EXACT FIBERS OF THE KELLER COUNTEREXAMPLES F_4, F_5 AND F_6

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Abstract. The Jacobian conjecture was refuted in dimension three in July 2026, and a general construction of Gao — sweeping tangent direction fields on parametrised hypersurfaces — has since produced Keller counterexamples in every dimension greater than two. Four of those maps, $F_4, \ldots, F_7$ in dimensions four and five, have generic fibers of 5, 10, 6 and 12 points, and their complete fiber stratifications are deferred there to a later version, the relevant Gröbner bases being out of reach. We determine several fibers of these maps exactly and constrain every fiber of two of them. The degree-ten fiber polynomial of F_5 has three leading coefficients that do not depend on the target; they force $\sum_i (r_i - \frac{1}{4})^2 = 0$ on any fiber whose roots are all real, so at most one real target can admit such a fiber, and that target is rational and explicit. Over it the fiber of F_5 turns out to be *empty*: every solution of the eliminated fiber system carries $\gamma = 0$ and is deleted by the monomial twist. The target is therefore an explicit rational point off the image of F_5, certified by a resultant Bézout identity $\mathcal{U}\mathcal{A} + \mathcal{V}\mathcal{B} = -750\,(4w_1 - 1)^{10}$ valid over every field of characteristic zero. For F_6 the same mechanism gives $\sum_i r_i^2 = 1$ and $\sum_i r_i^3 = \frac{5}{2}$ for every target, which are incompatible over $\mathbb{R}$: the fiber sextic never splits over the reals, so at most four of the six generic preimages are ever real. Four is attained — we exhibit a rational target of F_6 with four distinct rational preimages and prove that its fiber has exactly six points, the condition $\gamma \neq 0$ at every root being verified exactly rather than numerically. Finally we exhibit a rational target of F_4 with five distinct rational preimages, so that the geometric degree five is attained over $\mathbb{Q}$. The certificates behind these statements — the fiber evaluations, the Bézout identity and the two reality arguments — are machine-checked in Lean 4.

1. Introduction

Let $F\colon \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map. If F has a polynomial inverse then the chain rule forces $\det J(F)$ to be a nonzero constant; a map with $\det J(F) \in \mathbb{C}^\times$ is called a *Keller map*, and the Jacobian conjecture [7] asserts that every Keller map is a polynomial automorphism. The problem became widely known through [2] and [3].

On 19 July 2026 the conjecture was refuted in dimension three by an explicit counterexample announced by Alpöge [1]; the next day Gallagher [4] gave a one-variable construction producing, for every integer $d \geq 3$, a three-dimensional counterexample of geometric degree d, and expositions by Tao [12] and Speyer [11] followed. Speyer isolated and named the mechanism: one sweeps the tangent lines of a plane curve — a map that is many-to-one by projective duality — and conjugates by monomial maps so that the Jacobian factor picked up by the sweep is cancelled exactly. The price of the cancellation is that ramification is pushed to infinity, so the resulting map is everywhere unramified but not proper, and injectivity fails through escape to infinity. Throughout, the *geometric degree* of a generically finite polynomial map is the number of points of its generic fiber.

Gao [5] generalises the tangent sweep from plane curves to tangent direction fields on parametrised hypersurfaces and obtains counterexamples in every dimension $n > 2$, constructing five explicit new maps. Four of these are the subject of the present paper:

map	source and target	$\det J$	geometric degree	component degrees
F_4	$\mathbb{C}^4 \to \mathbb{C}^4$	$-44/9$	5	$4, 11, 12, 21$
F_5	$\mathbb{C}^4 \to \mathbb{C}^4$	$160/29$	10	$3, 12, 14, 16$
F_6	$\mathbb{C}^5 \to \mathbb{C}^5$	-290	6	$7, 38, 40, 42, 44$
F_7	$\mathbb{C}^5 \to \mathbb{C}^5$	119377	12	$7, 86, 89, 92, 95$

For the two three-dimensional maps it treats [5] gives the complete fiber stratification. For F_4–F_7 it gives only the generic counts, and says so in its introduction:

> "for F_4–F_7, whose Gröbner bases are out of reach, we summarize the univariate elimination data behind the generic counts, the complete stratifications being deferred to a subsequent version."

The obstruction is real: the graph ideals of these maps live in 8 or 10 variables with generators of degree up to 95. But the univariate elimination data are available, and they are enough to decide several fibers outright and to constrain, uniformly in the target, every fiber of two of these maps. That is what we do here.

Results. Write R_Y for the univariate fiber polynomial attached to a target (Section 2); its roots with $\gamma \neq 0$ are in bijection with the fiber. We prove the following, all over $\mathbb{Q}$ or over an arbitrary field of characteristic zero.

(A) *A totally rational fiber of* F_4 (Theorem 4.1). The rational target $(1, -\frac{13}{6}, -\frac{5}{18}, \frac{97}{18})$ has five distinct rational preimages, so the geometric degree 5 of F_4 is attained over $\mathbb{Q}$. The fiber exhibited in [5] is numerical, with "residuals below 10^{-9}".

(B) *Reality rigidity for* F_5 (Theorem 5.1). If all ten roots of the fiber polynomial of F_5 over a real target are real, then all ten equal $\frac{1}{4}$; hence at most one real target — an explicit rational one — can have this property.

(C) *A rational point off the image of* F_5 (Theorem 5.4). At that target the fiber of F_5 is *empty*: every solution of the eliminated fiber system carries $\gamma = 0$, so the monomial twist deletes it and nothing is left. The certificate is a resultant Bézout identity valid over every field of characteristic zero. No empty fiber, and no point off the image, is exhibited for F_5 in [5].

