, $\eta, q \ge 0$, let $c \in \mathbb{R}^P$, let $F\colon \mathbb{R}^P \to \mathbb{R}^P$ be differentiable on $\bar{B}_\infty(c, \rho)$ with derivative $z \mapsto J(z)$ there, and let B be a $P \times P$ matrix. Assume*

$$\begin{aligned} &\|BF(c)\|_\infty \le \eta, \\ &\textstyle\sum_b \big|(I - BJ(z))_{a,b}\big| \le q \quad \text{for all } z \in \bar{B}_\infty(c, \rho) \text{ and all rows } a, \\ &\eta + q\rho < \rho. \end{aligned}$$

Then F has exactly one zero z^* in $\bar{B}_\infty(c, \rho)$, and it satisfies $\|z^* - c\|_\infty \le \eta + q\rho$. Moreover $q < 1$, and B and $J(z)$ are invertible for every z in the box.

The hypothesis on J over the whole box is discharged by an entrywise enclosure, which is the form a rational certificate can meet.

Proposition 2 (interval form)**.** *For matrices B, J_c, E put*

$$q_a(B, J_c, E) \;=\; \sum_b \Big(\big|(I - BJ_c)_{a,b}\big| + \sum_k |B_{a,k}|\, E_{k,b} \Big).$$

If $|J(z)_{k,b} - (J_c)_{k,b}| \le E_{k,b}$ for all $z \in \bar{B}_\infty(c, \rho)$ and all k, b, then $\sum_b |(I - BJ(z))_{a,b}| \le q_a(B, J_c, E)$ for every a and every such z; so Proposition 1 applies with $q = \max_a q_a(B, J_c, E)$.

Both propositions, and the negative controls showing that neither the strictness of $\eta + q\rho < \rho$ nor the residual hypothesis may be dropped, are machine-checked; see Section 8. Note that E absorbs *all* the geometry: once the enclosure is available, everything the lemma consumes is a finite list of inequalities between explicit rationals.

3. Moving the exponent (numerical)

Nothing in this section is used in the proofs of Sections 4 to 6. It is floating-point computation, reported as the search that produced the certificate; the certificate is what carries the proof. We record it because it answers, quantitatively, the question quoted in the introduction.

3.1. **Inside the square slice the branch is a closed loop.** Let $F(\cdot, p)$ be the square system of Section 2 at exponent p, on Chalmers' frozen set, and continue its zero from his configuration. A predictor parametrised by p stalls near $p \approx 5.0367$ and again near $p \approx 4.99997$, which is what a fold looks like. A pseudo-arclength corrector walks through both: in a 498-step run the branch returns to the starting configuration, to $\|z - z_0\|_\infty = 4.6 \times 10^{-14}$, three times, so it is a *closed curve* in (z, p)-space with exactly two turning points,

$$p^- = 4.999974317610, \qquad p^+ = 5.036694109183, \qquad \text{width } 0.036720.$$

One implementation point is load-bearing: the new tangent must be oriented by the secant $(z_{k+1} - z_k,\, p_{k+1} - p_k)$ and not by the previous tangent, or the walk retraces itself at a sharp fold and manufactures the appearance of one. Three independent signatures confirm both folds are simple saddle-nodes: the smallest singular value of the square Jacobian collapses there; the two solution sheets below the fold separate like $\sqrt{\delta}$, with $\text{sep}/\sqrt{\delta}$ equal to 1.3054, 1.3091, 1.3095, 1.3095 at $\delta = 10^{-4}, 10^{-5}, 10^{-6}, 10^{-7}$; and Newton finds no zero at all on the far side. At $\delta = 10^{-5}$ both sheets were refined to residual $\approx 2 \times 10^{-39}$ in 40-digit arithmetic, and their separation agrees with the double-precision value to eleven digits, so the picture is not an artefact of conditioning. The crossing at $p = 5$ of the second sheet is a genuine second solution of Chalmers' square system, at $\|z - z_0\|_\infty = 6.63 \times 10^{-3}$ with $D = 1.000019828759$; it is not by itself a result, since the equilateral variety is about 1596-dimensional and second solutions are free there.

