EXACT RATIONAL COUNTEREXAMPLES TO THE TRIANGLE INEQUALITY FOR THE KOMÁLOVICS–MOLNÁR DISTANCES $d_{1/k}$, $2 \leq k \leq 8$

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Abstract. For $p > 0$ and positive semidefinite matrices X, Y, Komálovics and Molnár set $X\kappa_p Y = (X^{p/4}Y^{p/2}X^{p/4})^{1/p}$ and $d_p(X,Y) = \big(\mathrm{Tr}((X+Y)/2 - X\kappa_p Y)\big)^{1/2}$, a one-parameter family through the quantum Hellinger ($p = 1$) and Bures ($p = 2$) distances, and asked (their Problem 2) whether d_p on $\mathbb{M}_n(\mathbb{C})_+$, $n \geq 2$, is a metric for each $1 < p < 2$ and fails to be one for each $0 \leq p < 1$, reporting numerical evidence for both halves. Zhang (arXiv:2602.11922) settled the single value $p = 1/2$ by an explicit triple in $\mathbb{M}_2(\mathbb{C})_{++}$. We settle $p = 1/3, 1/4, 1/5, 1/6, 1/7, 1/8$, and give an independent, entirely rational witness at $p = 1/2$. The mechanism is that at $p = 1/k$ the outer exponent is the natural number k: if $X = X_1^{4k}$ and $Y = Y_1^{4k}$ with X_1, Y_1 rational and positive semidefinite, then $\mathrm{Tr}(X\kappa_{1/k}Y) = \mathrm{Tr}\big((X_1Y_1^2X_1)^k\big)$ is a rational number produced by ring arithmetic alone, and the triangle inequality becomes a comparison of two rationals. Each triple is built from one rational reflection W of $\mathbb{R}^2$: $A_1 = \mathrm{diag}(1, 1/10)$, $B_1 = WA_1W$, and C_1 a polynomial in W with two rational eigenvalues chosen for each k, so that $d_p(A,C) = d_p(C,B)$ and the failure reads $d_p(A,B)^2 - 4\,d_p(A,C)^2 > 1/10$. The same triples satisfy the triangle inequality at $p = 1$, the failure is by a factor below 2, and a rank-one triple satisfies the inequality at each $p = 1/k$, as Komálovics and Molnár's Proposition 8 predicts. All statements are verified in Lean 4 against *Mathlib*. The half $1 < p < 2$ of Problem 2 is untouched here; we record, as an unproved reduction, a two-variable inequality which would settle the whole range $0 < p < 1$ at once.

1. Introduction

Let $\mathcal{A}$ be a unital C^*-algebra with a faithful tracial positive linear functional τ, and let $\mathcal{A}_+$ be its positive cone. For $p > 0$ and $A, B \in \mathcal{A}_+$ Komálovics and Molnár [1] define

$$(1)\qquad A\kappa_p B := \big(A^{p/4}B^{p/2}A^{p/4}\big)^{1/p}, \qquad A\nabla B := \frac{A+B}{2}, \qquad d_p^\tau(A,B) := \big(\tau(A\nabla B - A\kappa_p B)\big)^{1/2},$$

the powers being taken by the continuous functional calculus on $[0,\infty)$. In the matrix case $\mathcal{A} = \mathbb{M}_n(\mathbb{C})$, $\tau = \mathrm{Tr}$, one writes d_p. The point of the family is that $\sqrt{2}\,d_1$ is the quantum Hellinger distance and $\sqrt{2}\,d_2$ the Bures (or Bures–Wasserstein) distance [3, 4], both true metrics on $\mathbb{M}_n(\mathbb{C})_+$ (at the C^*-level, [1] attributes the metric property of d_2^τ to Farenick and Rahaman [7]); the κ_p are described in [1] as variants of the Kubo–Ando geometric mean [8], with which they coincide for commuting arguments. (For the parallel one-parameter families of matrix power means see [9]; for order properties of means outside the Kubo–Ando class in operator algebras see [11]; the κ_p also occur in the study of quantum Rényi relative entropies [10].) Komálovics and Molnár prove that $\tau(A\nabla B - A\kappa_p B) \geq 0$ for $0 \leq p \leq 2$ in every such $\mathcal{A}$, and for every $p \geq 0$ when $\mathcal{A} = \mathbb{M}_n(\mathbb{C})$, so that d_p^τ is defined there; Zhang [2, Theorem 3.3] extends the nonnegativity to all $p > 0$ in every C^*-algebra with a faithful trace, answering their Problem 1. The question this paper is about is their Problem 2, which we quote from the article (p. 18):

> "**Problem 2.** Given a positive integer $n \geq 2$, is that true that d_p on $\mathbb{M}_n(\mathbb{C})_+$ is a metric for each $1 < p < 2$ and it is not a metric for each $0 \leq p < 1$? Concerning the case of C^*-algebras, is it true that d_p^τ is a metric if $1 < p < 2$?
>
> As for the first two questions, using mathematical software, computer simulations convince us to believe that the answers to those questions are indeed affirmative."

(Here d_0 uses $A\kappa_0 B = \exp((\log A + \log B)/2)$, the log-Euclidean mean, on $\mathcal{A}_{++}$.) So the authors had numerical evidence at every p and no proof at any p other than 1 and 2; the word "counterexample" does not occur in the article. What they do prove about the triangle inequality is confined to pure states: their Proposition 8 (p. 15) says that on the rank-one projections of a Hilbert space H with

$\dim H > 1$, d_p is a metric if and only if $p \le 2$, with the explicit formula $d_p(P,Q) = (1-\cos^{2/p}\alpha)^{1/2}$, α the angle between the ranges. They also remark, immediately after Problem 2, that a negative answer for some $0 < p < 1$ would carry over to C^*-algebras: "if, for $0 < p < 1$, d_p is really not a metric on $\mathbb{M}_n(\mathbb{C})_+$ ($n \ge 2$), then using the method presented in the proof of Proposition 2, one could prove that if d_p^τ is a true metric on the positive cone $\mathcal{A}_+$ of a C^*-algebra $\mathcal{A}$ for some $0 < p < 1$, then $\mathcal{A}$ is necessarily commutative."

Zhang [2] restates the question as his Problem 1.2 ("Let $n \ge 2$. Is d_p^{Tr} on $\mathbb{M}_n(\mathbb{C})_+$ a metric for each $1 < p < 2$, and is it *not* a metric for each $0 < p < 1$? More generally, if $\mathcal{A}$ is a C^*-algebra with a faithful trace τ, is d_p^τ a metric for $1 < p < 2$?") and settles exactly one value. His Proposition 4.2, in the section titled "On Problem 1.2: an explicit counterexample for $p = \frac{1}{2}$", exhibits

$$A = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 5/2 & 3/2 \\ 3/2 & 5/2 \end{pmatrix}, \quad C = \begin{pmatrix} 25/8 & 3/4 \\ 3/4 & 13/8 \end{pmatrix} \in \mathbb{M}_2(\mathbb{C})_{++}$$

with $d_{1/2}(A,B) > d_{1/2}(A,C) + d_{1/2}(C,B)$, the violation reducing to the integer inequality $1089 > 1088$; his Remark 4.3 reads: "Proposition 4.2 shows that d_p need not be a metric for $0 < p < 1$. It does not address the case $1 < p < 2$, which remains open in general." (Statement numbers of [2] are those of the e-print compiled from the arXiv source of version 1, the only version as of 2026-09-07; those of [1] are article pages of the published layout.) So before the present work the table read: $p = 1$ and $p = 2$, metrics; $p > 2$, not a metric already on pure states ([1], Proposition 8); $p = 1/2$, not a metric on $\mathbb{M}_2(\mathbb{C})_{++}$ ([2], Proposition 4.2); and nothing at any other $p < 1$.

Results. We fill in six more values of the negative half.

- **Six new values** (Theorems 4.2–4.7). For each $p \in \{1/3, 1/4, 1/5, 1/6, 1/7, 1/8\}$ there is an explicit triple of real symmetric positive definite 2×2 matrices with rational entries at which the triangle inequality for d_p fails strictly; hence d_p is not a metric on $\mathbb{M}_2(\mathbb{R})_{++}$, and, by the padding argument of Remark 5.1 (on paper, not formalised), not on $\mathbb{M}_n(\mathbb{C})_+$ for any $n \ge 2$.
- **An independent witness at** $p = 1/2$ (Theorem 4.1). The value is Zhang's; the triple is new and entirely rational, with all three squared distances rational numbers, whereas his computation runs through $\mathbb{Q}(\sqrt{2}, \sqrt{17})$. Its margin $d_{1/2}(A,B)^2 - 4d_{1/2}(A,C)^2 \approx 0.113$ compares with $3\sqrt{34}/4 - \sqrt{17} - 1/4 \approx 1.08 \cdot 10^{-4}$ for his.
- **The mechanism** (Proposition 2.3). At $p = 1/k$ the outer exponent of (1) is the natural number k and the inner ones are $4k$-th and $2k$-th roots, so if the vertices are chosen as $4k$-th powers of rational positive semidefinite matrices every root the definition asks for is again rational, $d_{1/k}(X,Y)^2$ is a rational number computed by ring arithmetic, and the triangle inequality is decided by comparing two rationals. No square root, interval arithmetic or floating point enters.
- **Controls** (Proposition 4.8). For each of the seven triples: the three squared distances are strictly positive; the margin exceeds $1/10$; the failure is by a factor below 2; the *same* triple satisfies the triangle inequality at $p = 1$, where d_1 is the Hellinger metric; and a rank-one triple satisfies it at the same p, in agreement with Proposition 8 of [1].

