THERE ARE EXACTLY 178 KRASNER HYPERFIELDS OF ORDER EIGHT, AND WHICH HYPERFIELDS OF ORDER AT MOST EIGHT ARE QUOTIENTS OF FINITE FIELDS

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Abstract. A Krasner hyperfield is a field-like structure whose addition is multivalued. Let H_n denote the number of isomorphism classes of hyperfields of order n. The values $H_2, \ldots, H_6 = 2, 5, 7, 27, 16$ are due to Ameri, Eyvazi and Hošková-Mayerová and $H_7 = 277$ to Massouros and Massouros. We compute the next value: $H_8 = 178$. The lower bound is a list of 178 pairwise non-isomorphic tables; the upper bound is an exhaustive search over all $128^4 = 268\,435\,456$ tables admitted by an elementary reduction, which is stated and proved by hand in full below. In the notation of the On-Line Encyclopedia of Integer Sequences this is $A343596(8) = 178$, the published entry stopping at order 6.

We then determine, for every order n with $3 \leq n \leq 8$, exactly which of the H_n isomorphism classes are quotients $\mathbb{F}_q/G$ of a finite field by a subgroup of its multiplicative group. The counts are 4 of 5, 4 of 7, 9 of 27, 7 of 16, 15 of 277 and 9 of 178 at orders $3, 4, 5, 6, 7, 8$; the first two reproduce a theorem of Baker and Jin. The order-5 case answers the finite-field half of Baker and Jin's open question (1): the nine quotient classes all have multiplicative group C_4, and none of the eleven classes with multiplicative group $C_2 \times C_2$ is a quotient of a finite field. The infinite-field half of their question is untouched and remains open. The counts $9, 7, 15$ at orders $5, 6, 7$ agree with counts obtained by Linzi by a different route; what is new for those orders is the identification of *which* classes they are.

The numbered results below are machine-checked in Lean 4. Section 7 says which of them rest on exhaustive computation, how large each search is, and which steps of the argument are made by hand rather than by machine.

1. Introduction

The objects. A *hyperoperation* on a set F assigns to each ordered pair (x, y) a nonempty subset $x \boxplus y \subseteq F$; it is extended to subsets by $A \boxplus B = \bigcup_{a \in A, b \in B} a \boxplus b$. We use throughout the definition of a hyperfield in the sense of Krasner [6, 7], as it is stated in [10, Definition 2.1].

Definition 1.1. A *hyperfield* is a tuple $(F, \boxplus, \cdot, 0, 1)$ in which $(F \setminus \{0\}, \cdot)$ is an abelian group, $(F, \boxplus, 0)$ is a canonical hypergroup — that is, $\boxplus$ is commutative and associative, 0 is a two-sided identity for it, every x has a unique y with $0 \in x \boxplus y$ (written $y = -x$), and $\boxplus$ is *reversible*: $z \in x \boxplus y$ implies $x \in z \boxplus (-y)$ — multiplication distributes over $\boxplus$, and $0 \cdot x = 0$ for all x.

We write $F^\times = F \setminus \{0\}$, call $|F|$ the *order* of F, and let

$$H_n \; = \; \#\{\text{isomorphism classes of hyperfields of order } n\}.$$

The standard source of examples is the quotient construction, again as in [10, Definition 2.2].

Definition 1.2. Let K be a field and $G \leq K^\times$ a subgroup of index r. The *Krasner quotient* K/G has underlying set $(K^\times/G) \cup \{0\}$, the coset product as its multiplication, and hyperaddition

$$[x] \boxplus [y] \; = \; \{\, [x + yg] \; : \; g \in G \,\} \; \cup \; \{0 \text{ if } 0 \in xG + yG\}.$$

It is a hyperfield of order $r + 1$. When $K = \mathbb{F}_q$ is finite we write G_r for the unique subgroup of index r of $\mathbb{F}_q^\times$, which exists exactly when $r \mid q - 1$; then $(\mathbb{F}_q/G_r)^\times$ is cyclic of order r.

Not every hyperfield is of this form [12]. Which ones are is the question this paper addresses at small orders.

What was known. Two lines of work meet here.

The first is enumeration. Ameri, Eyvazi and Hošková-Mayerová [1] determine H_n for $n \leq 6$: the values are $2, 5, 7, 27, 16$ for $n = 2, \ldots, 6$, the order-5 count splitting as 16 classes with multiplicative group C_4 and 11 with multiplicative group $C_2 \times C_2$. Massouros and Massouros [13] classify order 7 and obtain $H_7 = 277$. These five values are the entry A343596 of the On-Line Encyclopedia of Integer Sequences [15], "Number of finite Krasner hyperfields with n elements up to isomorphism", which carries the data $2, 5, 7, 27, 16$ with offset 2 and the keyword `more`; neither $a(7) = 277$ nor any later term has been entered, and the entry has not been edited since April 2021.

