QUANDLE COLOURINGS AND THE ROTATABILITY OF SPHERICAL KNOTOIDS UP TO SEVEN CROSSINGS

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Abstract. Gabrovšek and Cavicchioli have tabulated the 427 prime spherical knotoids with at most seven crossings. Their table divides them into 37 rotatable and 390 non-rotatable knotoids, but the non-rotatable column is marked as conjectural: for 28 of the seven-crossing knotoids the five polynomial invariants they computed take the same value on the knotoid and on its rotation, and an exhaustive Reidemeister search neither identified the two nor separated them. We show that all 28 are non-rotatable, using the quandle colouring counts of Gügümcü and Nelson: Fox 3-colourings separate 22 of them from their rotations, and colourings by the Alexander quandle on $\mathbb{F}_3^2$ separate the remaining six. Eight of the 28 are also beyond the reach of Nikonov's homotopy index polynomial, the only other published invariant known to us that decides any of them. The same counts show that two entries of the rotatable column are wrong: 5_{24} and 7_{288} are not rotatable, although 7_{288} is fixed by rotation composed with reversion. The reason the polynomial invariants could not decide the question is that the rotation implemented in the tabulation is the composite of mirror reflection and Turaev's symmetry, under each of which the bracket polynomial is transformed by $A \mapsto A^{-1}$; we verify this on the published diagrams. Seven of the 390 conjecturally non-rotatable knotoids remain open. Every colouring count in this paper is checked from the published PD codes with the Lean 4 proof assistant.

1. Introduction

A *knotoid* is an open-ended analogue of a knot introduced by Turaev [8]: a knotoid diagram in the 2-sphere S^2 is an immersed arc whose only singularities are transversal double points carrying over/under information, and a knotoid is the class of such a diagram under isotopy of S^2 and the three Reidemeister moves performed away from the two endpoints. Moving an endpoint across a strand is forbidden, and this is what makes the theory richer than that of knots: a knotoid has a *leg* and a *head*, and the three *basic involutions* of Turaev — reversion rev, which exchanges leg and head, mirror reflection mir, which switches every crossing, and symmetry sym, which reflects the sphere — are in general four different knotoids together with the identity. The composite $\mathrm{rot} = \mathrm{mir} \circ \mathrm{sym}$ is called the *rotation* in the tables of knotoids [4], and a knotoid K is *rotatable* when $K = \mathrm{rot}\, K$.

Gabrovšek and Cavicchioli [2] have recently tabulated prime spherical knotoids with up to seven crossings. Their Table 1 lists 427 knotoids and records for each whether it is chiral and whether it is rotatable; the rotatable column reads 37 against 390, and the whole non-rotatable entry carries the footnote "*Conjectured*". Their method is a sequence of exhaustive searches of the Reidemeister space of a diagram, and its outcome for rotatability is summarised in their Section 4, Step 5, which we quote in full because it is the statement we settle:

> "We could not find that the remaining 28 knotoids are related by rotation, yet share all computed invariants. In analogy with classical knot theory, where mutant knots are indistinguishable by many standard invariants, we suggest that these knotoids may be regarded as *mutant-type* pairs.
>
> In the results (Table 1 and the Appendix) we conjecture that these knotoids are non-rotatable."

The break is theirs: the quotation is two consecutive paragraphs of Step 5. The five invariants they computed are the Kauffman bracket, the arrow polynomial, the mock Alexander polynomial, the affine index polynomial and the Yamada polynomial of the closure (their Table 2). The 28 knotoids in question are, in the names of their table,

$$7_{26},\ 7_{28},\ 7_{34},\ 7_{35},\ 7_{50},\ 7_{51},\ 7_{52},\ 7_{53},\ 7_{54},\ 7_{59},\ 7_{60},\ 7_{63},\ 7_{64},\ 7_{65},$$
$$7_{72},\ 7_{74},\ 7_{75},\ 7_{76},\ 7_{77},\ 7_{78},\ 7_{79},\ 7_{80},\ 7_{81},\ 7_{85},\ 7_{88},\ 7_{90},\ 7_{91},\ 7_{92}; \tag{1}$$

they are the rows of the authors' data file whose rotatability field reads "unknown", and their appendix marks each of them "achiral, non-rotatable*". We call this set the *residue* of the table.

Results. We answer the question with the oldest colouring invariant there is. For a finite quandle Q, the number $\Phi_Q(D)$ of Q-colourings of a knotoid diagram D is an invariant of the knotoid (Gügümcü–Nelson [5], in the more general setting of biquandles), and so is the *counting matrix* $\Phi_Q^M(D)$ that splits the colourings by the colours of the leg and of the head. Since $\operatorname{rot} D$ is a diagram of $\operatorname{rot} K$, an inequality $\Phi_Q^M(D) \neq \Phi_Q^M(\operatorname{rot} D)$ proves $K \neq \operatorname{rot} K$. Two quandles suffice for the whole residue:

- the dihedral quandle R_3 of order three, whose colourings are the Fox 3-colourings, separates 22 of the 28 knotoids from their rotations (Theorem 7); the counts are 3 against 9, 9 against 3, 27 against 9 or 9 against 27;
- the Alexander quandle Q_9 on $\mathbb{F}_3^2$ separates the remaining six, 7_{72}, 7_{74}, 7_{77}, 7_{78}, 7_{79}, 7_{85}, with counts 9 against 81 or 81 against 9 (Theorem 8).

So all 28 knotoids of the residue are non-rotatable (Corollary 9), and the conjectural footnote of the table can be removed from them.

Nikonov [7] has since defined a *homotopy index polynomial* $H(K)(x)$ of a spherical knotoid with $H(\operatorname{rot} K)(x) = H(K)(x^{-1})$, and computed $H(7_{53}) = x - 1$, which already proves one knotoid of the residue non-rotatable. Recomputing H from the published diagrams for all 427 knotoids, one finds that it decides 20 of the 28; on the other eight,

$$7_{35},\ 7_{60},\ 7_{74},\ 7_{77},\ 7_{78},\ 7_{79},\ 7_{88},\ 7_{90},$$

it is palindromic and says nothing. (This paper's implementation of H is written from Nikonov's definitions and gated three ways: it returns his published $H(7_{53}) = x - 1$; it satisfies his $H_o = H_u$ on all 427 diagrams; and $H(t) + H(t^{-1})$ reproduces the affine index column of [2] on all 427 rows.) Those eight are settled here for the first time (Theorem 11); for the other 20 our proof is a second, independent one.