Everything above except the passages marked as on paper is verified in Lean 4 against *Mathlib*; Section 6 records the modules and the axioms, and Appendix A reproduces the formal statements verbatim. Section 5 records, labelled as computations outside the formal development, the reduction of the whole range $0 < p < 1$ to a two-variable inequality (5), the numerical evidence for it, and why no single triple of the present shape can serve every k. Nothing here bears on the positive half $1 < p < 2$ of Problem 2, which remains open.

What is new and what is not. The statement "$d_{1/2}$ is not a metric on $\mathbb{M}_2(\mathbb{C})_{++}$" is Zhang's; what is ours at $p = 1/2$ is the witness and its certification. The two-line closed form for $\mathrm{Tr}(X\kappa_p Y)$ on 2×2 matrices (Lemma 2.1) is elementary — trace cyclicity and multiplicativity of the determinant — and is implicit in Zhang's computation; we claim nothing for it, and use it only for checks. Zhang's Lemma 4.1, the closed form $M^{1/2} = (M + \sqrt{\det M}\, I)/\sqrt{\mathrm{Tr}\, M + 2\sqrt{\det M}}$ for $M \in \mathbb{M}_2(\mathbb{C})_{++}$, is what makes his single example exact. (He derives it from a result of [5] that he cites as its Proposition 4.1.12; we have not consulted the book, and note only that the closed form is two lines of Cayley–Hamilton,

$(M+\delta I)^2 = (\operatorname{Tr} M + 2\delta)M$ for $\delta = \sqrt{\det M}$.) It does not extend to other $p = 1/k$, because the roots that the definition needs at $p = 1/k$ — of his $A = \operatorname{diag}(4,1)$ and of his d_1-midpoint $C = ((\sqrt{A} + \sqrt{B})/2)^2$ — generate number fields that grow with k (Remark 5.2). What we claim is the observation that $4k$-th powers of rational matrices make $d_{1/k}$ exactly rational, the triples themselves, which were found by a small search over two rational parameters (Section 3), and the six values. The nearest prior negative results we know of are Proposition 8 of [1] for $p > 2$ on pure states, the two examples of Bhatia, Gaubert and Jain [3] showing that $(\operatorname{Tr}(A+B) - 2\operatorname{Tr}(A\#B))^{1/2}$ and its log-Euclidean analogue fail the triangle inequality on 2×2 matrices (numerical; the first is described there as a small modification of one suggested to those authors by S. Sra), and the remark of [1] after Problem 2 that the Kubo–Ando version $(\tau(A\nabla B - A\#B))^{1/2}$ is a metric only on commutative algebras. For the positive side, [1] (p. 14) records that for $1 \le p \le 2$, d_p was shown in [6] to be a quantum divergence ([6, Definition 2.1 and Theorem 3], as cited there; we have not consulted that paper directly), and Sra's S-divergence [12] is the standing example of a log-determinant quantity whose square root is a true metric. As of 2026-09-07, the arXiv listing of [2] shows a single version; OpenAlex lists no work citing it and six works citing [1] (on central positive elements, barycenters [13], Heron-type means, the canonical trace, Tsallis relative entropies, and geometric-mean preservers), none about the metric property of d_p; and a full-text sweep of the 890,739 arXiv e-prints in our corpus (through 2609.04199) for the title or DOI of [1], the identifier or title of [2], and the surname Komálovics in its accented and TEX-escaped forms, re-run for this paper, returned exactly five e-prints. Only two of them match the title or the DOI of [1]: [2] itself, and the survey [13], which cites [1] in its bibliography and contains no statement about the triangle inequality for d_p. The other three match the surname alone, through a paper of Á. Komálovics and P. Bálint on non-uniformly hyperbolic dynamical systems that they cite or are by; none of them mentions d_p.

Verification. Theorems 4.1–4.7 and Proposition 4.8, together with Lemma 2.2, Proposition 2.3 and Lemma 2.4, rest on declarations checked by the Lean 4 kernel. Lemma 2.1, Lemma 3.1, Remark 5.1 and the whole of Section 5 are on paper or are numerical computations and are labelled as such; the formal development does not depend on them (in particular the equality $d_p(A,C) = d_p(C,B)$ of Lemma 3.1 is not used formally — both sides are computed and found equal).

2. Preliminaries

Throughout, $\mathbb{M}_2(\mathbb{R})_+$ and $\mathbb{M}_2(\mathbb{R})_{++}$ are the real symmetric positive semidefinite, respectively positive definite, 2×2 matrices. A real symmetric matrix is a complex Hermitian one, and the functional calculus of a real symmetric matrix by a real function computed over $\mathbb{R}$ agrees with the one computed over $\mathbb{C}$, so d_p on $\mathbb{M}_2(\mathbb{R})_+$ is the restriction of d_p on $\mathbb{M}_2(\mathbb{C})_+$; this is the only sense in which the results below are "real". For $X \in \mathbb{M}_2(\mathbb{R})_+$ and $r > 0$ we write X^r for the power by the continuous functional calculus (formally `mrpow X r`, defined as `cfc (fun t => t ^ r) X`), and we use (1) with $\tau = \operatorname{Tr}$: formally `kappa p X Y`, `dsq p X Y` for $d_p(X,Y)^2 = \operatorname{Tr}\big((X+Y)/2 - X\kappa_p Y\big)$, and `dd p X Y` $= \sqrt{\texttt{dsq p X Y}}$, where Mathlib's `Real.sqrt` is 0 on negative arguments (a convention that Proposition 4.8(i) makes harmless).

Lemma 2.1 (the 2×2 closed form; on paper)**.** *Let $X, Y \in \mathbb{M}_2(\mathbb{R})_+$, $p > 0$, and $P = X^{p/4}Y^{p/2}X^{p/4}$. Then $\operatorname{Tr} P = \operatorname{Tr}(X^{p/2}Y^{p/2})$ and $\det P = (\det X \det Y)^{p/2}$, so the eigenvalues $\mu_1, \mu_2 \ge 0$ of P are the roots of $t^2 - (\operatorname{Tr} P)\,t + \det P$, and*

$$\operatorname{Tr}(X\kappa_p Y) = \mu_1^{1/p} + \mu_2^{1/p}, \qquad d_p(X,Y)^2 = \frac{\operatorname{Tr} X + \operatorname{Tr} Y}{2} - \mu_1^{1/p} - \mu_2^{1/p}.$$

If $X = \psi\psi^*$ and $Y = \varphi\varphi^*$ are rank-one projections with $\langle\psi,\varphi\rangle = \cos\alpha$, then $P = \cos^2\!\alpha\ X$ and $d_p(X,Y)^2 = 1 - \cos^{2/p}\alpha$.

Proof. Cyclicity of the trace and multiplicativity of the determinant; the rank-one case is $X^r = X$, $Y^r = Y$ and $XYX = \langle\psi,\varphi\rangle^2 X$. The last formula is the one printed in [1] before Proposition 8. ☐

The lemma is the whole $n = 2$ theory and is what we used to re-derive, exactly in $\mathbb{Q}(\sqrt{2},\sqrt{17})$, every printed value of [2, Proposition 4.2] before any new value was claimed: $\operatorname{Tr}(P^2) = \frac{1}{4} + 3\sqrt{2}$, $d_{1/2}(A,B)^2 = \frac{19}{4} - 3\sqrt{2}$, $\operatorname{Tr}(Q^2) = \frac{29}{8} - \frac{\sqrt{17}}{4} + \frac{3\sqrt{2}}{4} + \frac{3\sqrt{34}}{16}$, $d_{1/2}(A,C)^2 = \frac{5}{4} + \frac{\sqrt{17}}{4} - \frac{3\sqrt{2}}{4} - \frac{3\sqrt{34}}{16}$, and $d_{1/2}(A,B)^2 - 4d_{1/2}(A,C)^2 = \frac{3\sqrt{34}}{4} - \sqrt{17} - \frac{1}{4} = 1.0829551631\ldots \cdot 10^{-4}$; the positivity of the last quantity is equivalent, after two squarings, to $1089 > 1088$ as in [2], or to $1225 > 1224$ if one squares $3\sqrt{34} - 1 > 4\sqrt{17}$ in the other order. The formal development does not use Lemma 2.1; it uses the following two statements instead.