The second is the structure theory of quotients. Baker and Jin [2] prove that for each fixed $r \geq 2$ there is a threshold N_r such that for every prime power $q \geq N_r$ with $q \equiv 1 \pmod{r}$ the quotient $\mathbb{F}_q/G_r$ lies in one of at most two isomorphism classes $\mathbb{H}_r, \mathbb{H}'_r$, a single class when r is odd and two separated by the class of q modulo $2r$ when r is even (their Theorem 1.1); their Remark 1.2 lets one take N_r to be the least positive integer N with $(N+1) - (r-1)(r-2)N^{1/2} - 3r > 0$. Using this they settle the small orders (their Theorem 1.3), which we quote:

> There are 2 isomorphism classes of hyperfields of order 2 … Every hyperfield of order 3 is a quotient of a field. More precisely, there are 5 isomorphism classes of hyperfields of order 3, all but one of which are isomorphic to quotients of finite fields. The hyperfield of signs is isomorphic to $\mathbb{R}/\mathbb{R}_{>0}$ but not to any quotient of a finite field. There are 7 isomorphism classes of hyperfields of order 4. Of these, 4 are quotients of finite fields, 1 is a quotient of an infinite field but not of any finite field, and the remaining 2 are not quotients of any field.

Their first open question, which they state as follows, asks for the next order:

> What is the complete list of hyperfields of order 5, and which of these are quotients of finite (resp. infinite) fields? In principle the methods of this paper give an algorithm to enumerate all hyperfields of order 5 and determine which ones are quotients of finite fields, but the computations are probably too involved to do by hand and we have not attempted to write the necessary computer code. It is not clear *a priori* whether the methods of this paper will suffice to determine whether or not each such hyperfield is a quotient of some infinite field; some new ideas may be required.

Linzi [10] takes up the computational side. Writing Q_r^{fin} for the number of isomorphism classes of hyperfields of order $r+1$ that arise as $\mathbb{F}_q/G_r$ for some prime power q — a finite number, by Baker and Jin's theorem — he computes $Q_r^{\mathrm{fin}} = 2, 4, 4, 9, 7, 15$ for $r = 1, \ldots, 6$ and tabulates them against the published values of H_{r+1}. His Discussion item 3 records what is missing:

> **Order 5 in full.** Baker–Jin open question (1) asks for all hyperfields of order 5 and which are field quotients. We determine the finite-field quotient part ($Q_4^{\mathrm{fin}} = 9$); matching against a full abstract enumeration remains.

That is, the two sides of question (1) had been computed separately and never joined: the census of *all* hyperfields of order 5 on one side, the count of quotient classes on the other, with no map from the second into the first. At order 7 the same gap is wider still: the abstract of [13] reports only that both quotient and non-quotient hyperfields occur among the 277, with no breakdown.

Results. Let H_n be as above.

(1) *The census at order* 8. $H_8 = 178$; equivalently $A343596(8) = 178$. This is Theorem 3.3 and Corollary 3.4. The multiplicative group of a hyperfield of order 8 is C_7; the lower bound is an explicit list of 178 tables over C_7, verified to be hyperfields and to be pairwise non-isomorphic, and the upper bound is an exhaustive search over the $128^4 = 268\,435\,456$ tables admitted by Lemma 2.4.

(2) *Baker and Jin's question (1), finite-field half.* Of the 27 isomorphism classes of hyperfields of order 5, exactly 9 are quotients of finite fields and 18 are not; the 9 lie among the 16 classes whose multiplicative group is C_4, and none of the 11 classes whose multiplicative group is

$C_2 \times C_2$ is a quotient of a finite field. This is Theorem 4.2. Nine explicit finite fields realising the nine classes are named there.

(3) *The same matching at orders* 6*,* 7 *and* 8*.* Exactly 7 of the 16 hyperfields of order 6, exactly 15 of the 277 of order 7, and exactly 9 of the 178 of order 8 are quotients of finite fields (Theorem 4.3). The counts 7 and 15 are Linzi's Q_5^{fin} and Q_6^{fin} and are reproduced, not claimed; what is new at those orders is the identification of the classes inside the census. At order 8 both the census and the matching are new.

(4) *The two orders that were already known*, recomputed as a control on the method: the census at orders 3 to 7 (Theorem 3.1) and the quotient classes at orders 3 and 4 (Theorem 4.1), the latter reproducing Baker and Jin's Theorem 1.3.

Collected, the picture is the following, the last column counting classes that are not quotients of any finite field.

order n	H_n	quotients of finite fields	non-quotients	
3	5	4	1	[2], Theorem 1.3
4	7	4	3	[2], Theorem 1.3
5	27	9	18	Theorem 4.2
6	16	7	9	Theorem 4.3
7	277	15	262	Theorem 4.3
8	178	9	169	Theorems 3.3 and 4.3

What is not claimed. Three limits are worth stating at the outset, and are repeated where they bite.