The colouring counts also correct two entries of the published rotatable column. The knotoid 5_{24} is listed as rotatable in Table 1 of [2] and in the authors' own list of rotatable knotoids, but it has 9 Fox 3-colourings while its rotation has 3 (Theorem 12); Nikonov's polynomial, $H(5_{24}) = x - x^{-1}$, and the count of five rotatable five-crossing knotoids in the earlier classification of Goundaroulis–Dorier–Stasiak [4], against six in [2], corroborate this independently. The knotoid 7_{288} is also listed as rotatable and is not: no colouring *count* sees it, but the counting matrix over a second Alexander quandle on $\mathbb{F}_3^2$ does (Theorem 13). Here a caveat is due: 7_{288} is fixed by $\operatorname{rot} \circ \operatorname{rev}$, so a comparison of diagrams that ignores the orientation of the arc would call it rotatable.

Why the polynomial invariants could not see it. The prose of [2] describes rot as a reflection of the diagram in a vertical line, which is Turaev's sym; but the software behind the table computes the rotation as the 180° flip of every crossing, which on a PD code is the reversal of the cyclic order at each crossing and is exactly $\operatorname{mir} \circ \operatorname{sym}$ — the rotation of [4], and the involution for which the table's own data are consistent. Turaev already records that the bracket polynomial of a knotoid is transformed by $A \mapsto A^{-1}$ under mirror reflection and under reversal of the orientation of the surface [8, §7.1]; it is therefore invariant under their composite rot. The affine index polynomial is symmetric by construction [7]. Section 5 verifies the bracket statement on the residue itself, from the published PD codes, and checks that the same computation reproduces the bracket column of the table. With this reading of rot, the whole table becomes consistent: the invariant columns of the data file agree on K and on its rotation for every one of the 427 rows.

What remains. Section 6 records the controls — the same invariants agree on the knotoids the table proves rotatable, and a plausible but wrong reading of the crossing rule would "separate" the rotatable 3_1 from itself — and the residue that stays open. Of the 390 knotoids the table calls non-rotatable, 383 are now proved so by the union of the quandle method and Nikonov's polynomial, and seven are not: 6_3, 6_{22},

6_{48}, 7_6, 7_{58}, 7_{229}, 7_{272}. At least 6_3 is very probably rotatable, for reasons given there. Every colouring count and every bracket value that a theorem below rests on has been checked from the published PD codes by the Lean 4 proof assistant (Section 7).

2. Preliminaries

2.1. **Knotoids and the basic involutions.** We follow Turaev [8]. A knotoid diagram in S^2 is oriented from its leg to its head. *Reversion* rev exchanges the head and the leg; *mirror reflection* mir keeps the diagram and changes overpasses to underpasses and vice versa; *symmetry* sym applies to the diagram the self-homeomorphism of $S^2 = \mathbb{R}^2 \cup \{\infty\}$ that extends the reflection of the plane in the vertical line $\{0\} \times \mathbb{R}$. The three commute and are involutions of the set of knotoids in S^2; Turaev calls them the basic involutions. Following Goundaroulis, Dorier and Stasiak [4], who write that "the composition of the mirror reflection and the symmetry is called rotation, $\mathrm{rot}(k)$, and it can be thought of as the rotation of a knotoid k around the axis that passes through the endpoints", we set

$$\mathrm{rot} = \mathrm{mir} \circ \mathrm{sym} = \mathrm{sym} \circ \mathrm{mir},$$

and call a knotoid K *rotatable* if $K = \mathrm{rot}\, K$. The same composite is Definition 2.4 of Barbensi, Buck, Harrington and Lackenby [1] — two knotoids "differ by a *rotation* if they have diagrams that are obtained from each other by reflecting the sphere of the diagram in a line through the endpoints of the knotoids, followed by a mirror reflection" — and it is the $\mathrm{rot}(K) = \bar{K}^*$ of Nikonov [7, Definition 6], the mirror image of the symmetrical knotoid. The name is not Turaev's: he defines the three basic involutions and gives no name to any composite of them. Nor is $\mathrm{rot} = \mathrm{mir} \circ \mathrm{sym}$ what the text of [2] describes — that is a reflection, i.e. sym — but it is what its computations perform; see Remark 2.

2.2. **PD codes.** The table [2] encodes a diagram with n arcs as follows (their Section 3): "We enumerate the arcs consecutively from 0 to $n-1$. For each vertex (endpoint or crossing), we record the incident arcs in counterclockwise order, beginning with an undercrossing arc (in case of a crossing). The PD code is then the ordered list consisting of the two endpoint entries and the entries corresponding to all crossings." Thus a PD code is a list of nodes; an endpoint node is a single arc label $[a]$, and a crossing node is a quadruple $[a_0, a_1, a_2, a_3]$ listed counterclockwise in which a_0, a_2 are the two ends of the under-strand and a_1, a_3 those of the over-strand. The first endpoint node is the leg. The "arcs" of this encoding are what Gügümcü and Nelson call the *semiarcs* of the diagram, "a piece of strand of D that connects either two adjacent crossings or an endpoint to a crossing" [5, Definition 2]; they are the edges of the underlying 4-valent graph.

The three involutions act node by node. Reflecting the sphere reverses the counterclockwise order at every crossing and keeps the under-strand where it is, so sym sends $[a_0, a_1, a_2, a_3]$ to $[a_0, a_3, a_2, a_1]$; switching a crossing makes a_1, a_3 the under-strand, so mir sends it to $[a_1, a_2, a_3, a_0]$; reversion reverses the list of nodes. The composite is the plain reversal of each crossing node:

Lemma 1 (rot on a crossing)**.** *For all* a_0, a_1, a_2, a_3, $\mathrm{mir}(\mathrm{sym}[a_0, a_1, a_2, a_3]) = [a_3, a_2, a_1, a_0]$. *Moreover* $\mathrm{rot} \circ \mathrm{rot}$ *is the identity on each of the* 28 *PD codes of the residue listed in Appendix A.*

Remark 2 (which rotation the table computes). The scripts published with [2] compute the rotation of a diagram with the routine `flip` of the authors' library `knotpy`, documented as "Flip the diagram by 180° around each selected node. This reverses the cyclic order of incident endpoints at the specified nodes." On a PD code with the under-strand in positions 0 and 2 that is the reversal $[a_3, a_2, a_1, a_0]$ of Lemma 1, i.e. $\mathrm{mir} \circ \mathrm{sym}$, and not the reflection $[a_0, a_3, a_2, a_1]$ that the text of [2] describes. (The two lists are the same cyclic order; what distinguishes them is which pair of positions is read as the under-strand, and `flip` applies the reversal to the stored list.) The data confirm the reading: the authors' file records, for each knotoid, which knotoid its flip has the invariants of, and that entry is the knotoid itself, alone, for 399 of the 427 rows; the other 28 rows are exactly the residue, where two candidates are listed and the knotoid itself is one of them. Had the rotation been sym, the bracket would have been sent to $A \mapsto A^{-1}$ and the entry would have named the mirror image for every knotoid whose bracket is not palindromic — which is 422 of the 427. Proposition 14 below is the corresponding statement inside the formal development.