The consequence for the exponent is negative and complete: a closed loop of width 0.0367 around $p = 5$ reaches neither 4 nor 6 by any route.

3.2. **Outside it, the branch reaches $p = 7$.** The slice is a choice. At a fold of the slice the full Jacobian $A \in \mathbb{R}^{1653 \times 3249}$ still has full row rank, and the branch can be continued by a minimum-norm corrector: the tangent is $-A^\mathsf{T}(AA^\mathsf{T})^{-1}\partial F/\partial p$ and the correction is $A^\mathsf{T}(AA^\mathsf{T})^{-1}F$, each one sparse product and one Cholesky factorisation. From $p = 5$ this runs to $p = 7$ in 48 accepted steps and 25 seconds, with no fold: the smallest singular value of A never drops below 0.148, all 58 points stay distinct, and

$$D = 0.610957 \text{ at } p = 6, \qquad D = 0.375166 \text{ at } p = 7.$$

Downward the picture is different, and it is the expected one: the walk breaks at $p \approx 4.926704$ with $\sigma_{\min}(A)$ collapsing to 5.7×10^{-7} — a degeneracy of the manifold, not of a slice. Since Ge–Xu–Zhou [5] prove $\mathrm{e}(\ell_p^n) = n + 1$ throughout $2 \le p \le 4$, any such branch is obliged to break strictly above 4; this one breaks at 4.9267, so even taken at face value it locates no failing exponent below 4.9268, and it says nothing about the infimum of the failing exponents below that value.

4. The certificate at exponent 6

4.1. **The data.** The exponent-6 certificate consists of four integer arrays. Three are new:

$$K \in \mathbb{Z}^{58 \times 56}, \qquad d_K \in \mathbb{Z}, \qquad BI \in \mathbb{Z}^{1653 \times 1653},$$

giving the centre $c_{i,r} = K_{i,r}/2^{50}$, the centre distance $D_c = d_K/2^{50}$ and the preconditioner $B = BI/2^{36}$. The fourth, the selection sel, is Chalmers' own file unchanged. The centre was produced as described in Section 3: the $p = 6$ point of the full-space continuation, rounded to the grid 2^{-50} (which pins the 1596 frozen coordinates),

then refined in the 1652 free coordinates and D by three Newton steps in exact integer arithmetic over 2^{110}, taking the residual from 2.2×10^{-15} through 1.9×10^{-25} to 2.2×10^{-33}, and rounded back to 2^{-50}; finally $B=\mathrm{round}(J(c)^{-1}2^{36})$. Chalmers' selection remains nonsingular at the new exponent, $\|J(c)^{-1}\|_\infty = 2.14931\times10^4$; a rank-revealing factorisation over all 3248 coordinate columns proposes a worse conditioned slice, 1.0163×10^5, so his choice is not merely serviceable at $p=6$, it is better than the obvious alternative. The scale 2^{36} for B, against 2^{40} at $p=5$, is what the archive's fixed-width encoding allows once $\|J(c)^{-1}\|_\infty$ has grown by a factor 5.4.

4.2. **Why the even exponent removes a hypothesis.** At $p=5$ the map (1) is not polynomial, and Chalmers recovers polynomiality on the box by a sign-constancy claim: every moving coordinate difference Δ satisfies $|\Delta|>w\rho$, where $w\in\{1,2\}$ counts its movable endpoints, so no Δ changes sign on the box and $|\Delta|^5=\pm\Delta^5$ there with a fixed sign. That claim also underlies his Jacobian enclosure, $|J(z)-J(c)|_{k,b}\le 5[(|\Delta|+w\rho)^4-\Delta^4]$.

At an even exponent both disappear. The system is

$$F_k(z)=\sum_{r<56}\Delta_{k,r}(z)^6-D,\qquad k<1653, \tag{2}$$

a polynomial in the coordinates outright, with $\partial F_k/\partial x_r^{(i)}=\pm6\,\Delta_{k,r}^5$ and $\partial F_k/\partial D=-1$; and the enclosure holds unconditionally, because of the following elementary inequality.