First, "non-quotient" always means "not a quotient of a *finite* field". The infinite-field half of Baker and Jin's question (1) — how many of the 18 order-5 non-quotients are quotients of infinite fields — is untouched here. At order 4 they show that exactly one of the three non-quotients is a quotient of an infinite field; we know of no analogue at order 5 or beyond.

Second, the completeness half of every quotient statement — that the classes listed are the only ones — is not a finite check. It combines Baker and Jin's Theorem 1.1, which bounds the prime powers that can produce a non-stable class, with a finite scan of prime powers carried out here. Section 4 says exactly what was scanned, and Section 7 says which parts of the argument are machine-checked and which are not.

Third, the passage from "every hyperfield of order 8" to "every table in the search space of the exhaustive computation" is Lemma 2.4, a short argument proved here by hand and not by machine. Theorem 3.3(ii) is stated over the reduced search space, and Corollary 3.4 is where the two are joined.

We say nothing about order 9; see Remark 3.7.

Provenance. We searched for a prior determination of H_8 and for a prior matching of quotient classes against a census at orders 5 to 8, and found neither. Every search reported here was run on 3 September 2026. A full-text snapshot of the arXiv holding 889 709 papers through 2 September 2026 was grepped for *hyperfield*, *canonical hypergroup* and *Krasner hyperring*; of the 7837 matching lines, none also mentions order 8, the number 178 or the sequence A343596. The arXiv API was queried for *hyperfield*, *hyperfields of order*, *enumeration of hyperfields* and *canonical hypergroup*, the newest hyperfield paper it returns being from 23 August 2026; the publication page of C. G. Massouros was read for a sequel to [13] and lists none. The forward citations of [2] were enumerated through two citation services, which return 5 and 12 works; none of them contains a list of the order-5 quotient classes, and the only source that mentions the gap is [10] itself, in the passage quoted above. The sequence $2, 5, 7, 27, 16, 277$ returns nothing from the Encyclopedia, while the control $1, 1, 2, 5, 14, 42$ returns the Catalan numbers, so the endpoint was working. Three neighbouring papers were checked and do not overlap: Hobby [3], which proposes a structural tool for exactly this kind of search and names small-hyperfield enumeration as future work; Kędzierski, Kuhlmann and

Stojałowska [5], which classifies the strictly smaller class of finite *real* hyperfields with cyclic positive cones; and Le and Lowen [8], which gives asymptotics rather than exact counts. A result published only in a journal and never posted as a preprint would be invisible to the first of these searches, and we record that residual risk.

2. Preliminaries

Tables. Let F be a hyperfield of order n and $M = F^\times$, an abelian group of order $m = n-1$. Distributivity gives, for $x \in M$ and any y,

$$x \boxplus y \;=\; x \cdot \bigl(1 \boxplus x^{-1}y\bigr), \tag{1}$$

and $0 \boxplus y = \{y\}$ by the identity axiom. So the whole hyperaddition is determined by the single vector

$$s(u) \;=\; 1 \boxplus u \subseteq F, \qquad u \in M,$$

of nonempty subsets of F. Conversely a group M together with a vector $s : M \to \mathcal{P}(F) \setminus \{\emptyset\}$ determines a candidate structure through (1), and it is a hyperfield exactly when the axioms of Definition 1.1 hold, each of which is a finite condition on s. This is the presentation used throughout: a hyperfield of order n is a pair (M, s) passing a finite list of tests.

Two constraints cut the space of vectors s before any search is run.

Lemma 2.1. *Let (M, s) be a hyperfield of order $m+1$ and put $Z = \{u \in M : 0 \in s(u)\}$. Then $Z = \{\varepsilon\}$ is a singleton, $-x = \varepsilon x$ for every $x \in M$, and $\varepsilon^2 = 1$.*

Proof. By (1), $0 \in x \boxplus y$ if and only if $0 \in 1 \boxplus x^{-1}y$, i.e. $x^{-1}y \in Z$. Uniqueness of additive inverses says that for each x there is exactly one y with $0 \in x \boxplus y$, so Z is a singleton $\{\varepsilon\}$ and $-x = x\varepsilon$. Applying this twice, $x = -(-x) = \varepsilon^2 x$, so $\varepsilon^2 = 1$. □

Lemma 2.2. *Let (M, s) be a hyperfield. Then $s(u) = u \cdot s(u^{-1})$ for every $u \in M$.*

Proof. Commutativity of $\boxplus$ and (1) give $1 \boxplus u = u \boxplus 1 = u \cdot (1 \boxplus u^{-1})$. □

So s is free only on one representative of each pair $\{u, u^{-1}\}$.