(D) *A reality obstruction for* F_6 (Theorem 6.1). The fiber sextic of F_6 never splits over $\mathbb{R}$, for any target; consequently at most four of the six generic preimages of a real target are ever real.

(E) *The bound is attained* (Theorem 6.3). The rational target $(1, -\frac{33}{8}, -1, -1, \frac{57}{8})$ of F_6 has four distinct rational preimages.

(F) *An exactly six-point fiber of* F_6 (Theorem 6.4). At that same target the sextic factors as $2w(w+1)(w-\frac{1}{2})(w-\frac{3}{2})(w^2+\frac{5}{4})$ and γ is nonzero at every one of its six roots — exactly, not numerically — so that fiber has exactly six points, four of them rational and two defined over $\mathbb{Q}(\sqrt{-5})$ and conjugate over $\mathbb{Q}$. In [5] the two conditions at a random target are "verified exactly for the squarefreeness and numerically for the roots", and no target is named.

Three different behaviours thus occur among three maps of one construction: F_4 has a totally rational, hence totally real, fiber; F_5 can have one only at a single target, where the fiber turns out to be empty; F_6 can have none at all.

We have found no later work on the fibers of F_4–F_7. A search of a full-text mirror of the arXiv LaTeX sources, complete through 27 August 2026, for the identifier `2608.00222` returns exactly one citing paper, [10], which takes from [5] only the tangent sweep and works with homogeneous Keller maps in a different setting; the symbols $F_4, \ldots, F_7$ do not occur in it, nor in either of the two neighbouring papers that report verification with Lean and *Mathlib*, on the separable Jacobian conjecture in characteristic two in dimension two [8] and on the quartic Hessian conjecture in dimension four [9]. An arXiv full-text query for "Jacobian conjecture" over the same window returns no paper about these maps. This is a not-found statement of 28 August 2026 and nothing more.

Method. The whole of (A)–(E) rests on one observation that [5] does not make. Each of these maps has a fiber polynomial with *constant* leading coefficient — that is the point on which [5] rests its generic counts, since it shows the root count never drops. But for F_4, F_5 and F_6 *several* of the leading coefficients are independent of the target, and each such coefficient pins one elementary symmetric function of the roots uniformly in Y. Newton's identities then pin the low power sums of the roots uniformly in Y, and a totally real root system with prescribed $\sum r_i$, $\sum r_i^2$, $\sum r_i^3$ is severely constrained — for F_6, impossibly so. The same inversion, run forwards, produces the rational targets of (A) and (E): one prescribes rational roots subject to the uniform constraints and reads off the target.

Organisation. Section 2 fixes the construction data of the four maps and the fiber dictionary, both as in [5]. Section 3 extracts the target-independent symmetric functions. Sections 4, 5 and 6 prove (A), (B)–(C) and (D)–(F). Section 7 records our independent reproduction of the published data that the new results use. Section 8 describes the formal verification.

The subsequent version announced in [5] has not appeared: as of 28 August 2026 the arXiv submission history of `2608.00222` lists v1 (31 July 2026) and nothing else. All numbered references to [5] below — theorems, propositions and appendix sections — are to that version.

2. The four maps and the fiber dictionary

Everything in this section is from [5]; we restate it in the form used below and record that we re-derived all of it in exact rational arithmetic (Section 7).

Each map is a five-arrow composition

$$\text{monomial} \longrightarrow \text{stage} \longrightarrow \text{scaling} \longrightarrow \text{padded sweep} \longrightarrow \text{monomial twist.}$$

Write x for the first source coordinate, γ for the first output of the stage, $w = (w_1, \dots)$ for the sweep parameters, X for the swept hypersurface, Δ for the tangent direction field, $S = X + \gamma\Delta$ for the sweep and $C = \gamma x$. With twist exponents $\mu_1, \mu_2, \dots$ the map is

$$F = \Big(C,\ \frac{S_1}{C^{\mu_1}},\ \frac{S_2}{C^{\mu_2}},\ \dots\Big)^{\mathsf{T}},$$

each quotient being a polynomial by the divisibility statements of the construction (Proposition 7.1 below).

The map F_4. On $\mathbb{C}^4 \ni (x, y, z, t)$ put

$$\gamma = 1 + xy + x^2t, \qquad u_1 = 1 + xz, \qquad u_2 = -\tfrac{7}{9} + \tfrac{22}{9}xy + \tfrac{8}{3}xz,$$

$w_1 = \gamma u_1$, $w_2 = \gamma u_2$, $C = \gamma x$. The swept surface is the non-cylindrical ruled surface $X = A(w_1) + w_2 B(w_1)$ with

$$A = \Big(\tfrac{1}{3}w_1 - \tfrac{4}{3}w_1^2,\ \tfrac{2}{3}w_1^2 - \tfrac{8}{9}w_1^3,\ 2w_1 + \tfrac{7}{9}w_1^3 - \tfrac{2}{3}w_1^4\Big)^{\mathsf{T}}, \qquad B = \big(w_1^2 - 1,\ w_1^3,\ w_1^4 + w_1^2 + 2\big)^{\mathsf{T}},$$

and $\Delta = (1, w_1, w_1^2)^{\mathsf{T}}$. With $S_i = A_i + w_2 B_i + \gamma\Delta_i$,

$$F_4 = \Big(C,\ \tfrac{S_1}{C},\ \tfrac{S_2}{C^2},\ \tfrac{S_3}{C}\Big)^{\mathsf{T}}.$$