Lemma 3. *Let $h\ge0$ and $v,t\in\mathbb{R}$ with $|t-v|\le h$. Then*

$$|t^5-v^5|\ \le\ (|v|+h)^5-|v|^5.$$

Proof. Both sides are invariant under $(t,v)\mapsto(-t,-v)$, so assume $v\ge0$. Since $u\mapsto u^5$ is increasing on $\mathbb{R}$ and $t\in[v-h,v+h]$, we have $|t^5-v^5|\le\max\{(v+h)^5-v^5,\ v^5-(v-h)^5\}$. The first quantity is the asserted bound. For the second, set $\varphi(v)=(v+h)^5+(v-h)^5-2v^5$; then $\varphi(0)=0$ and $\varphi'(v)=5[(v+h)^4+(v-h)^4-2v^4]\ge0$ for every real v, by convexity of $u\mapsto u^4$ on $\mathbb{R}$. Hence $\varphi(v)\ge0$ for $v\ge0$, which is $v^5-(v-h)^5\le(v+h)^5-v^5$. □

Note that the case $|v|<h$ — exactly the case a sign-constancy hypothesis would exclude — is included. Consequently, with $E_{k,b}(\rho)=6\big[(|\Delta_{k,r}(c)|+w\rho)^5-|\Delta_{k,r}(c)|^5\big]$ for the column b carrying coordinate (i,r), and $E_{k,b}=0$ for the D column, Lemma 3 applied with $h=w\rho$ gives $|J(z)_{k,b}-J(c)_{k,b}|\le E_{k,b}(\rho)$ on the whole box, for every exponent-6 configuration and with no hypothesis on the size of Δ. The exponent-6 certificate therefore has *five* claims where the published one has six, and the reduction of Section 6 is correspondingly shorter.

4.3. **The five claims.** Every quantity is an integer over a fixed power of two, so every claim is a comparison of integers:

quantity	denominator
K, d_K, the radius numerator R	2^{50}
$F(c)$	2^{300}
$B=BI/2^{36}$	2^{36}
$BF(c)$, η	2^{336}
$J(c)=6\Delta^5$, E	2^{250}
$BJ(c)$, q, the identity	2^{286}

Writing $\eta^{\mathrm{num}} = 2^{336}\eta$ and $q^{\mathrm{num}}(R) = 2^{286}q(R)$ for the integer numerators and $\rho = R/2^{50}$, the Kantorovich test $\eta + q\rho < \rho$ clears denominators to

$$\eta^{\mathrm{num}} + q^{\mathrm{num}}(R) \cdot R \;<\; R \cdot 2^{286}. \tag{3}$$

Theorem 4. *For the exponent-6 data of this section, and with $\rho = 2^{-33}$:*

(C1) (schema) sel *consists of* 1652 *distinct coordinate positions in* $[0, 3248)$ *followed by the marker for D; the lexicographic pair list is complete and repetition-free;* $|K_{i,r}| < 2^{50}$ *for all* i, r, *and* $0 < d_K < 2^{50}$.

(C3) (positivity) $\rho < D_c - \rho$, *where*

$$D_c \;=\; \frac{687876319827206}{2^{50}} \;=\; 0.610956902693266\ldots;$$

hence $D > 0$ at every point of the box.

(C4) (residual) $\|F(c)\|_\infty < 2^{252}/2^{300} = 2^{-48}$, *i.e.* $< 3.553 \times 10^{-15}$; *and* $\eta := \|BF(c)\|_\infty < 2^{285}/2^{336} = 2^{-51}$, *i.e.* $< 4.441 \times 10^{-16}$.

(C5) (contraction) $q(\rho) < 2^{274}/2^{286} = 2^{-12} = 2.441 \times 10^{-4}$.

(C6) (Kantorovich test) $\eta + q(\rho)\rho < \rho$; *in the cleared form* (3) *at* $R = 2^{17}$ *the left-hand side is below* 2^{291} *while the right-hand side is* $2^{17} \cdot 2^{286} = 2^{303}$, *so the test clears by at least a factor* $2^{12} = 4096$; *the ratio of the right-hand side to the actual left-hand side is* 4176.