Isomorphism. An isomorphism of hyperfields is a bijection preserving 1, multiplication and hyperaddition. For two structures presented over the *same* multiplicative group M, an isomorphism restricts to an automorphism of M, and conversely each $\varphi \in \mathrm{Aut}(M)$ transports s to $s^\varphi(u) = \varphi(s(\varphi^{-1}(u)))$. Hence two presentations over M are isomorphic hyperfields if and only if some $\varphi \in \mathrm{Aut}(M)$ carries one vector to the other, and the orbit of s under $\mathrm{Aut}(M)$ is a complete isomorphism invariant *within a fixed M*. Structures over non-isomorphic groups are never isomorphic, so the classification at each order splits over the abelian groups of order $n-1$ and the two halves are counted separately. For $n \le 8$ the group is cyclic except at $n = 5$, where both C_4 and $C_2 \times C_2$ occur.

Concretely, we fix a numbering of M, encode each $s(u)$ as a subset of $\{0, \dots, m\}$, read the vector s as an integer, and take the least integer over the $\mathrm{Aut}(M)$-orbit as the canonical key of the class. Two presentations over M are isomorphic if and only if their canonical keys agree, provided the automorphism list used is complete; completeness of the list is itself checked, by comparing its length against a brute-force count of the identity-fixing bijections of M that are homomorphisms.

Remark 2.3. The key is a complete invariant only within one multiplicative group. At order 5 five of the eleven classes over $C_2 \times C_2$ have a key numerically equal to that of one of the sixteen classes over C_4; a count that pools the two groups returns 22 instead of 27. All counting below is done one group at a time.

The reduction at order 8.

Lemma 2.4. *Let F be a hyperfield of order 8. Then $M = F^\times$ is cyclic of order 7; fix a generator g and label the elements of M by their discrete logarithms $0, \dots, 6$, so that label 0 is 1, and write 7 for the additive zero. Encode each $s(u)$ as a subset of $\{0, \dots, 7\}$, i.e. as an integer below 2^8. Then there are integers $a, b, c, d \in \{0, \dots, 127\}$ such that*

$$s(0) = a + 128, \qquad s(1) = b, \quad s(2) = c, \quad s(3) = d, \qquad s(4) = g^4 \cdot d, \quad s(5) = g^5 \cdot c, \quad s(6) = g^6 \cdot b,$$

where $g^k \cdot (-)$ denotes the shift of labels $j \mapsto j + k \bmod 7$ acting on the first seven bits and fixing the eighth. In particular F is isomorphic to the structure presented by one of the $128^4 = 268\,435\,456$ quadruples (a, b, c, d).

Proof. M has order 7, a prime, hence is cyclic. By Lemma 2.1 there is a unique $\varepsilon \in M$ with $\varepsilon^2 = 1$ and $0 \in s(u)$ if and only if $u = \varepsilon$; since $|M| = 7$ is odd, M has no element of order 2, so $\varepsilon = 1$, which is the label 0. Thus the additive zero lies in $s(0)$ and in no other $s(u)$: the eighth bit of $s(0)$ is set and the eighth bit of $s(1), \dots, s(6)$ is clear, which is the displayed form with a recording the first seven bits of $s(0)$. Finally Lemma 2.2 gives $s(u) = u \cdot s(u^{-1})$, and in additive labels u^{-1} is $-u \bmod 7$, so $s(6) = g^6 s(1)$, $s(5) = g^5 s(2)$ and $s(4) = g^4 s(3)$, leaving a, b, c, d free. □

3. The census

We write c_n for the list of tables exhibited at order n in the accompanying development. Each list is verified to have three properties: every entry satisfies all the axioms of Definition 1.1; every entry is the least element of its own $\mathrm{Aut}(M)$-orbit, so the entries are canonical representatives; and no automorphism of M carries one entry to another. The third property gives pairwise non-isomorphism and hence a lower bound on H_n.

Theorem 3.1. *For each $n \in \{3, 4, 5, 6, 7\}$ the list c_n consists of pairwise non-isomorphic Krasner hyperfields of order n in canonical form, of the following lengths:*

$$|c_3| = 5, \quad |c_4| = 7, \quad |c_5| = 16 \ (\textit{over } C_4) \ + \ 11 \ (\textit{over } C_2 \times C_2) = 27, \quad |c_6| = 16, \quad |c_7| = 277.$$

In particular $H_3 \geq 5$, $H_4 \geq 7$, $H_5 \geq 27$, $H_6 \geq 16$ and $H_7 \geq 277$.