The map F_5. On $\mathbb{C}^4 \ni (x, y, z, t)$ put $(v_1, v_2, v_3) = (xy, x^2z, x^3t)$ and

$$\gamma = 1 + \tfrac{20}{29}v_1, \quad u_1 = 1 - \tfrac{49}{29}v_1 + 8v_2, \quad u_2 = 1 - \tfrac{69}{29}v_1 + 9v_2 + v_3 + \tfrac{4197}{1682}v_1^2,$$

$w_1 = \gamma u_1$, $w_2 = \gamma^2 u_2$, $C = \gamma x$; the last coefficient makes the stage a unipotent nonlinear automorphism which absorbs the residual quadratic side condition. The swept surface is

$$\begin{aligned}
X_1 &= \tfrac{160}{29}w_1^3 - \tfrac{60}{29}w_1^2 - \tfrac{240}{29}w_1w_2 + \tfrac{51}{29}w_1 + \tfrac{60}{29}w_2,\\
X_2 &= \tfrac{60}{29}w_1^4 - \tfrac{20}{29}w_1^3 - \tfrac{120}{29}w_2^2 + \tfrac{51}{29}w_2,\\
X_3 &= -\tfrac{48}{29}w_1^5 + \tfrac{15}{29}w_1^4 + \tfrac{240}{29}w_1^3w_2 - \tfrac{17}{29}w_1^3 - \tfrac{60}{29}w_1^2w_2 - \tfrac{240}{29}w_1w_2^2 + \tfrac{51}{29}w_1w_2 + \tfrac{30}{29}w_2^2,
\end{aligned}$$

with $\Delta = (1, w_1, w_2)^{\mathsf{T}}$, $S_i = X_i + \gamma\Delta_i$ and $F_5 = \big(C,\ S_1/C,\ S_2/C^2,\ S_3/C^3\big)^{\mathsf{T}}$.

The map F_6. On $\mathbb{C}^5 \ni (x, y, z_1, z_2, z_3)$ put $(v_1, v_2, v_3, v_4) = (xy, x^2z_1, x^3z_2, x^4z_3)$ and

$$\gamma = 1 - 29v_1 + 999v_1^2 + 355v_1v_2 - 41553v_1^3 + v_4, \quad u_1 = 1 + 27v_1 - 5v_2,$$

$$u_2 = v_1 - 12v_2 + \tfrac{2128}{5}v_1^2 + v_3, \qquad u_3 = -4v_1 - 10v_2,$$

$w_1 = \gamma u_1,\ w_2 = \gamma^3 u_2,\ w_3 = \gamma^4 u_3,\ C = \gamma x$. The swept threefold is built from the potentials $G_2 = w_2 + w_1 w_3 + w_1^2 - w_1^3$, $G_3 = 2w_1 G_2 + w_2$, $G_4 = w_3 - G_2^2 + 2w_1^4 - 2w_1^5$ through $X_1 = -\det J(G_2, G_3, G_4)$ and $X_{i+1} = G_{i+1} + \delta_{i+1} X_1$, which gives

$$X_1 = -w_3 + 2w_2 - 2w_1 + 2w_1 w_3 + 5w_1^2 - 2w_1^3 + 8w_1^4 - 10w_1^5,$$
$$X_2 = w_2 + 2w_1 w_2 - w_1^2 + 2w_1^2 w_3 + 4w_1^3 - 2w_1^4 + 8w_1^5 - 10w_1^6,$$
$$X_3 = w_2 + 2w_1 w_2 + w_1^2 w_3 + 2w_1^2 w_2 + 2w_1^3 w_3 + 3w_1^4 - 2w_1^5 + 8w_1^6 - 10w_1^7,$$
$$X_4 = w_3 - w_2 w_3 + w_2^2 - 2w_1 w_2 + 3w_1^2 w_2 - w_1^2 w_3^2 - 2w_1^3 w_3 + w_1^4 + 2w_1^4 w_3 + 8w_1^4 w_2 - 10w_1^5 w_2 - w_1^6.$$

With $\Delta = (1, w_1, w_1^2, w_2)^{\mathsf{T}}$, $S_i = X_i + \gamma \Delta_i$ and $F_6 = \big(C,\ S_1/C,\ S_2/C^2,\ S_3/C^3,\ S_4/C^4\big)^{\mathsf{T}}$.

The fiber dictionary. A *target* is a point $v \in \mathbb{C}^n$ with $v_1 = C_0 \neq 0$. Setting $Y_i = C_0^{\mu_i} v_{i+1}$ turns the equation $F(p) = v$ into the sweep system $S(\gamma, w) = Y$ together with $x = C_0/\gamma$. For all four maps this system triangularises: $S_1 = Y_1$ solves for $\gamma = Y_1 - X_1$, and eliminating the remaining transverse parameters leaves one univariate *fiber polynomial* $R_Y(w_1)$, whose degree is the geometric degree and whose leading coefficient does not depend on Y.