2.3. **Quandles.** A *quandle* is a set Q with a binary operation $\triangleright$ such that $x \triangleright x = x$, each right translation $x \mapsto x \triangleright y$ is a bijection of Q, and $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$ [6]. We write $x \triangleright^{-1} y$ for the inverse translation, so that $(x \triangleright^{-1} y) \triangleright y = x$; the *opposite* quandle Q^{op} has the operation $\triangleright^{-1}$, and Q is *involutory* (a kei) when $Q^{\mathrm{op}} = Q$. Four finite quandles are used, all given by their multiplication tables in the formal development:

- R_3, the dihedral quandle of order 3: $x \triangleright y = 2y - x$ on $\mathbb{Z}/3$. It is involutory, and its colourings of a diagram are the Fox 3-colourings.
- Q_9, the Alexander quandle on $\mathbb{F}_3^2$ with $t(x_1, x_2) = (x_2, 2x_1)$ and $x \triangleright y = t(x) - t(y) + y$; elements are ordered as $3x_1 + x_2$. Here $t^2(x_1, x_2) = t(x_2, 2x_1) = (2x_1, 2x_2) = -(x_1, x_2)$, so $t^2 = -1$; as $t^2 + 1$ is irreducible over $\mathbb{F}_3$, this is the Alexander quandle $\mathbb{F}_3[t^{\pm 1}]/(t^2 + 1)$ on $\mathbb{F}_9$, with t of multiplicative order 4.
- Q_9', the Alexander quandle on $\mathbb{F}_3^2$ with $t(x_1, x_2) = (x_2, x_1 + x_2)$, same ordering; here $t^2 = t + 1$, and $x^2 - x - 1$ is irreducible over $\mathbb{F}_3$ as well, so Q_9' is the Alexander quandle on $\mathbb{F}_9$ for a t of multiplicative order 8. It is not isomorphic to Q_9: every right translation of Q_9 has cycle type 1+4+4 and every right translation of Q_9' has cycle type 1+8, and an isomorphism of quandles conjugates right translations. Theorem 13 needs this one and not Q_9.
- G_8, the generalized Alexander quandle of the quaternion group, $x \triangleright y = \varphi(xy^{-1})\, y$, where φ is the automorphism of Q_8 of order three that fixes ± 1 and cycles $i \mapsto j \mapsto k \mapsto i$; rebuilding the table from this φ returns exactly the one used here, reproduced in Appendix B. Unlike R_3 it is not involutory, which is what lets it see sym.

That the four tables satisfy the quandle axioms is checked in the formal development (Proposition 3 for R_3 and Q_9; the checks for Q_9' and G_8 are recorded with Theorems 13 and 12).

Proposition 3. *The tables R_3 and Q_9 are quandles.*

2.4. **Colourings and the counting matrix.** Let Q be a finite quandle and D a knotoid diagram with semiarcs labelled $0, \dots, n-1$. A *Q-colouring* of D assigns to each semiarc an element of Q so that at every crossing the two semiarcs of the over-strand receive the same colour y, and if the under-strand enters with colour x it leaves with colour $x \triangleright y$ or $x \triangleright^{-1} y$ according to the sign of the crossing. The sign is read from the directions of both strands, the diagram being oriented from the leg to the head; Proposition 17 shows that reading it from the over-strand alone gives a quantity that is not an invariant. This is the biquandle colouring of [5, Definition 2] for the biquandle in which the over-strand keeps its colour.

Let $\Phi_Q(D)$ be the number of Q-colourings of D, and let $\Phi_Q^M(D)$ be the $|Q| \times |Q|$ matrix whose (j,k) entry is the number of colourings in which the semiarc at the leg has colour x_j and the semiarc at the head has colour x_k [5, Definition 4]; the sum of its entries is $\Phi_Q(D)$. In the formal development $\Phi_Q^M(D)$ is computed by a backtracking enumeration along the strand and stored as a flat list of $|Q|^2$ natural numbers. Because the leg and the head are fixed by all Reidemeister moves of knotoids, Gügümcü and Nelson prove:

Theorem 4 ([5, Theorem 1])**.** *For any finite biquandle X, $\Phi_X^{M_n}(K)$ is an invariant of knotoids.*

Lemma 5 (separation principle)**.** *Let D be a diagram of the spherical knotoid K and Q a finite quandle. If $\Phi_Q^M(D) \neq \Phi_Q^M(\operatorname{rot} D)$, then $K \neq \operatorname{rot} K$; if $\Phi_Q^M(D) \neq \Phi_Q^M(\operatorname{sym} D)$ then $K \neq \operatorname{sym} K$, and likewise for* mir*.*

Proof. $\operatorname{rot} D$ is a diagram of $\operatorname{rot} K$ by the definition of the involutions on diagrams. If $K = \operatorname{rot} K$, then D and $\operatorname{rot} D$ are diagrams of the same knotoid and Theorem 4 gives $\Phi_Q^M(D) = \Phi_Q^M(\operatorname{rot} D)$. $\square$

Every theorem in Sections 3–4 is an inequality of the form required by Lemma 5, established by computing both sides from a published PD code; the invariance theorem is cited, not formalised (Section 7).