Lemma 2.2 (a positive semidefinite root is the power; formal: `mrpow_inv_natCast`, `mrpow_natCast`)**.** *Let $n \geq 1$. If $R \in \mathbb{M}_2(\mathbb{R})_+$ and $R^n = X$, then $X^{1/n} = R$. If $P \in \mathbb{M}_2(\mathbb{R})_+$, then P^n computed by the functional calculus is the ordinary n-th power.*

Proof. With $X = R^n = \mathrm{cfc}(t \mapsto t^n)(R)$, composition of the functional calculus gives $X^{1/n} = \mathrm{cfc}\big(t \mapsto (t^n)^{1/n}\big)(R)$, and $(t^n)^{1/n} = t$ on the spectrum of R, which lies in $[0,\infty)$. The second statement is the identity $\mathrm{cfc}(t \mapsto t^n)(P) = P^n$. □

Proposition 2.3 (exact evaluation of $d_{1/k}$; formal: `kappaTrace_eq_of_roots`, `dsq_eq_of_roots`)**.** *Let $k \geq 1$ and let $X_1, Y_1 \in \mathbb{M}_2(\mathbb{R})_+$ with $X = X_1^{4k}$ and $Y = Y_1^{4k}$. Then*

$$\mathrm{Tr}\big(X\kappa_{1/k}Y\big) = \mathrm{Tr}\big((X_1Y_1^2X_1)^k\big), \qquad d_{1/k}(X,Y)^2 = \frac{\mathrm{Tr}\,X + \mathrm{Tr}\,Y}{2} - \mathrm{Tr}\big((X_1Y_1^2X_1)^k\big).$$

In particular, if X_1 and Y_1 have rational entries, $d_{1/k}(X,Y)^2$ is a rational number obtained from them by ring operations.

Proof. At $p = 1/k$ the exponents of (1) are $p/4 = 1/(4k)$, $p/2 = 1/(2k)$ and $1/p = k$. By Lemma 2.2, $X^{1/(4k)} = X_1$ and $Y^{1/(2k)} = Y_1^2$ (since $Y_1^2 \in \mathbb{M}_2(\mathbb{R})_+$ and $(Y_1^2)^{2k} = Y$); the product $X_1Y_1^2X_1$ is positive semidefinite, so its k-th power by the functional calculus is the ordinary one, again by Lemma 2.2. Take traces. □

The case $k = 1$ is the value at $p = 1$ and needs fourth roots; it is used for the control of Proposition 4.8(iv). The last step from rational squared distances to a verdict on the triangle inequality is a comparison of square roots.

Lemma 2.4 (formal: `sqrt_add_sqrt_lt`, `sqrt_le_sqrt_add`, `sqrt_le_sqrt_add_sqrt`, `sqrt_lt_two_sqrt_add`)**.** *Let $q_1, q_2, q_3 \geq 0$. Then: $4q_2 < q_1$ implies $\sqrt{q_2} + \sqrt{q_2} < \sqrt{q_1}$; $q_1 \leq 4q_2$ implies $\sqrt{q_1} \leq \sqrt{q_2} + \sqrt{q_2}$; $q_1 \leq q_2 + q_3$ implies $\sqrt{q_1} \leq \sqrt{q_2} + \sqrt{q_3}$; and $q_1 < 16q_2$ implies $\sqrt{q_1} < 2(\sqrt{q_2} + \sqrt{q_2})$.*

3. The construction

All seven triples are built from one rational reflection of $\mathbb{R}^2$. Put

$$\psi = \begin{pmatrix}1\\0\end{pmatrix}, \quad \varphi = \begin{pmatrix}4/5\\3/5\end{pmatrix}, \quad W = \begin{pmatrix}4/5 & 3/5\\3/5 & -4/5\end{pmatrix}, \qquad W^2 = I, \quad W\psi = \varphi, \tag{2}$$

so that $\gamma := \langle\psi,\varphi\rangle = 4/5$, and for $k \geq 1$ and rational $\lambda_1, \lambda_2 > 0$ let

$$A_1 = \mathrm{diag}(1, \tfrac{1}{10}), \qquad B_1 = WA_1W = \frac{1}{250}\begin{pmatrix}169 & 108\\108 & 106\end{pmatrix}, \qquad C_1 = \frac{\lambda_1+\lambda_2}{2}\,I + \frac{\lambda_1-\lambda_2}{2}\,W, \tag{3}$$

$$A = A_1^{4k}, \qquad B = B_1^{4k}, \qquad C = C_1^{4k}.$$

A_1, B_1 and C_1 are rational, symmetric and positive definite (C_1 has the eigenvalues λ_1, λ_2, on the eigenvectors of W), hence so are A, B, C, and Proposition 2.3 applies to each pair with the roots A_1, B_1, C_1.

Lemma 3.1 (symmetry; on paper)**.** *$WAW = B$ and $WCW = C$; consequently $d_p(A,C) = d_p(C,B)$ for every $p > 0$.*

Proof. $WA_1^{4k}W = (WA_1W)^{4k} = B$ since $W^2 = I$, and C_1 is a polynomial in W. The functional calculus commutes with conjugation by the orthogonal matrix W and the trace is invariant under it, so $d_p(C,B) = d_p(WCW, WAW) = d_p(C,A) = d_p(A,C)$ (the last equality because $\mathrm{Tr}(X\kappa_pY) = \mathrm{Tr}(Y\kappa_pX)$ by Lemma 2.1, or directly because $X^{p/4}Y^{p/2}X^{p/4}$ and $Y^{p/4}X^{p/2}Y^{p/4}$ have the same nonzero spectrum). □

The triangle inequality for the triple therefore holds or fails according as $d_p(A,B)^2 \le 4\,d_p(A,C)^2$ or not. The formal development does not use Lemma 3.1: it computes $d_{1/k}(A,C)^2$ and $d_{1/k}(C,B)^2$ separately and finds the same rational number.

How the parameters were chosen. The shape is dictated by two facts. A_1 interpolates between the projection $\psi\psi^*$ and the identity; the entry $1/10$ keeps the triple strictly inside the open cone while staying near the rank-one limit, where the failure is largest. C has to be of full rank: if all three vertices are rank-one projections, Proposition 8 of [1] excludes any violation at $p \le 2$, and Proposition 4.8(v) is that statement in an instance. With $x = \lambda_1^2$, $y = \lambda_2^2$, $m_\pm = (1 \pm \gamma)/2$ and $\rho = xm_+ + ym_-$, the rank-one limit ($A_1 \to \psi\psi^*$, $B_1 \to \varphi\varphi^*$) of the margin $d_{1/k}(A,B)^2 - 4d_{1/k}(A,C)^2$ is, by Lemma 2.1,

$$4\rho^k - 2x^{2k} - 2y^{2k} - 1 - \gamma^{2k}, \tag{4}$$

a two-parameter function that is cheap to maximise; the rational pairs (λ_1, λ_2) of Table 1 were read off a grid search on (4) and then confirmed by exact rational evaluation of the actual triple with $1/10$ in place of 0. (This paragraph describes the search; nothing in it is part of the proofs, which are the exact evaluations.) The last column of the table shows why the $p = 1$ control is informative: at $p = 1$ the same triples have a negative margin.

k	p	λ_1	λ_2	$q_1 = d_p(A,B)^2$	$q_2 = d_p(A,C)^2$	$q_1 - 4q_2$	margin at $p = 1$
2	1/2	19/20	13/20	0.581249324	0.117035354	0.113107907	−0.026920
3	1/3	19/20	3/4	0.729022483	0.130642250	0.206453481	−0.066550
4	1/4	1	17/20	0.824647883	0.143325951	0.251344078	−0.079506
5	1/5	1	17/20	0.886527985	0.150318911	0.285252340	−0.038769
6	1/6	1	9/10	0.926571185	0.148421372	0.332885697	−0.086561
7	1/7	1	9/10	0.952483519	0.151613532	0.346029390	−0.053162
8	1/8	1	9/10	0.969251635	0.159201882	0.332444106	−0.034553

TABLE 1. The seven triples. q_1, $q_2 = q_3 = d_p(C,B)^2$ and the margin $q_1 - 4q_2$ are exact rational numbers (Appendix A), shown here to nine decimals; the last column is the margin $d_1(A,B)^2 - 4d_1(A,C)^2$ of the same triple at $p = 1$, where it is negative. The ratio $d_p(A,B)/(d_p(A,C) + d_p(C,B)) = \sqrt{q_1/(4q_2)}$ lies between 1.11 and 1.26 for the seven rows.

4. The results

In each theorem, A, B, C are the matrices (3) for the stated k and (λ_1, λ_2), and q_1, q_2 are the rationals of Table 1, whose exact values are the literals of Appendix A. "d_p is not a metric on $\mathbb{M}_2(\mathbb{R})_{++}$" means precisely that the triangle inequality fails at some triple of positive definite matrices; formally, `exists_violation` of the block `Pk`.