Proposition 2.1 ([5, Prop. 4.11 for F_7; the proofs of Thms. 4.3, 4.4 and 4.5 for F_4, F_5 and F_6])**.** *For a target v with $C_0 \neq 0$, the fiber $F^{-1}(v)$ is in bijection with the set of roots w_1 of R_Y at which $\gamma \neq 0$, the bijection being $x = C_0/\gamma$ followed by the inverse of the stage. In particular a fiber can be smaller than the geometric degree only through multiple roots of R_Y or through roots at which γ vanishes.*

The reason is that the stage is a polynomial automorphism and the monomial conjugation is bijective where $x \neq 0$, so the only points lost are those the twist deletes, namely those with $\gamma = 0$. We use Proposition 2.1 as stated in [5] and do not reprove it. Both the bijection and its inverse are given by formulas with rational coefficients: $w_1 = \gamma u_1$ is a polynomial in the source coordinates over $\mathbb{Q}$, and conversely $x = C_0/\gamma$, the closed forms for the remaining sweep parameters and the inverse of the stage are all defined over $\mathbb{Q}$. Consequently, for a target whose coordinates lie in a subfield $L \subseteq \mathbb{C}$, a fiber point has its coordinates in L if and only if its root w_1 does; we use this below for $L = \mathbb{R}$ and $L = \mathbb{Q}$ and $L = \mathbb{Q}(\sqrt{-5})$.

Remark 2.2 (division-free form)*.* All fiber statements below are written without division: $F(p) = (1, T_1, \dots)$ is expressed as $C(p) = 1$ together with $S_j(p) = T_j$, which is equivalent because $C(p) = 1$ makes $S_j/C^{\mu_j} = S_j$, and because the quotients S_j/C^{μ_j} are polynomials (Proposition 7.1). This is what makes an exact fiber check a finite computation in $\mathbb{Q}$ rather than a computation with the fully expanded components — though we also carried it out against those.

The fiber polynomials we need are the following, all from [5]. For F_4 it is the *tangency quintic* of the swept ruled surface,

$$R(w_1; Y) = \det\big(A(w_1) - Y,\ B(w_1),\ \Delta(w_1)\big) = \tfrac{2}{9}w_1^5 + \tfrac{2}{9}w_1^4 + \big(\tfrac{8}{9} + Y_1\big)w_1^3 - \big(\tfrac{4}{3} + 2Y_2\big)w_1^2 + (2Y_1 + Y_3)w_1 - 2Y_2. \tag{1}$$

For F_5 it is the w_2-resultant of the two equations left after eliminating γ, which after the rescaling by $-29^4/30$ used in [5] is

$$\begin{aligned} R(w_1; Y) = {} & 20480w_1^{10} - 51200w_1^9 + 57600w_1^8 + (222720Y_1 - 126720)w_1^7 \\ & + (-204160Y_1 - 742400Y_2 + 234480)w_1^6 \\ & + (-119712Y_1 + 890880Y_2 + 890880Y_3 - 155448)w_1^5 + \cdots, \end{aligned} \tag{2}$$

the omitted terms being the coefficients of $w_1^4, \dots, w_1^0$, which are quadratic and cubic in Y and are displayed in full in [5]. For F_6 the triangular collapse gives

$$R_Y(w_1) = 2w_1^6 - 2w_1^5 - w_1^3 + (Y_1 + 1)w_1^2 + \big(Y_4 + Y_2^2 - Y_1 Y_3 - 2Y_2 + Y_1\big)w_1 + (Y_3 - Y_2), \tag{3}$$

obtained by substituting the closed forms $w_2 = Y_3 - 2w_1 Y_2 + w_1^2 Y_1$ and $w_3 = Y_4 + Y_2^2 - Y_1 Y_3 + 2w_1^5 - 2w_1^4$ into the one potential equation not yet used. For F_7, writing $p_2 = Y_4 - 3Y_2 w_1^2 + 2Y_1 w_1^3 - w_1^4$ and

$$p_1 = Y_3 - 2Y_2w_1 + Y_1w_1^2 - w_1^3 - w_1^2p_2,$$
$$R_Y(w_1) = p_1^2 + p_1 + w_1p_2 + w_1^3 + Y_1w_1 - Y_2. \tag{4}$$

3. Target-independent symmetric functions

Let $q(w) = c_d\, w^d + c_{d-1}w^{d-1} + \cdots$ have $c_d \neq 0$ and roots $r_1, \dots, r_d$ in an extension field, listed with multiplicity, and write e_k for the elementary symmetric functions and $s_k = \sum_i r_i^k$ for the power sums. Vieta gives $e_k = (-1)^k c_{d-k}/c_d$, and Newton's identities give

$$s_1 = e_1, \qquad s_2 = e_1^2 - 2e_2, \qquad s_3 = e_1s_2 - e_2s_1 + 3e_3. \tag{5}$$

The observation this paper turns on is that for three of the four maps the top coefficients of the fiber polynomial do not involve Y, so (5) pins s_1, s_2, s_3 *uniformly over all targets.*

Lemma 3.1. *Let Y be arbitrary and let $r_1, \dots, r_d$ be the roots of the fiber polynomial, with multiplicity. Then*

- *(i) for F_4 ($d = 5$): $e_1 = -1$ and $e_5 = e_3 - 6$;*
- *(ii) for F_5 ($d = 10$): $e_1 = \frac{5}{2}$ and $e_2 = \frac{45}{16}$, hence $s_1 = \frac{5}{2}$ and $s_2 = \frac{5}{8}$;*
- *(iii) for F_6 ($d = 6$): $e_1 = 1$, $e_2 = 0$ and $e_3 = \frac{1}{2}$, hence $s_1 = 1$, $s_2 = 1$ and $s_3 = \frac{5}{2}$;*
- *(iv) for F_7 ($d = 12$): $e_1 = 4Y_1$ and $e_2 = 4Y_1^2 + 6Y_2$, so that $s_1 = 4Y_1$ and $s_2 = 8Y_1^2 - 12Y_2$; neither is target-independent, and s_2 is unbounded on $\mathbb{R}^4$.*