Proof. Each assertion is the evaluation of a decision procedure on explicit data: for a list of length ℓ over a group of order m it runs the axiom tests on each entry, the $|\mathrm{Aut}(M)|$ canonicity comparisons on each entry, and the $\ell^2 |\mathrm{Aut}(M)|$ pairwise comparisons. The largest instance is $n = 7$, where $\ell = 277$, $m = 6$ and $|\mathrm{Aut}(C_6)| = 2$. The lists themselves were produced by an exhaustive enumeration of the vectors s over each group, organised as in Section 2; that enumeration is a search, and only its output is certified here. □

Remark 3.2. The five lengths reproduce the published counts of [1] and [13]; the first four are the terms $a(3), \dots, a(6)$ of [15], which does not carry the fifth. The point of Theorem 3.1 is not the numbers, which are known, but the lists, which are what the quotient matching of Section 4 is matched against; equality $H_n = |c_n|$ for $n \leq 7$ is quoted from those papers and is not re-proved here. An independent third source for H_n, $n \leq 5$, is [11].

Theorem 3.3. *Let C_7 be labelled as in Lemma 2.4.*

(i) *The list c_8 consists of 178 pairwise non-isomorphic Krasner hyperfields of order 8 in canonical form. In particular $H_8 \geq 178$.*
(ii) *For every quadruple $(a, b, c, d) \in \{0, \dots, 127\}^4$, if the vector $s = s(a, b, c, d)$ of Lemma 2.4 satisfies all the axioms of Definition 1.1, then the canonical key of s is the key of one of the 178 entries of c_8.*

Corollary 3.4. *$H_8 = 178$. Equivalently, A343596(8) = 178.*

Proof. Theorem 3.3(i) gives $H_8 \geq 178$. For the converse, let F be a hyperfield of order 8. By Lemma 2.4, F is isomorphic to the structure presented by some quadruple $(a, b, c, d) \in \{0, \ldots, 127\}^4$, whose vector therefore satisfies the axioms; by Theorem 3.3(ii) its canonical key is one of the 178, and by Section 2 the canonical key determines the isomorphism class over the fixed group C_7. So F is isomorphic to an entry of c_8, and $H_8 \leq 178$. □

Sketch of proof of Theorem 3.3. Part (i) is the same certification as Theorem 3.1, with $\ell = 178$, $m = 7$ and $|\operatorname{Aut}(C_7)| = 6$.

Part (ii) is an exhaustive search over all $128^4 = 268\,435\,456$ quadruples. Run flat, it is too slow to be checked inside a proof assistant, and it is cut by a single family of consequences of reversibility, applied at the multiplicative identity. Write rc for the implication

$$z \in 1 \boxplus y \implies 1 \in z \boxplus (-y) = z \boxplus y,$$

the last equality because $-y = y$, established in the proof of Lemma 2.4. In labels this reads: if bit z of $s(y)$ is set then bit $-z$ of $s(y - z)$ is set. Six instances of rc involve only $s(0), s(1), s(6)$ and so depend only on (a, b); twenty involve only $s(0), s(1), s(2), s(5), s(6)$ and so depend only on (a, b, c). Each instance is a consequence of the axioms — this is proved, not assumed — so a branch of the search that fails one of them contains no hyperfield and may be abandoned. The resulting depth-first search visits 128 nodes at depth 1, 16 256 at depth 2, 256 032 at depth 3 and 504 952 leaves, that is 777 368 nodes in place of 268 435 456, a 519-fold reduction; 3926 leaves survive reversibility, 932 satisfy all the axioms, and their canonical keys fall into the 178 classes of c_8. The search is executed in eight blocks of 16 values of a, and four inductions recombine the blocks into the universally quantified statement. The statement proved is about the full axiom set, not about the pruned predicate. □

Remark 3.5. Corollary 3.4 uses Lemma 2.4, which is proved above by hand; the machine check covers Theorem 3.3 and stops at the boundary of the reduced search space. We state this explicitly because it is the only step between the exhaustive computation and the value $H_8 = 178$.

Remark 3.6 (a computation made outside the formal development). The order-8 enumeration was first run as a standalone program, which examined all 262 193 024 candidate vectors — the quadruples of Lemma 2.4 with every $s(u)$ nonempty — in about 5.5 minutes, a rate of roughly 8×10^5 candidates per second, and returned 3926 reversible candidates, 932 hyperfields and 178 classes; the pruned depth-first version returned the same three numbers from 777 368 nodes. These are measurements of the computation, not theorems, and nothing below depends on them.

Remark 3.7. Order 9 is out of reach of the same method as it stands. Three abelian groups of order 8 occur, and for the cyclic one alone the analogous space has $256 \cdot 255^3 \cdot 15 + 255^4 \cdot 16 \approx 1.3 \times 10^{11}$ candidates, about 45 hours at the rate measured in Remark 3.6. What would change that estimate is a stronger prune; the "blocks" of [3] are a proposal for one, put forward there with the remark that exploiting them "would greatly speed up future computer searches for small hyperfields". We make no claim about H_9.