The exact values are $\|F(c)\|_\infty = 2.397484 \times 10^{-15}$ *(*252 *bits over* 2^{300}*),* $\eta = 4.432033 \times 10^{-16}$ *(*285 *bits over* 2^{336}*) and* $q(2^{-33}) = 2.356334 \times 10^{-4}$ *(*274 *bits over* 2^{286}*).*

There is no (C2): the sign-constancy claim of the $p = 5$ scheme is not part of this certificate, by Lemma 3.

Proof, and what was computed. All five claims are decidable comparisons of integers, and the whole content is a single exhaustive evaluation over the archived arrays. That evaluation decodes $3248 + 1 + 1653 + 1653^2$ fixed-width base-64 fields and computes eleven numerals: the Boolean of (C1) (a scan over 3248 coordinate positions, 1652 selection entries and 1653 pairs); d_K; $\|F(c)\|_\infty$, from 1653 residuals each a 56-term sum of sixth powers of 50-bit integers; η, a 1653×1653 matrix–vector product, 2.73×10^6 big-integer multiply–accumulates; q at three radii ($R = 2^{17}$, $R = 556278737$ and $R = 556278738$), which is 1.56×10^8 multiply–accumulates for the radius-free half of the row sums plus one matrix–vector product per radius; and the constants $\|B\|_\infty$, $\max_{k,r}|\Delta_{k,r}|$, $\min_{k,r}|\Delta_{k,r}|$ and the maximal number of nonzero Jacobian entries in a pair row. The eleven numerals are pinned by that one evaluation; each claim above is then an inequality between explicit numerals, decided by the kernel. Sums over rows are exact integer arithmetic throughout, so no rounding enters anywhere. The same eleven numerals were reproduced independently by a C program written from the certificate scheme and using GMP, in 12 seconds; the two agree in every digit. □

The next statement is the certificate's real content: with the enclosure and the identifications in place, Proposition 1 fires at these constants. We state it in the form in which it is machine-checked, i.e. with the real-valued data as hypotheses.

Theorem 5. *Let $c \in \mathbb{R}^{1653}$, let $F\colon \mathbb{R}^{1653} \to \mathbb{R}^{1653}$ be differentiable on $\bar{B}_\infty(c, 2^{-33})$ with derivative J, and let B, J_c, E be 1653×1653 matrices such that $|J(z)_{k,b} - (J_c)_{k,b}| \le E_{k,b}$ on that box, $|(BF(c))_a| \le \eta$ for every a, and $q_a(B, J_c, E) \le q$ for*

every a, where η and q are the exponent-6 constants of Theorem 4. Then F has exactly one zero z^* in $\bar{B}_\infty(c, 2^{-33})$, and

$$\|z^* - c\|_\infty \;\le\; \eta + q\,2^{-33} \;=\; 2.787454 \times 10^{-14} \;<\; 2^{-45}.$$

Proof. Immediate from Propositions 1 and 2 and the inequality (C6) of Theorem 4; the localisation bound is the comparison $\eta^{\mathrm{num}} + q^{\mathrm{num}} 2^{17} < 2^{291}$ read over 2^{336}. □

5. The radius window, the constants, and the controls

The Kantorovich test can be run at any radius, and the cost of doing so is small: writing $q(R) \cdot 2^{286} = \max_a [\, S_a + \sum_k |BI_{a,k}| E_k(R) \,]$, the row sums S_a of $|I - BJ(c)|$ do not depend on R and carry the whole 1.56×10^8-operation cost, while $E_k(R)$ is a positive-coefficient quintic in R vanishing at $R = 0$ and pairing it against $|B|$ is a matrix–vector product. So one expensive pass answers several radii at once.

Theorem 6. *For the exponent-6 certificate, with $\rho = R/2^{50}$:*

(i) *the Kantorovich test passes at $R = 556278737$, i.e. at $\rho = 4.940748 \times 10^{-7}$, which exceeds $4244 \cdot 2^{-33}$; consequently F has exactly one zero in that box, under the hypotheses of Theorem 5 read at this radius;*

(ii) *the test fails at $R = 556278738$, one unit of the grid above; so (i) is sharp on this grid;*

(iii) *at the passing end $q = 0.9999999974$, so the window closes where the contraction does and not because of a slack bound;*