Proof. Divide the fiber polynomial by its leading coefficient and read off Vieta. For F_4, (1) scaled by $\frac{9}{2}$ is $w^5 + w^4 + (4 + \frac{9}{2}Y_1)w^3 - (6 + 9Y_2)w^2 + (9Y_1 + \frac{9}{2}Y_3)w - 9Y_2$, so $e_1 = -1$, $e_3 = 6 + 9Y_2$ and $e_5 = 9Y_2$, whence $e_5 = e_3 - 6$; the coefficients pinning e_2 and e_4 do involve Y. For F_5, (2) gives $e_1 = 51200/20480 = \frac{5}{2}$ and $e_2 = 57600/20480 = \frac{45}{16}$, and $s_2 = e_1^2 - 2e_2 = \frac{25}{4} - \frac{45}{8} = \frac{5}{8}$. For F_6, (3) scaled by $\frac{1}{2}$ is $w^6 - w^5 + 0 \cdot w^4 - \frac{1}{2}w^3 + \cdots$, so $e_1 = 1$, $e_2 = 0$, $e_3 = \frac{1}{2}$, and (5) gives $s_2 = 1$ and $s_3 = 1 \cdot 1 - 0 \cdot 1 + \frac{3}{2} = \frac{5}{2}$. For F_7, expanding p_1 in (4) gives $p_1 = w^6 - 2Y_1w^5 + 3Y_2w^4 - w^3 + (Y_1 - Y_4)w^2 - 2Y_2w + Y_3$, so R_Y is monic of degree 12 with w^{11}-coefficient $-4Y_1$ and w^{10}-coefficient $4Y_1^2 + 6Y_2$; both move with the target, and $s_2 = e_1^2 - 2e_2 = 8Y_1^2 - 12Y_2$ takes every real value. □

Part (iv) is the reason F_7 carries no reality result below: the mechanism of Sections 5 and 6 needs a bounded power sum, and for F_7 already the second power sum is unbounded, so the argument has nothing to grip.

4. F_4: the geometric degree is attained over $\mathbb{Q}$

[5] establishes the geometric degree 5 of F_4 by exact elimination and exhibits one fiber numerically: "explicit fibers with residuals below 10^{-9}, e.g. the target of $(0.7, -0.4, 0.3, 0.5)$ has five preimages, three real and one conjugate pair". No exact fiber of F_4 appears there. The constraints of Lemma 3.1(i) make exact rational fibers easy to search for, and there are many.

Theorem 4.1. *Let $v^* = \left(1, -\frac{13}{6}, -\frac{5}{18}, \frac{97}{18}\right) \in \mathbb{Q}^4$. Then the five points*

$$\left(\tfrac{4}{21},\ \tfrac{903}{88},\ -\tfrac{31}{4},\ \tfrac{44541}{704}\right), \qquad \left(-2,\ \tfrac{59}{44},\ -\tfrac{1}{2},\ \tfrac{13}{44}\right), \qquad \left(\tfrac{12}{5},\ \tfrac{599}{264},\ -\tfrac{11}{12},\ -\tfrac{19895}{19008}\right),$$
$$\left(\tfrac{4}{15},\ \tfrac{129}{44},\ -\tfrac{7}{4},\ \tfrac{19485}{704}\right), \qquad \left(-\tfrac{6}{7},\ -\tfrac{301}{132},\ \tfrac{13}{6},\ -\tfrac{1666}{297}\right)$$

of $\mathbb{Q}^4$ are pairwise distinct and each satisfies $F_4(p) = v^*$.

Corollary 4.2. *The fiber of F_4 over v^* consists of exactly these five points; in particular it has the generic cardinality, so the geometric degree 5 of F_4 is attained by a fiber all of whose points are rational.*

Proof. By Proposition 2.1 the fiber injects into the root set of the tangency quintic (1), which has at most five elements; five are exhibited. □

Proof of Theorem 4.1. Distinctness is immediate. For each of the five points the assertion $F_4(p) = v^*$ is, by Remark 2.2, the conjunction of the four rational identities $C(p) = 1$, $S_1(p) = -\frac{13}{6}$, $S_2(p) = -\frac{5}{18}$, $S_3(p) = \frac{97}{18}$, each an evaluation of an explicit polynomial in four rational arguments; the twenty evaluations are exact arithmetic in $\mathbb{Q}$ and were carried out in the kernel of a proof assistant (Section 8). We also evaluated the fully expanded components of F_4, of degrees $4, 11, 12, 21$ and $3, 32, 39, 114$ terms, at all five points independently.

The target was found, not guessed. By Lemma 3.1(i) a rational fiber must have $e_1 = -1$ and $e_5 = e_3 - 6$; choosing three rational roots r_1, r_2, r_3 with $a = e_1(r_1, r_2, r_3)$, $b = e_2(r_1, r_2, r_3)$, $c = e_3(r_1, r_2, r_3)$, the remaining two roots are the roots of $w^2 - sw + p$ with $s = -1-a$ and $p = (c+bs-6)/(c-a)$ forced, and they are rational exactly when $s^2 - 4p$ is a square. A finite search over $r_i \in \{p/q : |p| \le 14,\ 1 \le q \le 3\}$ returned 104 targets meeting these conditions; the one displayed has roots $w_1 \in \{-\frac{5}{2}, -1, -\frac{1}{2}, 1, 2\}$, and it is the one of smallest height among them. The search is a heuristic for *finding* the target and no claim rests on its exhaustiveness; what the theorem asserts is verified pointwise. ∎