4. Quotients of finite fields

How a class is certified. Fix $r \geq 2$ and a prime power q with $r \mid q - 1$. We model $\mathbb{F}_{p^k}$ as $\mathbb{F}_p[x]/(f)$ with f monic of degree k, elements being the naturals below p^k read as vectors of base-p coefficients. Rather than assume that f is irreducible, we certify it: for a candidate g we check that $1, g, g^2, \ldots, g^{q-2}$ are $q - 1$ distinct nonzero elements and that $g^{q-1} = 1$. This forces every nonzero element of the commutative ring $\mathbb{F}_p[x]/(f)$ to be a unit, hence f irreducible and g primitive. With g in hand, G_r is $\langle g^r \rangle$, the cosets are indexed by discrete logarithm modulo r, and Definition 1.2 yields the vector $s(j) = 1 \boxplus [g^j]$, $j \in \mathbb{Z}/r$, which is exactly the presentation of Section 2 over $M = C_r$.

To certify that exactly N classes of order $r + 1$ are quotients we therefore need two things. The first, which is a finite check, is a list of N prime powers whose quotients are hyperfields, are pairwise non-isomorphic, and each occur in the census list c_{r+1}. The second, which is not a finite check, is that no prime power produces a further class. This is where Baker and Jin's Theorem 1.1 enters: beyond the threshold N_r every $\mathbb{F}_q/G_r$ falls into one of the at most two stable classes, so only prime powers below N_r can contribute anything else, and those are finitely many and were enumerated. Concretely we scanned every prime power $q \equiv 1 \pmod{r}$ up to twice the bound of their Remark 1.2, which for $r = 2, \dots, 8$ is $6, 17, 56, 171, 434, 940, 1810$, and found no class outside the listed ones and no exceptional q at or above the bound itself. That scan is a program run outside the formal development; Section 7 records the split.

Theorem 4.1. *Exactly* 4 *of the* 5 *isomorphism classes of hyperfields of order* 3*, and exactly* 4 *of the* 7 *of order* 4*, are quotients of finite fields. Representatives are* $\mathbb{F}_q/G_2$ *for* $q \in \{3, 5, 7, 9\}$ *at order* 3*, and* $\mathbb{F}_q/G_3$ *for* $q \in \{4, 7, 13, 19\}$ *at order* 4*. In particular exactly* 3 *of the* 7 *classes of order* 4 *are not quotients of finite fields.*

Proof. For each listed q the field certificate is checked, the quotient is verified to be a hyperfield, the four quotients at each order are verified to be pairwise non-isomorphic, and each is located in the census list c_3 respectively c_4 of Theorem 3.1; the count of census classes missed at order 4 is computed to be 3. Completeness is the scan described above, with $r = 2$ and $r = 3$. Baker and Jin obtain the same two counts in their Theorem 1.3 by a case analysis of the possible hyperaddition tables, closing the non-quotient half at order 4 by checking $q \le 16$, a bound they take from their Remark 1.2 and whose verification they omit as straightforward; their four order-4 quotients are $\mathbb{F}_4$, $\mathbb{F}_7/G_3$, $\mathbb{F}_{13}/G_3 \cong \mathbb{F}_{16}/G_3$ and $\mathbb{F}_q/G_3$ for $q \ge 19$, which are the four listed above. The statement is theirs; it is recomputed here as a control on the certification method used at the higher orders, where nothing is available to check against. □

Theorem 4.2 (Baker–Jin question (1), finite-field half)**.** *Of the* 27 *isomorphism classes of hyperfields of order* 5*, exactly* 9 *are quotients of finite fields and* 18 *are not. The* 9 *are realised by* $\mathbb{F}_q/G_4$ *for*

$$q \in \{5,\ 9,\ 13,\ 17,\ 25,\ 29,\ 37,\ 41,\ 49\},$$

one field per class, and all 9 *lie among the* 16 *classes whose multiplicative group is* C_4*. None of the* 11 *classes whose multiplicative group is* $C_2 \times C_2$ *is a quotient of a finite field.*

Proof. For each of the nine listed q the modulus and the primitive element are certified, the quotient is verified to be a hyperfield of order 5 over C_4, the nine quotients are verified to be pairwise non-isomorphic, and each is located in the list c_5 of the 16 classes over C_4. That is the finite half, and it gives 9 as a lower bound.

For the Klein classes, $(\mathbb{F}_q/G_r)^\times \cong \mathbb{F}_q^\times/G_r$ is a quotient of the cyclic group $\mathbb{F}_q^\times$, hence cyclic; and $C_2 \times C_2$ is not cyclic — no identity-fixing injection $C_2 \times C_2 \to C_4$ is a homomorphism, which is checked over all 4^3 identity-fixing maps. So no class over $C_2 \times C_2$ is a finite-field quotient, and by Theorem 3.1 there are 11 of them.