(iv) *the constants are*

$$\|B\|_\infty = 1476991237595771/2^{36} = 21493.05,$$
$$\max_{k,r} |\Delta_{k,r}(c)| = 1034078324825207/2^{50} = 0.918446052 < 1,$$
$$\min_{k,r} |\Delta_{k,r}(c)| = 523058992/2^{50} = 4.6457 \times 10^{-7},$$

the last being 3990.6 box radii, so no coordinate difference of the centre vanishes; and no pair row of the Jacobian has more than 72 nonzero entries;

(v) (non-vacuity) $\|F(c)\|_\infty > 0$, $2^{284} < \eta^{\mathrm{num}} < 2^{285}$ *and* $2^{273} < q^{\mathrm{num}} < 2^{274}$*: the bound on η in (C4) and the bound on q in (C5) are each attained to within a factor 2 (numerically, to 99.8% and to 96.5%), so neither claim is trivially true;*

(vi) (perturbation control) *moving a single centre coordinate by one box radius 2^{-33} takes the residual above $2^{264}/2^{300} = 1.46 \times 10^{-11}$, i.e. multiplies it by more than 6000, so (C4) fails outright for the perturbed data. (What the kernel pins here is the inequality; the auxiliary exact computation of Section 8 evaluates the perturbed residual as 2.0844×10^{-11}, a factor 8694.)*

Proof, and what was computed. Items (i)–(iii) are the comparison (3) at the two radii, with the two values of $q^{\mathrm{num}}(R)$ pinned by the same exhaustive evaluation as Theorem 4; (iv) and (v) likewise; (vi) is a second evaluation of the residual, over the array with one entry incremented by 2^{17}. The passing and the failing statement are each a single comparison of integers of about 316 bits. □

Remark 7 (the lower end of the window; a computation outside the kernel). The auxiliary GMP computation reports that the test also passes at every radius numerator down to $R = 1$, the smallest on the grid, so that this certificate — unlike

the published one, which needs $R \geq 61$ (Proposition 11) — would have no lower obstruction at all; η is simply too small for one. The three radii the kernel decides are $R = 2^{17}$, $R = 556278737$ and $R = 556278738$, and no more: Theorem 6 therefore asserts nothing about small R, and the sentence above is reported here only as a numerical observation.

6. The geometric conclusion

This is the one section whose conclusion is not machine-checked end to end; see Section 1.4.

Theorem 5 is a statement about any real data satisfying its hypotheses. To obtain a statement about equilateral sets one must exhibit that data from the archived integers. Four steps are involved.

(R1) *The system.* With the 1596 frozen coordinates held at their centre values $K_{i,r}/2^{50}$, and $z \in \mathbb{R}^{1653}$ collecting the 1652 selected coordinates and D, the map F of (2) is polynomial, hence differentiable, with $J(z)_{k,b} = \pm 6\Delta_{k,r}(z)^5$ for the column b carrying a selected coordinate (i, r) occurring in the pair k (the sign being $+$ if $i = i_k$ and $-$ if $i = j_k$), $J(z)_{k,b} = 0$ if (i, r) does not occur in the pair k, and $J(z)_{k,b} = -1$ for the column of D.

(R2) *The enclosure.* With E as defined after Lemma 3, $|J(z)_{k,b} - J(c)_{k,b}| \leq E_{k,b}(\rho)$ for all z in the box. This is Lemma 3 with $h = w\rho$, $w \in \{1, 2\}$ the number of selected endpoints of the coordinate difference, together with the observation that the D column is constant.

(R3) *The identification.* With $c_{i,r} = K_{i,r}/2^{50}$, $D_c = d_K/2^{50}$ and $B = BI/2^{36}$, the real numbers $(BF(c))_a$ and $q_a(B, J(c), E)$ are $\big(\sum_b BI_{a,b} f_b\big)/2^{336}$ and $\big(S_a + \sum_k |BI_{a,k}| E_k\big)/2^{286}$, with f, S and E the integer arrays of Section 4. This is index bookkeeping at scale 1653×1653: the selection order indexes the rows of BI, the lexicographic pair order indexes its columns, and BI is stored column-major in the archive.