5. F_5: REALITY RIGIDITY AND A POINT OFF THE IMAGE

5.1. At most one real target can have a totally real fiber polynomial.

Theorem 5.1. *Let $Y \in \mathbb{R}^3$ and suppose that all ten roots $r_1, \ldots, r_{10}$ of the fiber polynomial* (2) *of F_5 over Y are real. Then $r_i = \frac{1}{4}$ for every i.*

Proof. By Lemma 3.1(ii), $\sum_i r_i = \frac{5}{2}$ and $\sum_i r_i^2 = \frac{5}{8}$ whatever Y is. Hence

$$\sum_{i=1}^{10} \Big(r_i - \tfrac{1}{4}\Big)^2 = \sum_i r_i^2 - \tfrac{1}{2} \sum_i r_i + 10 \cdot \tfrac{1}{16} = \tfrac{5}{8} - \tfrac{5}{4} + \tfrac{5}{8} = 0,$$

a sum of ten real squares. Each vanishes. ∎

Corollary 5.2. *If the fiber polynomial of F_5 over a real target splits over $\mathbb{R}$, then $R(w_1; Y) = 20480\,(w_1 - \frac{1}{4})^{10}$ and*

$$Y = Y^* := \Big(\tfrac{23}{58},\ \tfrac{1709}{9280},\ \tfrac{1409}{37120}\Big).$$

Proof. The polynomial has degree 10 and leading coefficient 20480 by (2), and all its roots are $\frac{1}{4}$ by Theorem 5.1, so it equals $20480(w_1 - \frac{1}{4})^{10}$. Comparing the coefficient of w_1^7 on both sides, $222720 Y_1 - 126720 = 20480\binom{10}{3}(-\frac{1}{4})^3 = -38400$, gives $Y_1 = \frac{23}{58}$; the coefficient of w_1^6 then gives $-742400 Y_2 = 16800 + 80960 - 234480$, i.e. $Y_2 = \frac{1709}{9280}$; and the coefficient of w_1^5 gives $Y_3 = \frac{1409}{37120}$. This is a hand computation with three of the coefficients printed in [5]; it is not part of the formal development (Section 8). ∎

Remark 5.3. Corollary 5.2 uses only three of the eleven coefficients of (2). As a consistency check on the whole degeneration we substituted Y^* into all eleven coefficient polynomials of the resultant as printed in [5] and compared them with those of $20480(w_1 - \frac{1}{4})^{10}$; all eleven agree, the last four being -150, $\frac{225}{16}$, $-\frac{25}{32}$, $\frac{5}{256}$. That comparison is exact rational arithmetic performed outside the formal development, and nothing below depends on it.

5.2. **The fiber over Y^* is empty.** Corollary 5.2 leaves a single candidate target, Y^*, and had its fiber polynomial genuinely split there the fiber could hold at most one point, all ten roots being equal. We show that it holds none, and by an argument that does not use the collapse. Eliminating $\gamma = Y_1 - X_1(w_1, w_2)$ from the sweep system at $Y = Y^*$ leaves the two equations $X_2 + \gamma w_1 = Y_2$ and $X_3 + \gamma w_2 = Y_3$; clearing denominators, put

$$\mathcal{A} := 9280\big(X_2 + \gamma w_1 - Y_2\big), \qquad \mathcal{B} := 37120\big(X_3 + \gamma w_2 - Y_3\big), \qquad \Gamma := 58\gamma,$$

which are the integral polynomials

$$\begin{aligned}\mathcal{A} = &-32000w_1^4 + 76800w_1^2w_2 + 12800w_1^3 - 38400w_2^2 - 19200w_1w_2\\ &-16320w_1^2 + 16320w_2 + 3680w_1 - 1709,\\ \mathcal{B} = &-61440w_1^5 + 102400w_1^3w_2 + 19200w_1^4 - 21760w_1^3 - 38400w_2^2 + 14720w_2 - 1409,\\ \Gamma = &\ 23 - 2\big(160w_1^3 - 60w_1^2 - 240w_1w_2 + 51w_1 + 60w_2\big).\end{aligned}$$

Theorem 5.4. *Let K be a field of characteristic zero. If $(w_1, w_2) \in K^2$ satisfies $\mathcal{A}(w_1, w_2) = \mathcal{B}(w_1, w_2) = 0$, then $\Gamma(w_1, w_2) = 0$. Consequently the fiber of F_5 over the rational point*

$$v^* = \Big(1,\ \tfrac{23}{58},\ \tfrac{1709}{9280},\ \tfrac{1409}{37120}\Big)$$

is empty, and $v^* \notin F_5(\mathbb{C}^4)$.

Proof. Two identities do the work. The first is a Bézout relation for the w_2-resultant of $\mathcal{A}$ and $\mathcal{B}$: with

$$\begin{aligned}\mathcal{U} = &-3072000w_1^6 + 2995200w_1^5 + 1152000w_1^3w_2 - 1152000w_1^4 - 864000w_1^2w_2\\ &-4800w_1^3 + 216000w_1w_2 + 147600w_1^2 - 18000w_2 - 41400w_1 + 3525,\\ \mathcal{V} = &\ 1612800w_1^5 - 1152000w_1^3w_2 - 1728000w_1^4 + 864000w_1^2w_2 + 964800w_1^3\\ &-216000w_1w_2 - 327600w_1^2 + 18000w_2 + 59400w_1 - 4275,\end{aligned}$$

one has, identically in (w_1, w_2) over K,

$$\mathcal{U}\mathcal{A} + \mathcal{V}\mathcal{B} = -750\,(4w_1 - 1)^{10}. \tag{6}$$

The second is the factorisation

$$\Gamma = (4w_1 - 1)\big(120w_2 - 80w_1^2 + 10w_1 - 23\big), \tag{7}$$

again an identity in (w_1, w_2) over K. Both are verified by expanding the two sides; both sides have integer coefficients, so both identities in fact hold over any commutative ring, but only the stated form is used. Now if $\mathcal{A} = \mathcal{B} = 0$ then (6) gives $750(4w_1 - 1)^{10} = 0$, so $4w_1 = 1$ because $\operatorname{char} K = 0$, and then (7) gives $\Gamma = 0$, i.e. $\gamma = 0$.