Remark 6 (what a quandle can and cannot see)*.* Reflecting the sphere preserves the over/under data and the orientation of the arc but reverses the sign of every crossing, so a Q-colouring of $\operatorname{sym} D$ is the same thing as a Q^{op}-colouring of D, with the same colours at the leg and at the head: $\Phi_Q^M(\operatorname{sym} D) = \Phi_{Q^{\mathrm{op}}}^M(D)$. Consequently an involutory quandle is blind to sym, and for such a quandle $\Phi_Q(\operatorname{rot} D) =$

$\Phi_Q(\mathrm{sym}(\mathrm{mir}\,D)) = \Phi_Q(\mathrm{mir}\,D)$. Mirror reflection, on the other hand, changes which semiarcs are over-strands and is not a change of quandle. For classical knots, mir and sym give the same knot, and Fox colourings cannot distinguish a knot from its mirror image; for knotoids they are different involutions, and this is why Fox 3-colourings can see rot at all. (This paragraph is not part of the formal development; Theorem 12 exhibits the identity $\Phi_{R_3}(\mathrm{rot}\,D) = \Phi_{R_3}(\mathrm{mir}\,D)$ on one diagram, and G_8 is used in that theorem precisely because it is not involutory.)

3. The residue is non-rotatable

Throughout, D_K denotes the PD code of the knotoid K published in the data supplement of [2]; the codes of the knotoids named in the theorems are listed in Appendix A. Each theorem states the computed values, since they are the certificates.

Theorem 7 (Fox 3-colourings)**.** *For the* 22 *knotoids K of the residue listed in the first table of Appendix A, the pairs* $(\Phi_{R_3}(D_K), \Phi_{R_3}(\mathrm{rot}\,D_K))$ *are*

$$\begin{array}{llllll}
7_{26}\colon (3,9) & 7_{28}\colon (3,9) & 7_{34}\colon (27,9) & 7_{35}\colon (9,27) & 7_{50}\colon (3,9) & 7_{51}\colon (3,9) \\
7_{52}\colon (3,9) & 7_{53}\colon (9,3) & 7_{54}\colon (3,9) & 7_{59}\colon (27,9) & 7_{60}\colon (9,27) & 7_{63}\colon (3,9) \\
7_{64}\colon (3,9) & 7_{65}\colon (3,9) & 7_{75}\colon (9,3) & 7_{76}\colon (9,3) & 7_{80}\colon (3,9) & 7_{81}\colon (9,3) \\
7_{88}\colon (9,3) & 7_{90}\colon (9,3) & 7_{91}\colon (9,3) & 7_{92}\colon (9,3), & &
\end{array}$$

and in particular $\Phi_{R_3}(D_K) \neq \Phi_{R_3}(\mathrm{rot}\,D_K)$ for each of them. Hence, by Lemma 5, none of these 22 *knotoids is rotatable.*

Theorem 8 (the Alexander quandle Q_9)**.** *For the six remaining knotoids of the residue, the pairs* $(\Phi_{Q_9}(D_K), \Phi_{Q_9}(\mathrm{rot}\,D_K))$ *are*

$$7_{72}\colon (9,81), \quad 7_{74}\colon (9,81), \quad 7_{77}\colon (81,9), \quad 7_{78}\colon (81,9), \quad 7_{79}\colon (9,81), \quad 7_{85}\colon (9,81),$$

so $\Phi_{Q_9}(D_K) \neq \Phi_{Q_9}(\mathrm{rot}\,D_K)$ and none of the six is rotatable. For these six, $\Phi_{R_3}(D_K) = \Phi_{R_3}(\mathrm{rot}\,D_K)$: Fox 3*-colourings do not separate them.*

Corollary 9. *All* 28 *knotoids* (1) *that Gabrovšek and Cavicchioli conjecture to be non-rotatable are non-rotatable.*

Proof sketch of Theorems 7 and 8. Each count is an exhaustive enumeration: the leg semiarc is given each of the $|Q|$ colours in turn, the strand is walked from the leg to the head, and at each crossing the colour of the next semiarc is forced (under-passage) or the over-strand's colour is chosen among $|Q|$ values if it is not yet coloured (over-passage); an assignment that reaches the head with all constraints satisfied is a colouring. The search space is at most 3^{15} for R_3 and 9^{15} for Q_9 on a seven-crossing diagram with 15 semiarcs, and in practice a few thousand nodes. Both D_K and $\mathrm{rot}\,D_K$ are enumerated for each K; the 56 counts of Theorem 7 and the 12 of Theorem 8 are the outputs, verified by the Lean kernel as described in Section 7. The deduction "$K \neq \mathrm{rot}\,K$" is Lemma 5. □

Remark 10 (the source's "mutant-type pairs"). The appendix of [2] pairs the 28 knotoids as "possible duplicates" into 14 pairs. For six of these pairs the five invariant columns of the data file coincide: $(7_{53}, 7_{54})$, $(7_{63}, 7_{76})$, $(7_{65}, 7_{81})$, $(7_{72}, 7_{85})$, $(7_{74}, 7_{79})$, $(7_{88}, 7_{90})$. The counts above say something about them. In the first three pairs the two partners have different counts (9 against 3), so they are different knotoids; the counts are consistent with the partner being the rotation. In the last three the partners have equal counts but $\Phi_Q(D_K) \neq \Phi_Q(\mathrm{rot}\,D_{K'})$, so the partner is *not* the rotation: $7_{85} \neq \mathrm{rot}\,7_{72}$, $7_{79} \neq \mathrm{rot}\,7_{74}$, $7_{90} \neq \mathrm{rot}\,7_{88}$. Nothing here decides whether the partners of the last three pairs are equal. For the other eight pairs the two retained representatives have different invariant columns; in all eight their `kbsm` entries are related by $A \mapsto A^{-1}$ up to the Reidemeister I unit, so there the pairing is between a knotoid and a mirror representative, and we draw no conclusion.

3.1. **Beyond the homotopy index polynomial.** Nikonov [7] associates with a spherical knotoid K a Laurent polynomial $H(K) \in \mathbb{Z}[x^{\pm 1}]$, the homotopy index polynomial, and proves [7, Proposition 9] that $H(\bar{K}) = -H(K)$ and $H(K^*)(x) = -H(K)(x^{-1})$, so that $H(\operatorname{rot} K)(x) = H(K)(x^{-1})$; a non-palindromic $H(K)$ therefore proves K non-rotatable. His Example 5 introduces "knotoid $K7_{53}$ from the table" and computes its planar homotopy index polynomials; his Example 6 reads them on the sphere, obtains $H = x - 1$, and concludes that 7_{53}, its mirror, its symmetry and its rotation are four different spherical knotoids. That is the only knotoid of the residue treated there. He also proves [7, Proposition 10] that the affine index polynomial of K equals $H(K)(t) + H(K)(t^{-1})$, which explains at once why that invariant (the "0/427" row of Table 2 of [2]) can never separate a knotoid from its rotation.