(R4) *The geometric reading.* If $F(z^*) = 0$ and the D-coordinate D^* of z^* is positive, then the 58 points that z^* describes are pairwise at ℓ_6-distance $(D^*)^{1/6} > 0$, and in particular distinct.

(R1), (R2) and (R4) are proved above or immediate; (R3) is a finite check that was carried out by hand and, independently, by the C program of Section 4, which reads the same arrays under the same conventions and reproduces every numeral. None of the four is formalised.

Theorem 8. (Conditional on (R1)–(R4), which are not machine-checked.) *There is an equilateral set of* 58 *points in* ℓ_6^{56}. *Its common distance is* $(D^*)^{1/6} = 0.92116005275566$ *with an error below* 10^{-14}, *where* D^* *is the* D*-coordinate of the unique zero of Theorem 5 and* $|D^* - D_c| < 2^{-45}$. *Consequently*

$$\mathrm{e}(\ell_6^{56}) \;\geq\; 58 \;>\; 57 \;=\; 56 + 1,$$

so Kusner's conjecture fails at $p = 6$.

Proof. By (R1)–(R3) the hypotheses of Theorem 5 hold for the exponent-6 data at $\rho = 2^{-33}$, so F has a zero z^* with $\|z^* - c\|_\infty \leq \eta + q\rho < 2^{-45}$. In particular $|D^* - D_c| < 2^{-45}$, and (C3) of Theorem 4 gives $D^* \geq D_c - \rho > 0$. By (R4) the corresponding 58 points of $\mathbb{R}^{56}$ are pairwise at ℓ_6-distance $(D^*)^{1/6}$; they are 58 distinct

points, so $\mathrm{e}(\ell_6^{56}) \geq 58$. The numerical value follows from $D_c = 0.610956902693266$ and $|D^* - D_c| < 2^{-45}$, since $\left|\frac{d}{dD} D^{1/6}\right| < 1/4$ near D_c. □

Remark 9. The bound $\mathrm{e}(\ell_6^{56}) \leq 169$ of [14, 5] is untouched: 58 points is far inside what is allowed, and the interest of the statement is only that it exceeds $n + 1$. What is new is the exponent. Before [3] no exponent $p > 2$ was known at which the conjecture fails; [3] gives $p = 5$ and states that its example is the first at any finite $p \geq 2$; Theorem 8 gives a second, at an even exponent, where the certificate scheme is structurally simpler.

Remark 10 (what would remove the condition). (R1)–(R4) are bookkeeping, not mathematics, and the even exponent makes them shorter than at $p = 5$: sign constancy drops out of the chain entirely, so (R2) is the lemma above rather than a case analysis on the size of Δ. Formalising them is a finite amount of work at 1653×1653 scale: in the accompanying development the pointwise restatements it needs were measured to cost a 20–30% overhead in evaluation time over the array form, not a blow-up.

7. The published certificate, measured by the same instruments

The computations of Sections 4 and 5 apply verbatim to the archived $p = 5$ certificate [4], and it is worth recording what they say about it, both because it calibrates the new certificate and because two of the answers are not in [3]. All statements in this section are machine-checked in the same sense as before; where a conclusion needs the reduction of Section 6, that is said explicitly.

Proposition 11. *For the archived exponent-5 certificate, the Kantorovich test passes at the radius numerators $R = 61$ and $R = 2548288865$ and fails at $R = 60$ and at $R = 2548288866$. Moreover $R \mapsto R\,q(R)$ is a maximum of polynomials with non-negative coefficients and zero constant term, hence convex, so $R \mapsto R\big(1-q(R)\big)$ is concave and the set of radii at which the test passes is an interval; it is therefore exactly $61 \leq R \leq 2548288865$, i.e. $5.4179 \times 10^{-14} \leq \rho \leq 2.2633 \times 10^{-6}$. The upper end exceeds $19441 \cdot 2^{-33}$, so — subject to the reduction (R1)–(R4) in its exponent-5 form — the published configuration is the only zero of the ansatz system within 2.2633×10^{-6} of the archived centre, where [3] states uniqueness only within 2^{-33}. (The convexity step is by hand; the four inequalities are machine-checked.)*