For the consequence, note that v^* has first coordinate $C_0 = 1 \neq 0$ and that $Y_i = C_0^{\mu_i} v_{i+1}^* = v_{i+1}^*$, so the sweep target is exactly Y^*. By Proposition 2.1 a point of $F_5^{-1}(v^*)$ would give a solution of the sweep system with $\gamma \neq 0$; its transverse coordinates (w_1, w_2) satisfy $\mathcal{A} = \mathcal{B} = 0$, and the first paragraph forces $\gamma = 0$, a contradiction. Hence the fiber is empty. ☐

Remark 5.5. The degeneration is structural rather than accidental. The coefficient of w_2 in X_1 is $\frac{60-240w_1}{29}$, which vanishes exactly at $w_1 = \frac{1}{4}$, and there $X_1(\frac{1}{4}, \cdot) \equiv \frac{23}{58}$ — precisely the value of Y_1 that Corollary 5.2 forces. So the line $w_1 = \frac{1}{4}$ singled out by (6) is exactly the line along which γ vanishes identically. We also record that γ is not identically zero: $\Gamma(0, 0) = 23$.

Corollary 5.6. *F_5 is not proper at v^*; that is, v^* lies in the non-properness set of F_5, the set studied by Jelonek* [6].

Proof. Call a map F *proper at v* if some open neighbourhood $U \ni v$ has $F^{-1}(\overline{U})$ compact. Since $\det J(F_5)$ is a nonzero constant, F_5 is a local biholomorphism, hence an open map; its image therefore contains a nonempty open set, so F_5 is dominant. By Chevalley's theorem the image is constructible, hence contains a Zariski-open dense subset, whose complement is a proper algebraic subset and so has empty interior; thus $F_5(\mathbb{C}^4)$ meets every nonempty open subset of $\mathbb{C}^4$. Suppose F_5 were proper at v^*, and take for U a ball as above. If $y \in U$ is a limit of points $F_5(p_k) \in U$, then every p_k lies in the compact set $F_5^{-1}(\overline{U})$, so a subsequence of (p_k) converges to some p with $F_5(p) = y$; hence $F_5(\mathbb{C}^4) \cap U$ is closed in U. It is also open in U, and nonempty, and U is connected, so $F_5(\mathbb{C}^4) \cap U = U \ni v^*$ — contradicting Theorem 5.4. ☐

Remark 5.7. Theorem 5.4 and Corollary 5.2 together exhibit one point of the deepest stratum of the stratification deferred in [5]: v^* is the only real target whose fiber polynomial can split over $\mathbb{R}$, and its fiber is empty. Since a fiber of F_5 has at most ten points, the empty fiber is the extreme case, and it occurs at an explicit rational point.

6. F_6: no totally real fiber, and an exact fiber of six points

6.1. The reality obstruction.

Theorem 6.1. *There is no target Y for which the fiber sextic* (3) *of F_6 splits over $\mathbb{R}$. Equivalently: there are no six real numbers $r_1, \dots, r_6$ with*

$$e_1(r) = 1, \qquad e_2(r) = 0, \qquad e_3(r) = \tfrac{1}{2}.$$

Proof. Suppose such r_i exist. By (5), $\sum_i r_i^2 = e_1^2 - 2e_2 = 1$ and $\sum_i r_i^3 = e_1 s_2 - e_2 s_1 + 3e_3 = \frac{5}{2}$. From $\sum_i r_i^2 = 1$ we get $r_i^2 \le 1$, hence $|r_i| \le 1$, hence $r_i^3 \le r_i^2$ for every i; summing, $\frac{5}{2} = \sum_i r_i^3 \le \sum_i r_i^2 = 1$, a contradiction. By Lemma 3.1(iii) the three hypotheses hold for the roots of (3) at every target, so the sextic never splits over $\mathbb{R}$. □

Three lines and no computation; what makes it work is that (3) has its w^5, w^4 and w^3 coefficients free of Y.

Corollary 6.2. *For every real target of F_6 the fiber sextic has a non-real root and at most four real roots; consequently at most four points of the fiber over a real target are real, and the fiber over a rational target contains at most four rational points.*

Proof. If five of the six roots of (3) were real, the sixth would be real as well, since the sum of all six is $e_1 = 1$; then the sextic would split over $\mathbb{R}$, contradicting Theorem 6.1. So at most four roots are real, and by Proposition 2.1 the fiber points inject into the roots, a real fiber point having a real w_1. The rational statement is the same argument with "real" replaced by "rational" throughout. □

The only exact fiber of F_6 in [5] is the one over the punctured axis $(C_0, 0, 0, 0, 0)$, where $R_0 = w_1^2(w_1 - 1)(2w_1^3 - 1)$, the double root $w_1 = 0$ carries $\gamma = 0$ and is deleted, and the surviving four points are $w_1 = 1$ and the three roots of $2w_1^3 = 1$ — of which only one is rational. The bound of Corollary 6.2 is nevertheless attained by four *rational* points.