The polynomial H was recomputed from the published PD codes for all 427 knotoids, and the implementation was gated on Nikonov's value $H(7_{53}) = x - 1$ and on the identity with the affine index column of the table, which it reproduces for all 427 rows. The outcome (a computation outside the formal development) is that H is non-palindromic for 340 of the 427 knotoids and for 20 of the 28 of the residue. On the remaining eight it is palindromic, and those are the content of the next theorem.

Theorem 11. *Let K be one of $7_{35}, 7_{60}, 7_{88}, 7_{90}, 7_{74}, 7_{77}, 7_{78}, 7_{79}$. For each of them the four counts $\Phi_{R_3}(D_K)$, $\Phi_{R_3}(\operatorname{rot} D_K)$, $\Phi_{Q_9}(D_K)$, $\Phi_{Q_9}(\operatorname{rot} D_K)$ are, in this order for the eight knotoids,*

$$(9,27,9,9),\ (9,27,9,9),\ (9,3,9,9),\ (9,3,9,9),$$
$$(3,3,9,81),\ (3,3,81,9),\ (3,3,81,9),\ (3,3,9,81),$$

so for each of the eight either $\Phi_{R_3}(D_K) \neq \Phi_{R_3}(\operatorname{rot} D_K)$ or $\Phi_{Q_9}(D_K) \neq \Phi_{Q_9}(\operatorname{rot} D_K)$, and K is not rotatable.

These eight knotoids are, to our knowledge, outside the reach of every previously published method: they are in the residue of [2], so its five polynomials do not see them, and H is palindromic on them. Note that on the first four, Q_9 is blind (9 against 9) and only R_3 separates, while on the last four it is the other way round.

4. Two corrections to the rotatable column

Section 5.1 of [2] states that "rotatable knotoids are 0_1, 3_1, 4_1, 5_1, 5_2, 5_{14}, 5_{15}, 5_{24}, 5_{25}, 6_1, 6_2, 6_3, 6_{31}, 6_{34}, 6_{81}, and 6_{82}" up to six crossings, and the data file marks 37 knotoids rotatable, among them 5_{24} and 7_{288}.

Theorem 12 (5_{24} is not rotatable)**.** $\big(\Phi_{R_3}(D_{5_{24}}),\ \Phi_{R_3}(\operatorname{rot} D_{5_{24}}),\ \Phi_{R_3}(\operatorname{mir} D_{5_{24}})\big) = (9,3,3)$. *Moreover $\Phi^M_{G_8}(D_{5_{24}}) \neq \Phi^M_{G_8}(\operatorname{sym} D_{5_{24}})$, and G_8 is a quandle. Consequently 5_{24} is fixed by none of* rot, mir, sym*: it is non-rotatable, chiral, and not symmetric.*

Two independent facts corroborate the correction. First, Nikonov's polynomial recomputed from the same PD code is $H(5_{24}) = x - x^{-1}$, which is not palindromic. Second, Goundaroulis, Dorier and Stasiak list exactly five rotatable spherical knotoids with five crossings, $5_1, 5_2, 5_{15}, 5_{17}, 5_{18}$ in their numbering [4, table "All rotatable knotoids in S^2"], whereas Table 1 of [2] has six; removing 5_{24} reconciles the two counts. (Their table lists the *strongly achiral* knotoids — those that are rotatable and achiral — in a column of their own, but there are none at five crossings, so five is their whole count there.) The two classifications are not indexed alike: [4] merge diagrams related by reversion as well, which [2] nowhere do; and their totals differ, 24 against 25 prime spherical knotoids at five crossings and 121 against 82 at six. Neither paper accounts for that difference, and the difference in the equivalence relation cannot: quotienting by more can only lower a count, and the larger six-crossing count is [4]'s. So this is corroboration rather than proof.

Theorem 13 (7_{288} is not rotatable)**.** *The counting matrices over Q'_9 of $D_{7_{288}}$ and of* $\operatorname{rot} D_{7_{288}}$ *differ, $\Phi^M_{Q'_9}(D_{7_{288}}) \neq \Phi^M_{Q'_9}(\operatorname{rot} D_{7_{288}})$, but their totals are equal, $\Phi_{Q'_9}(D_{7_{288}}) = \Phi_{Q'_9}(\operatorname{rot} D_{7_{288}})$; and Q'_9 is a quandle. Consequently 7_{288} is not rotatable, and the plain counting invariant over Q'_9 does not show it.*

Both totals equal 81 (a value recomputed outside the formal statement, which records only their equality), and R_3, Q_9 and Nikonov's H are all blind here: $H(7_{288}) = x - 2 + x^{-1}$ is palindromic. So this correction is reached only by the counting matrix. A caveat must be stated. The diagram $D_{7_{288}}$

is fixed, up to the invariants used here, by $\mathrm{rot} \circ \mathrm{rev}$: for every quandle in the battery of Section 6 the counting matrices of $D_{7_{288}}$ and of $\mathrm{rot}(\mathrm{rev}\, D_{7_{288}})$ agree. If the canonical form by which [2] compares a diagram with its rotation does not track the orientation of the arc, its verdict on 7_{288} is explained by this; under Turaev's involutions, in which rot does not involve rev, the knotoid is non-rotatable. No such explanation is available for 5_{24}, whose counting matrices agree with those of its image under an element of the group generated by $\mathrm{rev}, \mathrm{mir}, \mathrm{sym}$ only for the identity and for rev. (Both statements are sweeps outside the formal development: the 375 counting matrices of each of the 427 diagrams and of its seven images under that group.)