Theorem 4.1 ($p = 1/2$). *Let $k = 2$ and $(\lambda_1, \lambda_2) = (19/20, 13/20)$, so that $C_1 = \begin{pmatrix} 23/25 & 9/100 \\ 9/100 & 17/25 \end{pmatrix}$ and*

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 10^{-8} \end{pmatrix}, \quad B = \begin{pmatrix} \frac{1600000009}{2500000000} & \frac{299999997}{625000000} \\ \frac{299999997}{625000000} & \frac{56250001}{156250000} \end{pmatrix}, \quad C = \begin{pmatrix} \frac{15366779809}{25600000000} & \frac{37893357}{200000000} \\ \frac{37893357}{200000000} & \frac{2432513953}{25600000000} \end{pmatrix}.$$

Then $A, B, C \in \mathbb{M}_2(\mathbb{R})_{++}$, $d_{1/2}(A,B)^2 = \frac{36328082769}{62500000000}$, $d_{1/2}(A,C)^2 = d_{1/2}(C,B)^2 = \frac{74902626729}{640000000000}$, and

$$d_{1/2}(A,C) + d_{1/2}(C,B) < d_{1/2}(A,B).$$

Hence $d_{1/2}$ is not a metric on $\mathbb{M}_2(\mathbb{R})_{++}$.

The value $p = 1/2$ is Zhang's [2, Proposition 4.2]; the theorem is a second, rational witness for it, and the control that the machinery agrees with the literature.

Theorem 4.2 ($p = 1/3$)**.** *Let $k = 3$ and $(\lambda_1, \lambda_2) = (19/20, 3/4)$, so that $C_1 = \begin{pmatrix} 93/100 & 3/50 \\ 3/50 & 77/100 \end{pmatrix}$. Then $A, B, C \in \mathbb{M}_2(\mathbb{R})_{++}$,*

$$d_{1/3}(A,B)^2 = \frac{11390976290229129}{15625000000000000}, \qquad d_{1/3}(A,C)^2 = d_{1/3}(C,B)^2 = \frac{66888832236057877}{512000000000000000},$$

and $d_{1/3}(A,C) + d_{1/3}(C,B) < d_{1/3}(A,B)$. Hence $d_{1/3}$ is not a metric on $\mathbb{M}_2(\mathbb{R})_{++}$.

Theorem 4.3 ($p = 1/4$)**.** *Let $k = 4$ and $(\lambda_1, \lambda_2) = (1, 17/20)$, so that $C_1 = \begin{pmatrix} 197/200 & 9/200 \\ 9/200 & 173/200 \end{pmatrix}$. Then $A, B, C \in \mathbb{M}_2(\mathbb{R})_{++}$,*

$$d_{1/4}(A,B)^2 = \frac{32212807938637887 53889}{390625000000000000000},$$
$$d_{1/4}(A,C)^2 = d_{1/4}(C,B)^2 = \frac{1174126193907709624 77393}{8192000000000000000000},$$

and $d_{1/4}(A,C) + d_{1/4}(C,B) < d_{1/4}(A,B)$. Hence $d_{1/4}$ is not a metric on $\mathbb{M}_2(\mathbb{R})_{++}$.

Theorem 4.4 ($p = 1/5$)**.** *Let $k = 5$ and $(\lambda_1, \lambda_2) = (1, 17/20)$ (the same C_1 as in Theorem 4.3). Then $A, B, C \in \mathbb{M}_2(\mathbb{R})_{++}$, $d_{1/5}(A,B)^2 = q_1 \approx 0.886527985$, $d_{1/5}(A,C)^2 = d_{1/5}(C,B)^2 = q_2 \approx 0.150318911$ with q_1, q_2 the rationals of Appendix A, and $d_{1/5}(A,C) + d_{1/5}(C,B) < d_{1/5}(A,B)$. Hence $d_{1/5}$ is not a metric on $\mathbb{M}_2(\mathbb{R})_{++}$.*

Theorem 4.5 ($p = 1/6$)**.** *Let $k = 6$ and $(\lambda_1, \lambda_2) = (1, 9/10)$, so that $C_1 = \begin{pmatrix} 99/100 & 3/100 \\ 3/100 & 91/100 \end{pmatrix}$. Then $A, B, C \in \mathbb{M}_2(\mathbb{R})_{++}$, $d_{1/6}(A,B)^2 = q_1 \approx 0.926571185$, $d_{1/6}(A,C)^2 = d_{1/6}(C,B)^2 = q_2 \approx 0.148421372$ with q_1, q_2 the rationals of Appendix A, and $d_{1/6}(A,C) + d_{1/6}(C,B) < d_{1/6}(A,B)$. Hence $d_{1/6}$ is not a metric on $\mathbb{M}_2(\mathbb{R})_{++}$.*

Theorem 4.6 ($p = 1/7$)**.** *Let $k = 7$ and $(\lambda_1, \lambda_2) = (1, 9/10)$. Then $A, B, C \in \mathbb{M}_2(\mathbb{R})_{++}$, $d_{1/7}(A,B)^2 = q_1 \approx 0.952483519$, $d_{1/7}(A,C)^2 = d_{1/7}(C,B)^2 = q_2 \approx 0.151613532$ with q_1, q_2 the rationals of Appendix A, and $d_{1/7}(A,C) + d_{1/7}(C,B) < d_{1/7}(A,B)$. Hence $d_{1/7}$ is not a metric on $\mathbb{M}_2(\mathbb{R})_{++}$.*

Theorem 4.7 ($p = 1/8$)**.** *Let $k = 8$ and $(\lambda_1, \lambda_2) = (1, 9/10)$. Then $A, B, C \in \mathbb{M}_2(\mathbb{R})_{++}$, $d_{1/8}(A,B)^2 = q_1 \approx 0.969251635$, $d_{1/8}(A,C)^2 = d_{1/8}(C,B)^2 = q_2 \approx 0.159201882$ with q_1, q_2 the rationals of Appendix A, and $d_{1/8}(A,C) + d_{1/8}(C,B) < d_{1/8}(A,B)$. Hence $d_{1/8}$ is not a metric on $\mathbb{M}_2(\mathbb{R})_{++}$.*

Proof of Theorems 4.1–4.7. The proofs are identical in shape and are exact finite computations; we describe the computation and say what is checked where. Positive definiteness of A_1, B_1, C_1 and of A, B, C is read off the leading minors (formal: `posDef_two`, with the minors discharged by `norm_num`). The identities $A_1^{4k} = A$, $B_1^{4k} = B$, $C_1^{4k} = C$ are verified by expanding the powers of the rational 2×2 matrices (formal: `A_pow`, `B_pow`, `C_pow`, by `norm_num` on `Matrix.mul_fin_two`). Proposition 2.3 then turns each squared distance into $(\operatorname{Tr} X + \operatorname{Tr} Y)/2 - \operatorname{Tr}((X_1 Y_1^2 X_1)^k)$, a rational expression that is evaluated exactly (formal: `dsq_AB`, `dsq_AC`, `dsq_CB`, each an equality with the rational literal of Appendix A); the two values $d_{1/k}(A,C)^2$ and $d_{1/k}(C,B)^2$ come out equal, as Lemma 3.1 predicts. Finally $4q_2 < q_1$ is a comparison of two rationals, and Lemma 2.4 gives $\sqrt{q_2} + \sqrt{q_2} < \sqrt{q_1}$, i.e. $d_{1/k}(A,C) + d_{1/k}(C,B) < d_{1/k}(A,B)$ (formal: `not_triangle`; the existential form is `exists_violation`). No search is part of any proof: the search of Section 3 produced the parameters, and the proof is the exact evaluation at those parameters. The largest literal is the 44-digit numerator of $d_{1/8}(A,B)^2$ (Appendix A), and the largest powers expanded are A_1^{32}, B_1^{32}, C_1^{32}. □

Proposition 4.8 (controls)**.** *For each $k \in \{2, \dots, 8\}$, with A, B, C the triple of the corresponding theorem and $p = 1/k$:*

(i) *(non-degeneracy) $d_p(A,B)^2 > 0$, $d_p(A,C)^2 > 0$ and $d_p(C,B)^2 > 0$;*
(ii) *(the margin) $d_p(A,B)^2 - 4\, d_p(A,C)^2 > 1/10$;*
(iii) *(the failure is bounded) $d_p(A,B) < 2\big(d_p(A,C) + d_p(C,B)\big)$;*
(iv) *(the same triple at $p = 1$) $d_1(A,B) \le d_1(A,C) + d_1(C,B)$;*

(v) *(a rank-one triple at the same p) with* $P = \psi\psi^* = \mathrm{diag}(1,0)$*,* $Q = \mathrm{diag}(0,1)$ *and* $R = \chi\chi^*$ *for* $\chi = (3/5, 4/5)$*,*
$$d_p(P,Q) \le d_p(P,R) + d_p(R,Q);$$
here $d_p(P,Q)^2 = 1$*,* $d_p(P,R)^2 = 1-(9/25)^k$ *and* $d_p(R,Q)^2 = 1-(16/25)^k$*.*