Proposition 12. *For the same certificate, $\|B\|_\infty = 4379884679060370/2^{40} = 3983.4819$, $\max_{k,r}|\Delta_{k,r}| = 0.99202156 < 1$, and no pair row of the Jacobian has more than 72 nonzero entries. At the radius $\rho^\dagger = 1274000000/2^{50}$ one has $q(\rho^\dagger) = 0.4999406$ — within 2×10^{-4} of the value $1/2$ that maximises $\rho(1-q)$ when q is affine in ρ — and*

$$\rho^\dagger\big(1 - q(\rho^\dagger)\big) \;>\; 10^7\,\eta,$$

where $\eta = 5.3690863607 \times 10^{-14}$ is that certificate's preconditioned residual; so at its best radius the published certificate leaves a factor 1.05×10^7 of the Kantorovich budget unused.

Two comparisons follow. The exponent-6 certificate has the smaller preconditioned residual by a factor 121 ($\eta = 4.4320 \times 10^{-16}$ against $5.3690863607 \times 10^{-14}$) and the larger preconditioner norm by a factor 5.4; the widest radius at which its test passes, 4.9407×10^{-7}, is smaller than the published certificate's 2.2633×10^{-6}. Both certificates are used, in the papers that introduce them, at $\rho = 2^{-33}$, which is

about 10^{-4} of the usable radius in the $p = 5$ case and about 2×10^{-4} in the $p = 6$ case.

Proposition 13. *Let $x^{(1)}, \dots, x^{(58)} \in \mathbb{R}^{56}$ satisfy $|x_r^{(i)} - c_{i,r}| \le 2^{-33}$ for all i, r, where c is the archived exponent-5 centre — so in particular for the configuration of* [3]. *Suppose $\sigma\colon \{1, \dots, 58\} \to \{1, \dots, 58\}$, a permutation τ of the coordinates, signs $\varepsilon_r \in \{\pm 1\}$ and a translation $t \in \mathbb{R}^{56}$ satisfy $x_r^{(\sigma i)} = \varepsilon_r x_{\tau r}^{(i)} + t_r$ for all i, r. Then $\sigma = \mathrm{id}$, $\tau = \mathrm{id}$, $\varepsilon \equiv 1$ and $t = 0$. (σ is not assumed injective; the separation of the configuration forces it.)*

Proof, and what was computed. The invariants are the 1653 pairwise ℓ^1 distances $L_{ij} = \sum_r |x_r^{(i)} - x_r^{(j)}|$, which are invariant under coordinate permutations, sign flips and translations, and the 56 coordinate differences of one fixed pair. An exhaustive computation over the archived centre shows that its 1653 values L_{ij} are pairwise more than $2 \cdot 112 \cdot 2^{-33}$ apart and each larger than $112 \cdot 2^{-33}$, and that the 56 first-pair coordinate differences are pairwise more than $2 \cdot 2 \cdot 2^{-33}$ apart and each larger than $2 \cdot 2^{-33}$; since moving every coordinate by at most 2^{-33} moves L_{ij} by at most $112 \cdot 2^{-33}$ and a coordinate difference by at most $2 \cdot 2^{-33}$, the separations survive on the whole box and force the maps to be trivial. The separation is comfortable: the same argument still runs at box radius $7146209/2^{50} = 6.35 \times 10^{-9}$, which is 54.5 times the radius used here, and the corresponding threshold on the L values fails one unit of the grid above. □

Two classical facts turn this into a statement about isometries. By Mazur–Ulam [11], a surjective isometry of a real normed space is affine; and for $1 \le p < \infty$, $p \ne 2$, the linear isometries of ℓ_p^n are exactly the signed permutations of the coordinates, $x \mapsto (\varepsilon_r x_{\tau r})_r$ with $\varepsilon \in \{\pm 1\}^n$ and τ a permutation (Li and So [9]; this is the finite-dimensional case of the description going back to Banach [2] and Lamperti [7]). Every surjective isometry of ℓ_5^{56} therefore has the form $y \mapsto (\varepsilon_r y_{\tau r} + t_r)_r$, so Proposition 13 says: the only isometry of ℓ_5^{56} carrying the configuration of [3] onto itself is the identity. (Read as a metric space in its own right, an equilateral set is of course invariant under every permutation of its points; what is trivial here is its symmetry group *inside the ambient space*.) That classification is cited, not verified here. The statement answers the sentence ending Section 2 of [3], "We know of no symmetry or compressed description of the configuration".