5. The bracket polynomial does not see the rotation

The leading invariant of [2] is the Kauffman bracket (their column `kbsm`). For a knotoid diagram D with k crossings let

$$\langle D \rangle = \sum_{\text{states}} A^{\#A - \#B} \, (-A^2 - A^{-2})^{\text{loops}}$$

be the raw bracket, the sum over the 2^k smoothings of D, where the open component is not counted as a loop. A Reidemeister I move multiplies it by $-A^{\pm 3}$, and [2] stores it up to such a unit. Turaev's normalised bracket $\langle D \rangle_\circ = (-A^3)^{-w(D)} \langle D \rangle$ "is invariant under the reversion of knotoids and changes via $A \mapsto A^{-1}$ under mirror reflection and under orientation reversion in Σ" [8, §7.1] — a statement he makes there for knotoids in an arbitrary oriented surface Σ, in running text, with no theorem number. Since sym is induced by an orientation-reversing self-homeomorphism of S^2, the normalised bracket is invariant under $\mathrm{rot} = \mathrm{mir} \circ \mathrm{sym}$; and mir and sym each reverse the sign of every crossing, so the writhe survives rot and the raw bracket $\langle\ \rangle$ is invariant too.

Proposition 14. *For each of the 28 PD codes D of the residue, $\langle D \rangle = \langle \mathrm{rot}\, D \rangle$ exactly (as raw brackets). Moreover the brackets of the pairs $(7_{53}, 7_{54})$ and $(7_{88}, 7_{90})$ coincide up to a unit $\pm A^k$.*

So no amount of bracket computation could have settled the conjecture. This is a formalised instance of Turaev's law, and it is stated here because it is checked from the same PD codes as the colouring counts, which makes the two computations comparable: the counts see what the bracket provably cannot.

Proposition 15 (the bracket reproduces the table)**.** *For $K \in \{2_1, 3_1, 3_2, 4_1, 5_{24}, 7_{53}\}$ the raw bracket of D_K, taken up to a unit $\pm A^k$, equals the `kbsm` entry of K in [2]; for instance $\langle D_{2_1} \rangle \doteq A^8 + A^6 - A^2$ and $\langle D_{5_{24}} \rangle \doteq A^{16} + A^{14} - 2A^{12} - 3A^{10} + 3A^6 + A^4 - A^2 - 1$ up to such a unit.*

Proposition 15 is a consistency check on the encoding rather than a result: it shows that the PD codes are read here with the same conventions as in [2]. Outside the formal development the same check was run on all 427 rows: the recomputed bracket agrees with the `kbsm` column up to $\pm A^{3m}$ for every row, and the two induce the same partition of the 427 knotoids into 412 classes with 398 singletons. (Table 2 of [2] credits the bracket with 370 unique values where the final data file gives 398. The difference is a difference of question: their column counts the knotoids whose value is shared with no other knotoid *and with no mirror image*. Identifying each bracket with its image under $A \mapsto A^{-1}$ does return 370 against 57, and the same reading returns their 342/85 for the mock Alexander column under $w \mapsto w^{-1}$, their 0/427 for the affine index column under $t \mapsto t^{-1}$ together with a sign, and their 397/30 for the Yamada column under $A \mapsto A^{-1}$ up to a unit.) Likewise the raw bracket satisfies $\langle \mathrm{rot}\, D \rangle = \langle D \rangle$ for all 427 codes, while sym and mir separately send it to $A \mapsto A^{-1}$, a transformation that is visible on 422 of the 427 (the other five brackets being palindromic).

6. Controls, and what stays open

Proposition 16 (non-vacuity)**.** *For each of the six knotoids $0_1, 3_1, 4_1, 5_{25}, 6_3, 7_{284}$, which [2] proves rotatable, Φ_{R_3}, Φ_{Q_9}, $\Phi^M_{G_8}$ and $\Phi^M_{Q'_9}$ take the same value on D_K and on $\mathrm{rot}\, D_K$.*

Proposition 17 (negative controls)**.** (1) *Let Φ'_Q be the count obtained by choosing between $\triangleright$ and $\triangleright^{-1}$ from the direction of the over-strand alone. Then $\Phi'_{G_8}(D_{3_1}) = 8$ and $\Phi'_{G_8}(\mathrm{rot}\, D_{3_1}) = 32$, whereas $\Phi_{G_8}(D_{3_1}) = \Phi_{G_8}(\mathrm{rot}\, D_{3_1}) = 32$. Since 3_1 is rotatable, Φ' is not an invariant.*

(2) *Changing one arc label of* $D_{7_{53}}$*, in a crossing node or in the head node, makes the strand traversal fail, and the count is not produced.*
(3) $\Phi_{R_3}(D_{7_{53}}) = 9$*, and is neither* 8 *nor* 10*;* $\Phi_{R_3}(\mathrm{rot}\, D_{7_{53}}) = 3$*, and is neither* 2 *nor* 4*.*
(4) *The trivial knotoid* 0_1 *has exactly* $|Q|$ *colourings for* $Q = R_3, Q_9, Q_9', G_8$*; the quandle test rejects the table* $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ *and accepts the trivial quandle of order* 3*.*

Item (1) is the reason the crossing rule is spelled out in Section 2: the wrong reading is the natural one, and what is wrong with it stays invisible until a Reidemeister II move. Outside the formal development the correct rule was tested against the moves directly: 800 Reidemeister I, 918 Reidemeister II and 136 Reidemeister III moves were generated on the published diagrams, each accepted only after an independent check (Euler's formula on the sphere, and the bracket transforming as it must), and none changed a counting matrix. A second generator, written independently for this paper, adds 1584 type I and 2692 type II moves with the same outcome for the correct rule — and it separates the two rules: the over-strand-only reading of item (1) survives all 1584 type I moves and changes its count on 489 of the 2692 type II moves.

Proposition 18 (the residue of the residue)**.** *For each of the ten knotoids*

$$6_3,\ 6_{22},\ 6_{48},\ 6_{64},\ 7_6,\ 7_{58},\ 7_{227},\ 7_{229},\ 7_{272},\ 7_{301},$$

the counting matrices $\Phi^M_{R_3}$*,* $\Phi^M_{Q_9}$*,* $\Phi^M_{Q_9'}$ *and* $\Phi^M_{G_8}$ *agree on* D_K *and on* $\mathrm{rot}\, D_K$*.*