Proof. (i) and (ii) are comparisons of the rational literals (formal: `dsq_pos`, `violation_margin`). (iii) is $q_1 < 16q_2$ and the last part of Lemma 2.4 (formal: `not_triangle_bounded`). (iv) uses Proposition 2.3 with $k=1$ and the fourth roots A_1^k, B_1^k, C_1^k of A, B, C, evaluates the three squared d_1-distances exactly (formal: `dsq_one_AB`, `dsq_one_AC`, `dsq_one_CB`), finds $d_1(A,B)^2 \le 4\,d_1(A,C)^2$ — the margin is the negative last column of Table 1 — and applies the second part of Lemma 2.4 (formal: `triangle_at_one`). This is a genuine check of the machinery: it runs the same evaluation lemma with $k=1$ and must change sign exactly at the endpoint where d_1 is the Hellinger metric. (v) A projection is its own $4k$-th root, so Proposition 2.3 gives $d_p(P,R)^2 = 1-\langle\psi,\chi\rangle^{2k}$ and $d_p(R,Q)^2 = 1-\langle\chi,e_2\rangle^{2k}$ (formal: `dsq_p_AB`, `dsq_p_AC`, `dsq_p_CB`), $1 \le (1-(9/25)^k) + (1-(16/25)^k)$ is a rational inequality, and the third part of Lemma 2.4 concludes (formal: `pure_state_triangle`). This is Proposition 8 of [1] in an instance: with $\cos\alpha = 3/5$ the values are its $1-\cos^{2/p}\alpha$ and $1-\sin^{2/p}\alpha$. □

5. Remarks

Remark 5.1 (transfer to $\mathbb{M}_n(\mathbb{C})_+$ and to C^*-algebras; not formalised). Every triple above is real symmetric, hence Hermitian, and positive definite, hence in $\mathbb{M}_2(\mathbb{C})_{++} \subset \mathbb{M}_2(\mathbb{C})_+$, the cones of [2] and of Problem 2. For $n \ge 3$, padding the three matrices by a common positive definite block D gives $d_p(A\oplus D, B\oplus D) = d_p(A,B)$ because κ_p and the trace respect block sums, so the same triple violates the triangle inequality in $\mathbb{M}_n(\mathbb{C})_{++}$ for every $n \ge 2$. Neither the passage from $\mathbb{R}$ to $\mathbb{C}$ nor the padding is part of the formal development. By the remark of [1] quoted in the introduction, each value $p = 1/k$ settled here would also give, "using the method presented in the proof of Proposition 2" of that paper, that d_p^τ is a metric on the positive cone of a C^*-algebra with a faithful trace only if the algebra is commutative; we quote the remark and do not reproduce its argument.

Remark 5.2 (why Zhang's triple does not generalise). Zhang's C is M_0^2 with $M_0 = (\sqrt{A}+\sqrt{B})/2 = \begin{pmatrix} 7/4 & 1/4 \\ 1/4 & 5/4 \end{pmatrix}$, the midpoint of A and B for the Hellinger distance $\sqrt{2}\,d_1$. His triple is not of the form of Section 3: already $A^{1/8} = \mathrm{diag}(2^{1/4}, 1)$ is irrational. His exactness at $p = 1/2$ comes instead from Lemma 2.1: with $C^{1/4} = M_0^{1/2}$ made explicit by his Lemma 4.1, the traces $\mathrm{Tr}(A^{1/4}B^{1/4})$, $\mathrm{Tr}(A^{1/4}C^{1/4})$ and the determinants land in $\mathbb{Q}(\sqrt{2},\sqrt{17})$, where the final comparison is decided. At $p = 1/k$ the same route needs $A^{1/(2k)} = \mathrm{diag}(2^{1/k}, 1)$ and $M_0^{1/k}$, and two exact statements block it. First, $x^k - 2$ is irreducible over $\mathbb{Q}$ by Eisenstein at 2, so $[\mathbb{Q}(2^{1/k}) : \mathbb{Q}] = k$: the entries of $A^{1/(2k)}$ generate a field of degree exactly k. Second, M_0 has no k-th root with entries in $\mathbb{Q}(\sqrt{2})$ for any $k \ge 2$: M_0 has the eigenvalues $\alpha = (6+\sqrt{2})/4$ and $\bar\alpha = (6-\sqrt{2})/4$ on eigenvectors $(1, \sqrt{2}-1)$ and $(1, -\sqrt{2}-1)$ already defined over $\mathbb{Q}(\sqrt{2})$, so such a root would have $\beta \in \mathbb{Q}(\sqrt{2})$ with $\beta^k = \alpha$; applying the nontrivial automorphism of $\mathbb{Q}(\sqrt{2})$ and multiplying, $N(\beta)^k = \alpha\bar\alpha = 17/8$ with $N(\beta) \in \mathbb{Q}$, while $17/8 = 17/2^3$ is not a rational k-th power for $k \ge 2$. So the example does not extend as an exact computation in one small field, and each new p needs a triple of the form of Section 3, for which every root is rational. At $p = 1/2$ his margin is $1.08 \cdot 10^{-4}$ and ours 0.113; the two constructions are different in kind, his a near-degenerate perturbation of the d_1-geodesic and ours a search-optimised triple. Evaluating his triple at other p (Lemma 2.1 at 90 significant digits) gives a positive margin $d_p(A,B)^2 - 4d_p(A,C)^2$ for every p on a grid in $(0,1)$, from $1.17 \cdot 10^{-4}$ at $p = 10^{-6}$ to $4.4 \cdot 10^{-10}$ at $p = 1 - 10^{-6}$, a value indistinguishable from 0 at $p = 1$, and a negative margin at every tested $p > 1$ ($-1.39 \cdot 10^{-3}$ at $p = 2$); this is a computation outside the formal development, and the exactness of $1089 > 1088$ is not available away from $p = 1/2$.

Remark 5.3 (a reduction of the whole range $0 < p < 1$; not proved). Take $A = \psi\psi^*$, $B = \varphi\varphi^*$ rank-one projections with $\gamma = \langle\psi,\varphi\rangle \in (0,1)$, and $C = M^2$ with $M = (A+B)/2$ — the d_1 midpoint, since $\sqrt{A} = A$ and $\sqrt{B} = B$. M has eigenvalues $m_\pm = (1 \pm \gamma)/2$ on $w_\pm = (\psi \pm \varphi)/\|\psi \pm \varphi\|$, and $\langle \psi, w_\pm \rangle^2 = m_\pm$, so $\langle \psi, M^p \psi \rangle = m_+^{p+1} + m_-^{p+1}$ and Lemma 2.1 gives, for every $p > 0$,

$$d_p(A,B)^2 = 1 - \gamma^{2/p}, \qquad d_p(A,C)^2 = d_p(C,B)^2 = \frac{3+\gamma^2}{4} - \big(m_+^{p+1} + m_-^{p+1}\big)^{1/p}.$$

(The first is the pure-state formula of [1]; the second is elementary.) Hence the triangle inequality fails for this triple at p if and only if

$$4\big(m_+^{p+1} + m_-^{p+1}\big)^{1/p} > 2 + \gamma^2 + \gamma^{2/p}, \qquad m_\pm = \tfrac{1\pm\gamma}{2}. \tag{5}$$

At $p = 1$, $m_+^2 + m_-^2 = (1+\gamma^2)/2$ makes (5) an equality for every γ — d_1 is a metric and C lies on its geodesic. If (5) holds for every $p \in (0,1)$ and some $\gamma \in (0,1)$, then d_p is not a metric on $\mathbb{M}_2(\mathbb{C})_+$ for every $p \in (0,1)$, and by Remark 5.1 the whole negative half of Problem 2 follows. We have not proved (5). Numerically (`decimal` arithmetic at 80 significant digits, re-run for this paper) its left side minus its right side is positive at every one of the 21×14 grid points with p at the values 10^{-6}, 10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}, 0.05, 0.1, 0.2, ..., 0.9, 0.95, 0.99, 0.999, 0.9999, 0.99999, 0.999999 and γ at the values 10^{-6}, 10^{-4}, 10^{-2}, 0.1, 0.25, 0.4, 0.5, 0.6, $1/\sqrt{2}$, 0.8, 0.9, 0.99, 0.9999, 0.999999, the smallest value being $1.0 \cdot 10^{-18}$ at $p = \gamma = 10^{-6}$; at $\gamma = 0.6$ the difference divided by $1-p$ tends to $0.14755\ldots$ as $p \to 1^-$, so the margin is of order $1-p$ and no margin uniform in p exists, which is why no finite certificate can cover an interval of p; and as $p \to 0^+$ the difference tends to $2(1+\gamma)^{(1+\gamma)/2}(1-\gamma)^{(1-\gamma)/2} - 2 - \gamma^2$. That limit is positive for every $\gamma \in (0,1)$, on paper: with $h(\gamma) = \frac{1}{2}\big[(1+\gamma)\log(1+\gamma) + (1-\gamma)\log(1-\gamma)\big] = \sum_{n\ge1} \gamma^{2n}/(2n(2n-1)) = \gamma^2/2 + \gamma^4/12 + \cdots$ one has $2e^h \ge 2(1 + h + h^2/2) > 2 + \gamma^2 + \gamma^4(\frac{1}{6} + \frac{1}{4}) > 2 + \gamma^2$. The grid values and the asymptotics are computation outside the formal development, and none of this is a proof of (5). Note that this C is positive definite but A and B are singular; a proof of (5) would settle the question on $\mathbb{M}_n(\mathbb{C})_+$, and, the inequality being strict and d_p continuous on the closed cone, also on $\mathbb{M}_n(\mathbb{C})_{++}$.