For the upper bound 9 among the classes over C_4: by Baker and Jin's Theorem 1.1, every $\mathbb{F}_q/G_4$ with $q \ge N_4$ lies in one of the two stable classes, and the scan described above covered every prime power $q \equiv 1 \pmod{4}$ up to twice their Remark 1.2 value $N_4 = 56$, producing no tenth class. Hence exactly 9 of the 27 are quotients, and $27 - 9 = 18$ are not. □

Theorem 4.3. *Exactly* 7 *of the* 16 *isomorphism classes of hyperfields of order* 6*, exactly* 15 *of the* 277 *of order* 7*, and exactly* 9 *of the* 178 *of order* 8 *are quotients of finite fields. Representatives, one field per class, are* $\mathbb{F}_q/G_5$ *for*

$$q \in \{11,\ 16,\ 31,\ 41,\ 61,\ 71,\ 81\}$$

at order 6*;* $\mathbb{F}_q/G_6$ *for*

$$q \in \{7,\ 13,\ 19,\ 25,\ 31,\ 37,\ 43,\ 49,\ 61,\ 67,\ 73,\ 97,\ 103,\ 109,\ 121\}$$

at order 7*; and* $\mathbb{F}_q/G_7$ *for*

$$q \in \{8,\ 29,\ 43,\ 64,\ 71,\ 113,\ 127,\ 169,\ 491\}$$

at order 8.

Proof. As in Theorem 4.2: for each listed q the field certificate is checked, the quotient is verified to be a hyperfield of the stated order, the quotients at each order are verified to be pairwise non-isomorphic, and each is located in the census list c_6, c_7 or c_8 of Theorem 3.1 and Theorem 3.3. Completeness is again Baker and Jin's Theorem 1.1 together with the scan, run at $r = 5, 6, 7$ up to twice the Remark 1.2 values $171, 434, 940$. □

Remark 4.4. The counts 7 at order 6 and 15 at order 7 are Q_5^{fin} and Q_6^{fin} of [10] and agree with the values reported there; they are reproduced, not claimed. What Theorem 4.3 adds at those orders is the location of the classes inside the census, which requires the census. At order 8 the count 9 agrees with a value computed after [10] appeared and released in its author's software repository (Remark 6.2); the matching against the order-8 census has no counterpart there, the census not being available.

Remark 4.5. At order 8, eight of the nine classes come from the eight exceptional prime powers $q = 8, 29, 43, 64, 71, 113, 127, 491$ and the ninth is the Baker–Jin stable class for the odd index $r = 7$, realised by every large $q \equiv 1 \pmod 7$ and represented above by $q = 169$. The smallest and the largest sporadic prime power at this index thus differ by a factor of more than sixty, while the threshold of Remark 1.2 of [2] is 940; the largest exceptional value is 491.

Remark 4.6. Of the 277 classes at order 7, only 15 are finite-field quotients, a share of about 5.4%; at order 8 the share is $9/178 \approx 5.1\%$. The last of Baker and Jin's open questions conjectures that the proportion Q_r/H_r of quotient hyperfields among all hyperfields of order $r+1$ tends to 0; Hobby [3] proves that most hyperfields of even order are not quotients, and Le and Lowen [8] state that they confirm the conjecture. The shares above are thus illustrations rather than evidence, and the comparison is imperfect in both directions: the numerator here counts only finite-field quotients, while the conjecture concerns quotients of arbitrary fields, and H_n is not monotone in n ($H_6 = 16 < H_5 = 27$).

5. Controls

The statements of this section are not results about hyperfields; they are checks that the tests used above are not vacuous and that the counts are not artefacts. Each refutes a specific too-large or too-small claim.

Proposition 5.1. *Over C_7, the vector with $1 \boxplus 1 = \{1\}$ and every other entry full fails the axioms, having no additive inverse for* 1. *The vector encoded* $[128, 8, 112, 34, 36, 28, 4]$ *over C_7 is commutative, has unique additive inverses and is reversible, but is not associative, hence is not a hyperfield. Adjoining a second copy of an entry of c_8 to it breaks the certification of Theorem 3.3(i), so the procedure that certifies* 178 *classes does not certify* 179. *Nor can the list be shortened: its first* 177 *entries pass the same certification, and its* 178*th entry is a hyperfield carried to none of them by any automorphism of C_7. The polynomial $x^2 - 1$ over $\mathbb{F}_3$ is rejected by the field certificate of Section 4 for every candidate primitive element. And $C_2 \times C_2 \ncong C_4$.*

Proof. Each is the evaluation of the relevant decision procedure on explicit data. □

Remark 5.2. The second control is the informative one. Of the 3926 order-8 candidates that survive reversibility in the search of Theorem 3.3(ii), 2994 fail associativity; so reversibility, which is what the prune uses, comes nowhere near carrying the census on its own, and the expensive associativity test is doing real work.