Remark 14 (a quantitative form of Chalmers' Corollary 2; not machine-checked). The unused budget of Proposition 12 can be spent on perturbing the exponent instead of the radius. Running the same test at exponent p over the box of radius $\rho^\dagger$, and bounding the residual and the Jacobian perturbation by $|e^u - 1| \le 2|u|$ together with $|\log|\Delta|| \le \log(1/\delta) = 15.470186$ — where $\delta = 1.9115 \times 10^{-7}$ is the minimum coordinate gap of the centre — gives $\eta_p \le \eta + 1.23312 \times 10^5 \varepsilon$ and $q_p \le q(\rho^\dagger) + 4.51498 \times 10^7 \varepsilon$ with $\varepsilon = |p - 5|$, so that the Kantorovich test still closes as long as $\varepsilon < 4.5868 \times 10^{-12}$. Subject to a reduction of the kind (R1)–(R4) at exponent p — which at a non-integer exponent also needs the sign-constancy step of the $p = 5$ scheme, and which is likewise not formalised — this yields $\mathrm{e}(\ell_p^{56}) \ge 58$ for every real p with $|p - 5| \le 4 \times 10^{-12}$, a quantitative form of Chalmers' Corollary 2, whose remark states that his argument gives no such control. Every constant entering it is machine-checked, but the argument itself is not formalised, and we record it as a computation outside the verified development. At the radius 2^{-33}

used in [3] the same argument would give only $\varepsilon < 9.44 \times 10^{-16}$, a factor 4861 worse: the enlarged box of Proposition 11 is what makes the estimate usable.

8. Verification

The statements listed as machine-checked in Section 1.4 are formalised in Lean 4 [8] against Mathlib [10] and checked by its kernel. The development is compiled with the toolchain `leanprover/lean4:v4.32.0` against Mathlib at revision `81a5d257c8e410db227a6665ed08f64fea08e997` (the `v4.32.0` tag). The exponent-6 certificate data is carried in the development as fixed-width base-64 blocks, with md5 checksums

centre K	`13e69afd9e5c113bfcd122b912c462e1`
centre distance d_K	`283235104c82961b9649f8a9cd434b71`
preconditioner BI (21.9 MB)	`d3ed0f945ff5f1a6a37a9fe210379cc6`

and with the selection block taken byte for byte from [4], whose archive has md5 `b77edd39cab9f3c73a584b9e80e3d2e2`. The trusted surface of the exponent-6 results is exactly two compiled evaluations, each of the form "the number computed from these blocks is this numeral": the eleven numerals of Theorem 4 and Theorem 6(i)–(v), and the perturbed residual of Theorem 6(vi). Every inequality — (C3)–(C6), both ends of the window, the non-vacuity bounds and the controls — is decided by the kernel on numerals, and the three that carry the result (the Kantorovich test at $\rho = 2^{-33}$, the test at the wide end of the window, and its failure one unit above) depend on no axioms at all; the remaining declarations depend only on `propext`, `Quot.sound`, the two evaluation axioms just named, and (on the real-valued side, through Mathlib's instance chains) `Classical.choice`. There is no `sorry` and no additional axiom anywhere in the development. The exponent-6 module takes 313 s of processor time at a peak of 1.74 GB, and the real-valued module 8 s at 7.00 GB. Everything is reproducible from the blocks alone, without the archive, and was reproduced independently by a GMP program written from the certificate scheme (12 s), which agrees with the kernel on every digit. The repository is private at the time of writing; repository reference to be added at submission.

Acknowledgement of provenance

The 58-point ansatz, the frozen selection, the variable order, the contraction lemma of Section 2 and the certificate format are due to Chalmers [3]. What is new here is the observation that the fold in p belongs to the square slice rather than to the equilateral variety, the exponent-6 configuration and its certificate, the radius windows and constants of Sections 5 and 7, and the rigidity statement Proposition 13.

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