Remark 5.4 (no single triple of the shape (3) serves every k; on paper). In the rank-one limit the margin is (4). If $\max(x,y) \le 1$ then $\rho \le 1$ and, as $k \to \infty$, the margin tends to a negative limit (-1 if $\rho < 1$ and $x, y < 1$ or $x = y = 1$, and -3 if exactly one of x, y equals 1); if $\max(x,y) > 1$ then $\max(x,y)^2 > \max(x,y) \ge \rho$ and the margin tends to $-\infty$. So, in the rank-one limit at least, the k-th counterexample genuinely needs its own (λ_1, λ_2), as Table 1 shows, and a statement quantified over all k would need a k-dependent family such as $\lambda_i^2 = 1 - a_i/k$, whose limit margin at the best a_i is $2m_+^{m_+} m_-^{m_-} - 1 > 0$ and whose proof would need bounds of $(1 - a/k)^k$ against e^{-a}; this is not done here.

Remark 5.5 (what remains). (a) The positive half of Problem 2, that d_p *is* a metric for $1 < p < 2$: nothing in this paper bears on it, and Remark 4.3 of [2] leaves it open. (b) Rational $p = j/k$ with $j \ge 2$: the same exactness holds whenever the j-th root of $X_1^j Y_1^{2j} X_1^j$ is rational; for $j = 2$ this is Zhang's Lemma 4.1 applied three times, i.e. $\det P$ a rational square and $\operatorname{Tr} P + 2\sqrt{\det P}$ a rational square simultaneously for the three pairs, a Diophantine search we did not run. (c) Values $p = 1/k$ with $k \ge 9$: the method applies verbatim, with literals growing linearly in k.

6. Verification

Lemma 2.2, Proposition 2.3, Lemma 2.4, Theorems 4.1–4.7 and Proposition 4.8 rest on declarations formally verified in Lean 4 against *Mathlib* [14, 15] (Lean 4.32.0, Mathlib at its `v4.32.0` tag), whose statements are reproduced verbatim in Appendix A. The development is two modules, 1494 lines in all. The first (185 lines; 1 abbreviation, 4 definitions, 11 theorems) defines X^r through Mathlib's continuous functional calculus `cfc` on real 2×2 matrices, $X \kappa_p Y$, $d_p(X,Y)^2$ and $d_p(X,Y)$ exactly as in (1), and proves Lemma 2.2, Proposition 2.3, the leading-minor criteria for positive (semi)definiteness of an explicit symmetric 2×2 matrix, and Lemma 2.4. The second (1309 lines; 63 definitions, 259 theorems) is generated by a script from the seven parameter pairs, because its literals run to 44 digits, and consists of seven blocks, one per $p = 1/k$; each block defines the roots A_1, B_1, C_1, the vertices A, B, C and the rank-one triple, proves positivity, the identities $A_1^{4k} = A$ etc., the nine exact squared distances (at $p = 1/k$, at $p = 1$, and for the rank-one triple), and the seven statements bound to the results of this paper: `not_triangle`, `exists_violation` (Theorems 4.1–4.7) and `dsq_pos`, `violation_margin`, `not_triangle_bounded`, `triangle_at_one`, `pure_state_triangle` (Proposition 4.8(i)–(v)). Every proof is checked by the Lean kernel; there is no compiled evaluation (`native_decide`), no `sorry`, and no axiom declared by hand; all rational identities and comparisons are discharged by `norm_num`. The axiom print (`#print axioms`) of each of the 49 bound theorems, re-run for this paper on the compiled modules, is `propext`, `Classical.choice`, `Quot.sound`, the baseline of any Mathlib development using the functional calculus. Outside the formal development, and labelled as such where they occur, are Lemma 2.1, Lemma 3.1, Remark 5.1, the search of Section 3, and the computations of Remarks 5.2 and 5.3; the exact re-derivation of Zhang's values after Lemma 2.1 and every number in Table 1 were recomputed independently for this paper in exact rational (respectively $\mathbb{Q}(\sqrt{2}, \sqrt{17})$) arithmetic. As recorded when the development was built, checking the second module took about 211 processor-seconds (47 seconds of wall time) with a peak resident set of about 7.9 GB, and the first about 8 seconds; the rational computations outside Lean took under one processor-minute. The development is currently in a private repository: repository reference to be added at submission.

Appendix A. The formal statements

The Lean statements corresponding to the results above, verbatim (statements only; proofs, docstrings and the binding attributes omitted). `Matrix (Fin 2) (Fin 2) ℝ` is Mathlib's type of real 2×2 matrices, `!![a, b; c, d]` the matrix with rows (a, b) and (c, d), `cfc f X` the continuous functional calculus of `X` applied to the real function `f`, `X.PosDef` and `X.PosSemidef` positive (semi)definiteness of a matrix, `Matrix.trace` the trace, `Real.sqrt` the square root on $\mathbb{R}$ (which is 0 on negative arguments), and `t ^ r` the real power t^r of `Real.rpow`. Everything lives in the namespace `KmpDpMetric`; the seven blocks are its sub-namespaces `P2`, …, `P8`, with `Pk` carrying the value $p = 1/k$. Long numerals are broken across lines by the typesetting only; each is a single literal in the source.

Definitions (module `Defs`).

```
abbrev M2 := Matrix (Fin 2) (Fin 2) ℝ

noncomputable def mrpow (X : M2) (r : ℝ) : M2 := cfc (fun t : ℝ => t ^ r) X

noncomputable def kappa (p : ℝ) (X Y : M2) : M2 :=
  mrpow (mrpow X (p / 4) * mrpow Y (p / 2) * mrpow X (p / 4)) (1 / p)

noncomputable def dsq (p : ℝ) (X Y : M2) : ℝ :=
  Matrix.trace ((2 : ℝ)⁻¹ • (X + Y) - kappa p X Y)

noncomputable def dd (p : ℝ) (X Y : M2) : ℝ := Real.sqrt (dsq p X Y)
```

The root lemma and the exact evaluation (Lemma 2.2, Proposition 2.3).

```
theorem mrpow_inv_natCast {n : ℕ} (hn : n ≠ 0) {X R : M2} (hR : R.PosSemidef)
    (h : R ^ n = X) : mrpow X ((n : ℝ)⁻¹) = R

theorem mrpow_natCast {P : M2} (hP : P.PosSemidef) (n : ℕ) : mrpow P ((n : ℝ)) = P ^ n

theorem kappaTrace_eq_of_roots {k : ℕ} (hk : k ≠ 0) {X Y X₁ Y₁ : M2}
    (hX₁ : X₁.PosSemidef) (hXp : X₁ ^ (4 * k) = X)
    (hY₁ : Y₁.PosSemidef) (hYp : Y₁ ^ (4 * k) = Y) :
    Matrix.trace (kappa (1 / (k : ℝ)) X Y) = Matrix.trace ((X₁ * Y₁ ^ 2 * X₁) ^ k)

theorem dsq_eq_of_roots {k : ℕ} (hk : k ≠ 0) {X Y X₁ Y₁ : M2}
    (hX₁ : X₁.PosSemidef) (hXp : X₁ ^ (4 * k) = X)
    (hY₁ : Y₁.PosSemidef) (hYp : Y₁ ^ (4 * k) = Y) :
    dsq (1 / (k : ℝ)) X Y
      = (Matrix.trace X + Matrix.trace Y) / 2 - Matrix.trace ((X₁ * Y₁ ^ 2 * X₁) ^ k)
```

Positivity of explicit 2×2 matrices and the square-root comparisons (Lemma 2.4).

```
theorem posDef_two {a b d : ℝ} (ha : 0 < a) (hdet : b * b < a * d) :
    (!![a, b; b, d] : M2).PosDef

theorem posSemidef_two {a b d : ℝ} (ha : 0 ≤ a) (hd : 0 ≤ d) (hdet : b * b ≤ a * d) :
    (!![a, b; b, d] : M2).PosSemidef

theorem sqrt_add_sqrt_lt {q₁ q₂ : ℝ} (h2 : 0 ≤ q₂) (h : 4 * q₂ < q₁) :
    Real.sqrt q₂ + Real.sqrt q₂ < Real.sqrt q₁

theorem sqrt_le_sqrt_add {q₁ q₂ : ℝ} (h2 : 0 ≤ q₂) (h : q₁ ≤ 4 * q₂) :
    Real.sqrt q₁ ≤ Real.sqrt q₂ + Real.sqrt q₂

theorem sqrt_le_sqrt_add_sqrt {q₁ q₂ q₃ : ℝ} (h2 : 0 ≤ q₂) (h3 : 0 ≤ q₃) (h : q₁ ≤ q₂ + q₃) :
    Real.sqrt q₁ ≤ Real.sqrt q₂ + Real.sqrt q₃

theorem sqrt_lt_two_sqrt_add {q₁ q₂ : ℝ} (h1 : 0 ≤ q₁) (h2 : 0 ≤ q₂) (h : q₁ < 16 * q₂) :
    Real.sqrt q₁ < 2 * (Real.sqrt q₂ + Real.sqrt q₂)
```