6. Computations outside the formal development

The following were carried out by standalone programs, are recorded for completeness, and are used nowhere above. They are stated as reproductions of published values, not as results.

Remark 6.1. Both printed tables of [10] were recomputed from its own definitions by an independently written program: the empirical stabilization thresholds $N_r^{\mathrm{emp}} = 6, 17, 42, 102, 278, 492, 762$ with $2, 4, 7, 7, 18, 8, 18$ exceptional prime powers for $r = 2, \dots, 8$, and $Q_r^{\mathrm{fin}} = 2, 4, 4, 9, 7, 15$ for $r = 1, \dots, 6$. All thirteen printed cells agree. So does the ancillary table of order-7 quotient classes with their characteristic and C-characteristic in the sense of [4], which is an independent check on our implementation of those invariants. Because the value of N_r^{emp} depends on Baker and Jin's Theorem 1.1 for the tail of the scan, it is not a finite check and is not part of the formal development.

Remark 6.2. The software repository accompanying [10] has continued past the preprint. Its commit `ae218703` of 12 August 2026 adds census results for $r = 11, 12$ and combined tables covering $r \leq 12$, that is, values beyond both printed tables: $N_r^{\mathrm{emp}} = 578, 1182, 1410, 1742$ with $15, 27, 11, 41$ exceptional prime powers for $r = 9, \dots, 12$, and $Q_r^{\mathrm{fin}} = 9, 17, 14, 27, 9, 36$ for $r = 7, \dots, 12$. All ten of the first group and all six of the second were reproduced here by the independent implementation of Remark 6.1. The repository also carries a file `AI_USE.md` disclosing that a large language model assistant was used in its development. The repository is live; `ae218703` was still its newest commit when we last read it, on 3 September 2026.

7. Formal verification

Every theorem and proposition of Sections 3 to 5 is formalized and machine-checked in Lean 4 [9] against Mathlib [14], with the exceptions named in this section; Corollary 3.4 is not among them, since it combines Theorem 3.3 with the by-hand Lemma 2.4 (Remark 3.5). The remarks of these sections are commentary, and those of them that report computations made outside the development say so where they appear. The development is four files: the table presentation of Section 2 with the axioms as decidable tests, the characteristic and C-characteristic of [4], the action of $\mathrm{Aut}(M)$ and the completeness check for the automorphism list; the census lists and their certification (Theorems 3.1 and 3.3(i)); the exhaustive search (Theorem 3.3(ii)); and the finite fields, the quotient construction and the matching (Theorems 4.1, 4.2 and 4.3). There is no incomplete proof anywhere in the development, and no statement is asserted as an axiom.

What rests on exhaustive computation, and how large each search is. Lemmas 2.1, 2.2 and 2.4 are proved by hand and are not formalized; they are the reduction that defines the search space, and Remark 3.5 says where they are used. Theorem 3.1 is six evaluations, one per group at orders 3 to 7, the largest being 277 tables over C_6. Theorem 3.3(i) is one evaluation on 178 tables over C_7. Theorem 3.3(ii) is the 268 435 456-point search, executed as eight blocks inside the compiled evaluator; the soundness of its prune — that each of the reversibility instances used to cut branches (six in the filter at the second level, twenty in the filter at the third) follows from the axioms, and that $-y = y$ throughout the enumerated family — is proved symbolically inside the development and not evaluated, and four inductions turn the pruned nested loop back into the universally quantified statement, which is stated about the full axiom set. Theorems 4.1, 4.2 and 4.3 are one evaluation per order on the listed fields, and Proposition 5.1 one evaluation per control. Each evaluation inside the compiled evaluator contributes one axiom asserting that the evaluator's verdict is correct; the headline statements carry propositional extensionality, those evaluation axioms, and — for Theorem 3.3(ii), through the arithmetic automation used in the four inductions — quotient soundness and choice, and nothing else.

Two things above are deliberately outside the formalization. The completeness half of Theorems 4.1, 4.2 and 4.3 — that no prime power yields a further class — combines Baker and Jin's Theorem 1.1 with the finite scan of prime powers described in Section 4, and that scan is a program run outside the development; what is machine-checked for those theorems is that the listed fields give the stated number of pairwise non-isomorphic quotients and that each is a class of the census, plus, at order 5, that no class over $C_2 \times C_2$ can be a quotient. And Remarks 3.6, 6.1 and 6.2 are computations made outside the development and are labelled as such where they appear. Before any of the Lean was written, the enumerator was validated by reproducing all five published values of H_n for $3 \leq n \leq 7$ and both printed tables of [10]; an earlier version of it reported $H_5 = 22$ because it pooled the two multiplicative groups of order 4, the error described in Remark 2.3, and no number in this paper comes from that version. The whole development elaborates in about twenty minutes, its cost dominated by the census at order 7 and by the search of Theorem 3.3(ii). The repository is private; repository reference to be added at submission.

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