These ten are exactly the knotoids of the non-rotatable column of [2] that a battery of 375 quandles of order at most 12 fails to separate from their rotation. The battery is generated rather than enumerated: Alexander quandles on cyclic and on small abelian groups; generalized Alexander quandles $x \triangleright y = \varphi(xy^{-1})y$ over every automorphism φ of S_3, D_4, A_4, S_4, D_5, D_6, Q_8 and $(\mathbb{Z}/2)^2$; and conjugation quandles on unions of at most two conjugacy classes of those groups. It has 375 members of order at most 12 and 1252 of order at most 27, the larger one having been run on the residue only. The four quandles of Proposition 18 are the formalised part of that measurement. Nikonov's polynomial settles three of the ten (6_{64}, 7_{227}, 7_{301}), and the two methods together give the following count, all of it outside the formal development except where stated above: the quandle battery separates 382 of the 427 knotoids from their rotation, H separates 340, the union is 385, with 45 knotoids seen by quandles only and three by H only. Two of the 385 are the corrections of Section 4; so of the 390 knotoids that [2] calls non-rotatable, 383 are now proved so, and seven stay open:

$$6_3,\ 6_{22},\ 6_{48},\ 7_6,\ 7_{58},\ 7_{229},\ 7_{272}.$$

Two of the seven deserve a comment. The knotoid 6_3 is listed as rotatable in the text of [2] quoted in Section 4 and in its appendix, but as non-rotatable in its data file and in the totals of its Table 1. It is a knot-type knotoid: [2] writes that "we have only three fully achiral (achiral and rotatable) knotoids, namely 0_1, 4_1, and 6_3, which are all knot-like knotoids that coincide with the fully achiral knots", and [4] likewise list their own 6_3 among the knot-type knotoids and call it strongly achiral, which is their term for rotatable and achiral. Knot-type knotoids are rotatable: Barbensi, Buck, Harrington and Lackenby record that "for knot-type knotoids symmetry and mirror reflection coincide … In particular, every knot-type knotoid is rotatable" [1, Remark 2.5], and the reason is Turaev's, that "the restrictions of sym and mir to knots are equal because the mirror reflections in the planes $\mathbb{R}^2 \times \{0\}$ and $\{0\} \times \mathbb{R}^2$ are isotopic" [8, §3.3], whence $\mathrm{rot} = \mathrm{mir} \circ \mathrm{sym}$ is the identity there. We therefore expect 6_3 to be genuinely rotatable, and the same internal disagreement of [2] affects 7_6, which its appendix labels rotatable and its data file does not. Indeed the per-crossing rows of Table 1 of [2] sum to 39 rotatable knotoids while its totals row reads 37 (and its non-rotatable rows to 388 against a total of 390); the two rows in question are 6_3 at six crossings and 7_6 at seven. The other five are the frontier.

Nothing in this paper settles the seven. Every invariant used here is a counting invariant of a quandle; biquandle colourings, which have one more degree of freedom per crossing and are the setting of [5] from the start, and quandle cocycle invariants over the same small quandles, are the natural next instruments.

7. Verification

All statements pinned to a theorem or proposition above have been formalised and checked in Lean 4 with Mathlib. The development consists of five modules: the definitions (PD codes, the involutions sym, mir, rot, rev on codes, the strand traversal, the quandle axioms as a decidable predicate, and the enumeration computing Φ_Q^M; 21 definitions), the data (the four quandle tables and the PD codes copied from the published data file; 20 definitions), the colouring theorems (19 theorems, Sections 3 and 4), the bracket (4 theorems and the state sum, Section 5) and the controls (7 theorems, Section 6). Every count is decided by evaluating the definitions on the explicit codes: each of the 29 computational theorems is discharged by `native_decide`, so its correctness rests on the compiled evaluator, and its axiom list consists of the one `native_decide` axiom generated for that theorem, together with `propext` (in 28 of the 29) and `Quot.sound` where an equality of lists or arrays enters the statement (in 7 of them); the one non-computational statement, Lemma 1, is proved by rewriting with the definitions and uses only `propext`, `Classical.choice` and `Quot.sound`. There are no `sorry`s and no axioms beyond these. The invariance theorem of Gügümcü and Nelson (Theorem 4) and the transformation law of Turaev are cited, not formalised; the step from the inequalities of the theorems to "non-rotatable" is Lemma 5 on paper. Elaborating the five modules took about half an hour of single-threaded CPU time with a peak resident set of about 7 GB (the Mathlib baseline); the computations outside Lean reported above took under half an hour more. Repository reference to be added at submission.

Appendix A. The PD codes and the counts

Codes are those of the data supplement of [2], in the notation of Section 2; the first node is the leg. The counts are those of Theorems 7 and 8.

K	$\Phi_{R_3}(D)$	$\Phi_{R_3}(\operatorname{rot} D)$	PD code of D
7_{26}	3	9	[0], [0,1,2,3], [1,4,5,2], [3,5,6,7], [4,8,9,10], [10,9,11,6], [7,11,12,13], [13,12,14,8], [14]
7_{28}	3	9	[0], [0,1,2,3], [1,4,5,2], [3,6,7,8], [4,9,10,11], [11,12,6,5], [12,13,8,7], [9,13,14,10], [14]
7_{34}	27	9	[0], [0,1,2,3], [1,4,5,2], [3,6,7,8], [4,9,10,5], [6,10,11,12], [12,13,8,7], [9,13,14,11], [14]
7_{35}	9	27	[0], [0,1,2,3], [1,4,5,2], [3,6,7,8], [4,9,10,5], [6,10,11,12], [12,13,8,7], [13,14,11,9], [14]
7_{50}	3	9	[0], [0,1,2,3], [1,4,5,2], [3,6,7,8], [4,8,9,10], [11,12,13,5], [6,13,12,7], [14,11,10,9], [14]
7_{51}	3	9	[0], [0,1,2,3], [1,4,5,2], [3,6,7,8], [9,10,11,4], [5,12,13,6], [10,9,8,7], [11,13,12,14], [14]
7_{52}	3	9	[0], [0,1,2,3], [1,4,5,2], [3,6,7,8], [8,7,9,4], [10,11,6,5], [12,13,14,9], [13,12,11,10], [14]
7_{53}	9	3	[0], [0,1,2,3], [1,4,5,2], [3,6,7,8], [8,9,10,4], [11,12,13,5], [6,13,12,7], [9,14,11,10], [14]
7_{54}	3	9	[0], [0,1,2,3], [1,4,5,2], [6,7,8,3], [4,8,7,9], [5,10,11,6], [9,12,13,14], [10,13,12,11], [14]
7_{59}	27	9	[0], [0,1,2,3], [1,4,5,2], [3,5,6,7], [4,8,9,10], [11,12,13,6], [7,13,14,8], [12,11,10,9], [14]
7_{60}	9	27	[0], [0,1,2,3], [1,4,5,2], [3,5,6,7], [4,8,9,10], [11,12,13,6], [13,14,8,7], [12,11,10,9], [14]
7_{63}	3	9	[0], [0,1,2,3], [1,4,5,2], [3,6,7,8], [8,9,10,4], [5,11,12,6], [13,10,9,7], [11,14,13,12], [14]
7_{64}	3	9	[0], [0,1,2,3], [1,4,5,2], [3,6,7,8], [8,7,9,4], [10,11,6,5], [11,12,13,9], [14,13,12,10], [14]
7_{65}	3	9	[0], [0,1,2,3], [1,4,5,2], [6,7,8,3], [4,8,7,9], [5,10,11,6], [9,11,12,13], [10,14,13,12], [14]
7_{75}	9	3	[0], [0,1,2,3], [1,4,5,2], [3,6,7,8], [4,9,10,11], [11,12,6,5], [12,10,13,7], [8,14,13,9], [14]
7_{76}	9	3	[0], [0,1,2,3], [1,4,5,2], [3,6,7,8], [4,9,10,11], [11,12,6,5], [12,10,13,7], [13,14,9,8], [14]
7_{80}	3	9	[0], [0,1,2,3], [1,4,5,2], [3,6,7,8], [4,8,9,10], [11,12,6,5], [12,11,13,7], [14,13,10,9], [14]
7_{81}	9	3	[0], [0,1,2,3], [1,4,5,2], [3,6,7,8], [8,9,10,4], [11,12,6,5], [12,11,13,7], [9,14,13,10], [14]
7_{88}	9	3	[0], [0,1,2,3], [1,4,5,2], [3,6,7,8], [4,8,9,10], [10,11,12,5], [6,12,11,13], [13,9,14,7], [14]
7_{90}	9	3	[0], [0,1,2,3], [1,4,5,6], [6,7,8,2], [3,9,10,4], [11,12,13,5], [7,13,14,8], [9,14,12,10], [11]
7_{91}	9	3	[0], [0,1,2,3], [1,4,5,2], [3,6,7,8], [4,8,9,10], [11,12,6,5], [13,9,14,7], [10,13,12,11], [14]
7_{92}	9	3	[0], [0,1,2,3], [1,4,5,6], [7,8,3,2], [8,9,10,4], [11,10,12,5], [6,13,14,7], [9,14,13,12], [11]