The block `P2`: $p = 1/2$ (Theorem 4.1, Proposition 4.8). The roots, the vertices, and the rank-one triple of the control:

```
noncomputable def A1 : M2 := !![1, 0; 0, (1/10)]

noncomputable def B1 : M2 := !![(169/250), (54/125); (54/125), (53/125)]

noncomputable def C1 : M2 := !![(23/25), (9/100); (9/100), (17/25)]

noncomputable def A : M2 := !![1, 0; 0, (1/100000000)]

noncomputable def B : M2 := !![(1600000009/2500000000), (299999997/625000000); (299999997/625000000),
    ↪ (56250001/156250000)]

noncomputable def C : M2 := !![(15366779809/25600000000), (37893357/200000000); (37893357/200000000),
    ↪ (2432513953/25600000000)]

noncomputable def Ap : M2 := !![1, 0; 0, 0]

noncomputable def Bp : M2 := !![0, 0; 0, 1]

noncomputable def Cp : M2 := !![9/25, 12/25; 12/25, 16/25]
```

The certificates (positivity, the 8-th powers, the exact squared distances at $p = 1/2$, at $p = 1$, and for the rank-one triple):

```
theorem A_pd : A.PosDef

theorem B_pd : B.PosDef

theorem C_pd : C.PosDef

theorem A_pow : A1 ^ (4 * 2) = A

theorem B_pow : B1 ^ (4 * 2) = B

theorem C_pow : C1 ^ (4 * 2) = C

theorem dsq_AB : dsq (1 / 2 : ℝ) A B = (36328082769/62500000000)

theorem dsq_AC : dsq (1 / 2 : ℝ) A C = (74902626729/640000000000)
```

```
theorem dsq_CB : dsq (1 / 2 : ℝ) C B = (74902626729/640000000000)

theorem dsq_one_AB : dsq (1 : ℝ) A B = (899820009/2500000000)

theorem dsq_one_AC : dsq (1 : ℝ) A C = (2475827217/25600000000)

theorem dsq_one_CB : dsq (1 : ℝ) C B = (2475827217/25600000000)

theorem dsq_p_AB : dsq (1 / 2 : ℝ) Ap Bp = 1

theorem dsq_p_AC : dsq (1 / 2 : ℝ) Ap Cp = (544/625)

theorem dsq_p_CB : dsq (1 / 2 : ℝ) Cp Bp = (369/625)
```

The seven bound statements:

```
theorem not_triangle : dd (1 / 2 : ℝ) A C + dd (1 / 2 : ℝ) C B < dd (1 / 2 : ℝ) A B

theorem exists_violation : ∃ X Y Z : M2, X.PosDef ∧ Y.PosDef ∧ Z.PosDef ∧
    dd (1 / 2 : ℝ) X Z + dd (1 / 2 : ℝ) Z Y < dd (1 / 2 : ℝ) X Y

theorem dsq_pos : 0 < dsq (1 / 2 : ℝ) A B ∧ 0 < dsq (1 / 2 : ℝ) A C ∧
    0 < dsq (1 / 2 : ℝ) C B

theorem violation_margin :
    (1 : ℝ) / 10 < dsq (1 / 2 : ℝ) A B - 4 * dsq (1 / 2 : ℝ) A C

theorem not_triangle_bounded :
    dd (1 / 2 : ℝ) A B < 2 * (dd (1 / 2 : ℝ) A C + dd (1 / 2 : ℝ) C B)

theorem triangle_at_one : dd (1 : ℝ) A B ≤ dd (1 : ℝ) A C + dd (1 : ℝ) C B

theorem pure_state_triangle :
    dd (1 / 2 : ℝ) Ap Bp ≤ dd (1 / 2 : ℝ) Ap Cp + dd (1 / 2 : ℝ) Cp Bp
```

The block `P3`: $p = 1/3$ (Theorem 4.2, Proposition 4.8). The seven bound statements and the certificates are word for word those of `P2` with `(1 / 2 : ℝ)` replaced by `(1 / 3 : ℝ)` and `(4 * 2)` by `(4 * 3)`; the rank-one triple `Ap`, `Bp`, `Cp` is the same. What changes is the data:

```
noncomputable def A1 : M2 := !![1, 0; 0, (1/10)]

noncomputable def B1 : M2 := !![(169/250), (54/125); (54/125), (53/125)]

noncomputable def C1 : M2 := !![(93/100), (3/50); (3/50), (77/100)]

noncomputable def A : M2 := !![1, 0; 0, (1/1000000000000)]

noncomputable def B : M2 := !![(16000000000009/25000000000000), (2999999999997/6250000000000);
    ↪ (2999999999997/6250000000000), (562500000001/1562500000000)]

noncomputable def C : M2 := !![(10024790304743037/20480000000000000), (390669108970413/2560000000000000);
    ↪ (390669108970413/2560000000000000), (1690515980040893/20480000000000000)]

theorem dsq_AB : dsq (1 / 3 : ℝ) A B = (11390976290229129/15625000000000000)

theorem dsq_AC : dsq (1 / 3 : ℝ) A C = (66888832236057877/512000000000000000)

theorem dsq_CB : dsq (1 / 3 : ℝ) C B = (66888832236057877/512000000000000000)

theorem dsq_one_AB : dsq (1 : ℝ) A B = (8999982000009/25000000000000)
```

```
theorem dsq_one_AC : dsq (1 : ℝ) A C = (2183934628434013/20480000000000000)

theorem dsq_one_CB : dsq (1 : ℝ) C B = (2183934628434013/20480000000000000)

theorem dsq_p_AB : dsq (1 / 3 : ℝ) Ap Bp = 1

theorem dsq_p_AC : dsq (1 / 3 : ℝ) Ap Cp = (14896/15625)

theorem dsq_p_CB : dsq (1 / 3 : ℝ) Cp Bp = (11529/15625)
```

The block `P4`: $p = 1/4$ (Theorem 4.3, Proposition 4.8). The seven bound statements and the certificates are word for word those of `P2` with `(1 / 2 : ℝ)` replaced by `(1 / 4 : ℝ)` and `(4 * 2)` by `(4 * 4)`; the rank-one triple `Ap`, `Bp`, `Cp` is the same. What changes is the data:

```
noncomputable def A1 : M2 := !![1, 0; 0, (1/10)]

noncomputable def B1 : M2 := !![(169/250), (54/125); (54/125), (53/125)]

noncomputable def C1 : M2 := !![(197/200), (9/200); (9/200), (173/200)]

noncomputable def A : M2 := !![1, 0; 0, (1/10000000000000000)]

noncomputable def B : M2 := !![(160000000000000009/250000000000000000), (29999999999999997/62500000000000000);
    ↪ (29999999999999997/62500000000000000), (5625000000000001/15625000000000000)]

noncomputable def C : M2 := !![(5946901191875666868481/65536000000000000000000),
    ↪ (1820096424372999394557/65536000000000000000000); (1820096424372999394557/65536000000000000000000),
    ↪ (1093310726881001816329/65536000000000000000000)]

theorem dsq_AB : dsq (1 / 4 : ℝ) A B = (3221280793863788753889/3906250000000000000000)

theorem dsq_AC : dsq (1 / 4 : ℝ) A C = (11741261939077096247739 3/81920000000000000000000)

theorem dsq_CB : dsq (1 / 4 : ℝ) C B = (117412619390770962477393/819200000000000000000000)

theorem dsq_one_AB : dsq (1 : ℝ) A B = (89999998200000009/250000000000000000)

theorem dsq_one_AC : dsq (1 : ℝ) A C = (7200865462629895260 21/65536000000000000000000)

theorem dsq_one_CB : dsq (1 : ℝ) C B = (720086546262989526021/65536000000000000000000)

theorem dsq_p_AB : dsq (1 / 4 : ℝ) Ap Bp = 1

theorem dsq_p_AC : dsq (1 / 4 : ℝ) Ap Cp = (384064/390625)

theorem dsq_p_CB : dsq (1 / 4 : ℝ) Cp Bp = (325089/390625)
```

The block `P5`: $p = 1/5$ (Theorem 4.4, Proposition 4.8). The seven bound statements and the certificates are word for word those of `P2` with `(1 / 2 : ℝ)` replaced by `(1 / 5 : ℝ)` and `(4 * 2)` by `(4 * 5)`; the rank-one triple `Ap`, `Bp`, `Cp` is the same. What changes is the data:

```
noncomputable def A1 : M2 := !![1, 0; 0, (1/10)]

noncomputable def B1 : M2 := !![(169/250), (54/125); (54/125), (53/125)]

noncomputable def C1 : M2 := !![(197/200), (9/200); (9/200), (173/200)]
```

```
noncomputable def A : M2 := !![1, 0; 0, (1/100000000000000000000)]

noncomputable def B : M2 := !![(16000000000000000009/25000000000000000000),
    ↪ (2999999999999999997/6250000000000000000); (2999999999999999997/6250000000000000000),
    ↪ (5625000000000000001/15625000000000000000)]

noncomputable def C : M2 := !![(947782631406647572522401601/1048576000000000000000000),
    ↪ (302380105780057282432795197/1048576000000000000000000);
    ↪ (302380105780057282432795197/1048576000000000000000000),
    ↪ (141435682659828152701614409/1048576000000000000000000)]

theorem dsq_AB : dsq (1 / 5 : ℝ) A B = (865749985588048943399693049/976562500000000000000000)

theorem dsq_AC : dsq (1 / 5 : ℝ) A C = (98513001748164579965025995061/655360000000000000000000000)

theorem dsq_CB : dsq (1 / 5 : ℝ) C B = (98513001748164579965025995061/655360000000000000000000000)

theorem dsq_one_AB : dsq (1 : ℝ) A B = (899999999820000000009/25000000000000000000)

theorem dsq_one_AC : dsq (1 : ℝ) A C = (104534979463574942830712901/1048576000000000000000000)

theorem dsq_one_CB : dsq (1 : ℝ) C B = (104534979463574942830712901/1048576000000000000000000)

theorem dsq_p_AB : dsq (1 / 5 : ℝ) Ap Bp = 1

theorem dsq_p_AC : dsq (1 / 5 : ℝ) Ap Cp = (9706576/9765625)

theorem dsq_p_CB : dsq (1 / 5 : ℝ) Cp Bp = (8717049/9765625)
```

The block `P6`: $p = 1/6$ (Theorem 4.5, Proposition 4.8). The seven bound statements and the certificates are word for word those of `P2` with `(1 / 2 : ℝ)` replaced by `(1 / 6 : ℝ)` and `(4 * 2)` by `(4 * 6)`; the rank-one triple `Ap`, `Bp`, `Cp` is the same. What changes is the data:

```
noncomputable def A1 : M2 := !![1, 0; 0, (1/10)]

noncomputable def B1 : M2 := !![(169/250), (54/125); (54/125), (53/125)]

noncomputable def C1 : M2 := !![(99/100), (3/100); (3/100), (91/100)]

noncomputable def A : M2 := !![1, 0; 0, (1/1000000000000000000000000)]

noncomputable def B : M2 := !![(1600000000000000000000009/2500000000000000000000000),
    ↪ (2999999999999999999999997/6250000000000000000000000); (2999999999999999999999997/6250000000000000000000000),
    ↪  (5625000000000000000000001/15625000000000000000000000)]

noncomputable def C : M2 := !![(9079766443076872509863361/1000000000000000000000000),
    ↪ (2760700670769382470409917/1000000000000000000000000);
    ↪ (2760700670769382470409917/1000000000000000000000000),
    ↪ (1717897987691852588770249/1000000000000000000000000)]

theorem dsq_AB : dsq (1 / 6 : ℝ) A B = (226213668225454079146314340182609/244140625000000000000000000000)

theorem dsq_AC : dsq (1 / 6 : ℝ) A C = (148421372129638098674566210079/100000000000000000000000000000)

theorem dsq_CB : dsq (1 / 6 : ℝ) C B = (148421372129638098674566210079/100000000000000000000000000000)

theorem dsq_one_AB : dsq (1 : ℝ) A B = (89999999999982000000000009/25000000000000000000000000)
```

```
theorem dsq_one_AC : dsq (1 : ℝ) A C = (111640267889982068348481/1000000000000000000000000)

theorem dsq_one_CB : dsq (1 : ℝ) C B = (111640267889982068348481/1000000000000000000000000)

theorem dsq_p_AB : dsq (1 / 6 : ℝ) Ap Bp = 1

theorem dsq_p_AC : dsq (1 / 6 : ℝ) Ap Cp = (243609184/244140625)

theorem dsq_p_CB : dsq (1 / 6 : ℝ) Cp Bp = (227363409/244140625)
```

The block `P7`**:** $p = 1/7$ **(Theorem 4.6, Proposition 4.8).** The seven bound statements and the certificates are word for word those of `P2` with `(1 / 2 : ℝ)` replaced by `(1 / 7 : ℝ)` and `(4 * 2)` by `(4 * 7)`; the rank-one triple `Ap`, `Bp`, `Cp` is the same. What changes is the data:

```
noncomputable def A1 : M2 := !![1, 0; 0, (1/10)]

noncomputable def B1 : M2 := !![(169/250), (54/125); (54/125), (53/125)]

noncomputable def C1 : M2 := !![(99/100), (3/100); (3/100), (91/100)]

noncomputable def A : M2 := !![1, 0; 0, (1/1000000000000000000000000000000)]

noncomputable def B : M2 := !![(1600000000000000000000000000009/2500000000000000000000000000000),
    ↪ (2999999999999999999999999999997/6250000000000000000000000000000);
    ↪ (2999999999999999999999999999997/6250000000000000000000000000000),
    ↪ (5625000000000000000000000000001/15625000000000000000000000000000)]

noncomputable def C : M2 := !![(90523347633027360537213511521/1000000000000000000000000000000),
    ↪ (28429957100917918388359465437/1000000000000000000000000000000);
    ↪ (28429957100917918388359465437/1000000000000000000000000000000),
    ↪ (14710128697246244834921603689/1000000000000000000000000000000)]

theorem dsq_AB : dsq (1 / 7 : ℝ) A B = (58134980390785151965243774724833198569/61035156250000000000000000000000000000)

theorem dsq_AC : dsq (1 / 7 : ℝ) A C = (15161353221640025770359566126905391/1000000000000000000000000000000000000)

theorem dsq_CB : dsq (1 / 7 : ℝ) C B = (15161353221640025770359566126905391/1000000000000000000000000000000000000)

theorem dsq_one_AB : dsq (1 : ℝ) A B = (899999999999998200000000000009/2500000000000000000000000000000)

theorem dsq_one_AC : dsq (1 : ℝ) A C = (10329058919640396794935462961/1000000000000000000000000000000)

theorem dsq_one_CB : dsq (1 : ℝ) C B = (10329058919640396794935462961/1000000000000000000000000000000)

theorem dsq_p_AB : dsq (1 / 7 : ℝ) Ap Bp = 1

theorem dsq_p_AC : dsq (1 / 7 : ℝ) Ap Cp = (6098732656/6103515625)

theorem dsq_p_CB : dsq (1 / 7 : ℝ) Cp Bp = (5835080169/6103515625)
```

The block `P8`**:** $p = 1/8$ **(Theorem 4.7, Proposition 4.8).** The seven bound statements and the certificates are word for word those of `P2` with `(1 / 2 : ℝ)` replaced by `(1 / 8 : ℝ)` and `(4 * 2)` by `(4 * 8)`; the rank-one triple `Ap`, `Bp`, `Cp` is the same. What changes is the data:

```
noncomputable def A1 : M2 := !![1, 0; 0, (1/10)]

noncomputable def B1 : M2 := !![(169/250), (54/125); (54/125), (53/125)]
```

```
noncomputable def C1 : M2 := !![(99/100), (3/100); (3/100), (91/100)]

noncomputable def A : M2 := !![1, 0; 0, (1/1000000000000000000000000000000000)]

noncomputable def B : M2 := !![(1600000000000000000000000000000009/25000000000000000000000000000000000),
    ↪ (2999999999999999999999999999999997/6250000000000000000000000000000000);
    ↪ (2999999999999999999999999999999997/6250000000000000000000000000000000),
    ↪ (562500000000000000000000000000001/1562500000000000000000000000000000)]

noncomputable def C : M2 := !![(903433683820292512484657849089281/1000000000000000000000000000000000),
    ↪ (289698948539122462546026452732157/1000000000000000000000000000000000);
    ↪ (289698948539122462546026452732157/1000000000000000000000000000000000),
    ↪ (130903154382632612361920641803529/1000000000000000000000000000000000)]

theorem dsq_AB : dsq (1 / 8 : ℝ) A B =
    ↪ (14789606249145131062413222186785852563156929/15258789062500000000000000000000000000000000)

theorem dsq_AC : dsq (1 / 8 : ℝ) A C =
    ↪ (1592018823741545631395642863016634426239/10000000000000000000000000000000000000000)

theorem dsq_CB : dsq (1 / 8 : ℝ) C B =
    ↪ (1592018823741545631395642863016634426239/10000000000000000000000000000000000000000)

theorem dsq_one_AB : dsq (1 : ℝ) A B = (899999999999999982000000000000009/2500000000000000000000000000000000)

theorem dsq_one_AC : dsq (1 : ℝ) A C = (98638217212944125746107545779841/1000000000000000000000000000000000)

theorem dsq_one_CB : dsq (1 : ℝ) C B = (98638217212944125746107545779841/1000000000000000000000000000000000)

theorem dsq_p_AB : dsq (1 / 8 : ℝ) Ap Bp = 1

theorem dsq_p_AC : dsq (1 / 8 : ℝ) Ap Cp = (152544843904/152587890625)

theorem dsq_p_CB : dsq (1 / 8 : ℝ) Cp Bp = (148292923329/152587890625)
```

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