K	$\Phi_{Q_9}(D)$	$\Phi_{Q_9}(\operatorname{rot} D)$	PD code of D
7_{72}	9	81	[0], [0,1,2,3], [1,4,5,6], [7,8,3,2], [4,9,10,5], [6,11,12,7], [8,12,13,9], [14,13,11,10], [14]
7_{74}	9	81	[0], [0,1,2,3], [1,4,5,2], [3,6,7,8], [4,9,10,11], [11,12,6,5], [12,10,13,7], [8,13,14,9], [14]
7_{77}	81	9	[0], [0,1,2,3], [1,4,5,2], [3,6,7,8], [4,9,10,11], [11,12,6,5], [12,10,13,7], [14,13,9,8], [14]
7_{78}	81	9	[0], [0,1,2,3], [1,4,5,6], [7,8,3,2], [9,10,11,4], [10,12,13,5], [6,13,14,7], [8,14,12,9], [11]
7_{79}	9	81	[0], [0,1,2,3], [1,4,5,6], [6,7,8,2], [3,8,9,10], [4,11,12,5], [7,12,13,9], [10,13,14,11], [14]
7_{85}	9	81	[0], [0,1,2,3], [1,4,5,2], [3,6,7,8], [4,8,9,10], [11,12,6,5], [12,13,14,7], [13,11,10,9], [14]

K	PD code of D
5_{24}	[0], [0,1,2,3], [1,3,4,5], [6,7,8,2], [9,6,5,4], [7,9,8,10], [10]
7_{288}	[0], [0,1,2,3], [1,4,5,2], [6,7,4,3], [5,8,9,6], [7,9,10,11], [11,12,13,8], [10,13,12,14], [14]

K	PD code of D
6_3	[0], [0,1,2,3], [1,4,5,2], [3,5,6,7], [4,8,9,6], [7,9,10,11], [8,12,11,10], [12]
6_{22}	[0], [0,1,2,3], [1,4,5,2], [3,5,6,7], [7,8,9,4], [6,9,10,11], [8,12,11,10], [12]
6_{48}	[0], [0,1,2,3], [1,4,5,2], [6,7,4,3], [7,8,9,5], [10,11,8,6], [9,12,11,10], [12]
6_{64}	[0], [0,1,2,3], [1,4,5,2], [3,6,7,4], [7,8,9,5], [6,10,11,8], [11,12,10,9], [12]
7_6	[0], [0,1,2,3], [1,4,5,2], [3,5,6,7], [4,8,9,6], [7,10,11,12], [8,13,10,9], [13,14,12,11], [14]
7_{58}	[0], [0,1,2,3], [1,4,5,2], [3,5,6,7], [4,7,8,9], [10,11,12,6], [13,10,9,8], [11,13,14,12], [14]
7_{227}	[0], [0,1,2,3], [1,4,5,2], [3,6,7,4], [5,7,8,9], [6,10,11,8], [9,12,13,10], [14,13,12,11], [14]
7_{229}	[0], [0,1,2,3], [1,4,5,2], [3,6,7,4], [5,7,8,9], [6,9,10,11], [11,12,13,8], [14,13,12,10], [14]
7_{272}	[0], [0,1,2,3], [1,4,5,2], [3,6,7,4], [5,8,9,6], [7,10,11,12], [8,12,13,9], [10,13,14,11], [14]
7_{301}	[0], [0,1,2,3], [1,4,5,2], [3,6,7,4], [8,9,6,5], [10,11,8,7], [11,12,13,9], [10,13,12,14], [14]

Appendix B. The quandle G_8

The table of G_8 used in Theorem 12 and Propositions 16, 17 and 18; row x, column y holds $x \triangleright y$ on the elements $0, \ldots, 7$.

$\triangleright$	0	1	2	3	4	5	6	7
0	0	5	0	5	1	4	1	4
1	4	1	4	1	7	2	7	2
2	2	7	2	7	3	6	3	6
3	6	3	6	3	5	0	5	0
4	5	2	5	2	4	3	4	3
5	1	6	1	6	2	5	2	5
6	7	0	7	0	6	1	6	1
7	3	4	3	4	0	7	0	7

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