6.2. Four rational preimages.

Theorem 6.3. *Let $v^* = \left(1, -\frac{33}{8}, -1, -1, \frac{57}{8}\right) \in \mathbb{Q}^5$. Then the four points of $\mathbb{Q}^5$*

$$\left(-\tfrac{8}{135}, \tfrac{127}{232}, \tfrac{3429}{928}, \tfrac{534328341}{430592}, -\tfrac{1369753965163395}{799178752}\right),$$
$$\left(\tfrac{8}{15}, -\tfrac{58817}{783000}, -\tfrac{6663}{116000}, -\tfrac{10677774936967}{1634904000000}, -\tfrac{55497623981500351}{936537600000000}\right),$$
$$\left(-\tfrac{4}{9}, \tfrac{947}{9396}, \tfrac{3}{232}, \tfrac{1537490707}{196188480}, -\tfrac{177100501775}{674307072}\right),$$
$$\left(\tfrac{4}{105}, -\tfrac{22921207}{26856900}, \tfrac{2546427}{284200}, -\tfrac{92921395109904721}{19627022520000}, \tfrac{14081796983626669084987907}{1349738223254400000}\right)$$

are pairwise distinct and each satisfies $F_6(p) = v^*$. By Corollary 6.2 this is the largest number of rational points a fiber of F_6 can have.

Proof. As in Theorem 4.1 the assertion $F_6(p) = v^*$ is, by Remark 2.2, the conjunction of $C(p) = 1$ and $S_j(p) = v^*_{j+1}$ for $j = 1, \dots, 4$: twenty exact evaluations of explicit polynomials at rational arguments. Here the arguments have height up to $1.4 \cdot 10^{25}$ and the components have degrees up to 44, so these evaluations were performed by compiled exact rational arithmetic and are the one place in this paper where the verification is by evaluation rather than inside the proof-assistant kernel (Section 8); they were also reproduced independently against the fully expanded 904-term component $F_{6,5}$.

The target was produced by inverting Lemma 3.1(iii). Prescribing three rational roots r_1, r_2, r_3 with symmetric functions a, b, c, the fourth rational root must satisfy

$$r^3 + (a-1)r^2 + (a^2 - a - b)r + \left(a^3 - a^2 - 2ab + b + c - \tfrac{1}{2}\right) = 0,$$

after which the remaining quadratic $w^2 - sw + p$ is determined by $s = 1 - E_1$ and $p = E_1^2 - E_1 - E_2$, where E_k is the k-th symmetric function of the four chosen roots. A finite search over $r_i \in \{p/q : |p| \le 8,\ 1 \le q \le 3\}$ returned 80 targets with four distinct rational roots, an irreducible quadratic factor and $\gamma \ne 0$ at all six roots. The one displayed uses the roots $\{-1, 0, \frac{1}{2}, \frac{3}{2}\}$. Again the search is only how the target was found; the theorem is verified pointwise. □

6.3. **That fiber has exactly six points.** For F_6, [5] states the generic count as follows: "For random rational targets R_Y is squarefree with all six roots satisfying $\gamma \ne 0$ (verified exactly for the squarefreeness and numerically for the roots), giving generic fiber size 6." At the target of Theorem 6.3 both halves can be settled exactly. Substituting $Y = (-\frac{33}{8}, -1, -1, \frac{57}{8})$ into (3) and into $\gamma = Y_1 - X_1$ along the fiber gives the two univariate polynomials

$$R(w) = 2w^6 - 2w^5 - w^3 - \tfrac{25}{8}w^2 + \tfrac{15}{8}w, \qquad \gamma(w) = -4w^6 + 16w^5 - 10w^4 + 2w^3 + \tfrac{13}{4}w^2 - 10w + \tfrac{15}{8},$$

the second obtained by substituting $w_2 = -\frac{33}{8}w^2 + 2w - 1$ and $w_3 = 2w^5 - 2w^4 + 4$, the closed forms of Section 2, into X_1.

Theorem 6.4. *Over any field K of characteristic zero,*

$$R(w) = 2w\,(w+1)\left(w - \tfrac{1}{2}\right)\left(w - \tfrac{3}{2}\right)\left(w^2 + \tfrac{5}{4}\right),$$

and $\gamma(w) \ne 0$ at every root of R. Consequently the fiber of F_6 over $v^* = \left(1, -\frac{33}{8}, -1, -1, \frac{57}{8}\right)$ has exactly six points, of which four are the rational points of Theorem 6.3 and the remaining two have their coordinates in $\mathbb{Q}(\sqrt{-5})$ and are conjugate over $\mathbb{Q}$.

Proof. The factorisation is a polynomial identity. Its four rational roots are distinct, and a root of $w^2 + \frac{5}{4}$ is none of them, since 0, -1, $\frac{1}{2}$, $\frac{3}{2}$ have squares $0, 1, \frac{1}{4}, \frac{9}{4} \ne -\frac{5}{4}$ in characteristic zero; so R has six distinct roots over an algebraic closure. For γ we evaluate at the four rational roots,

$$\gamma(0) = \tfrac{15}{8}, \qquad \gamma(-1) = -\tfrac{135}{8}, \qquad \gamma\left(\tfrac{1}{2}\right) = -\tfrac{9}{4}, \qquad \gamma\left(\tfrac{3}{2}\right) = \tfrac{105}{4},$$

all nonzero. At a root of the quadratic factor we reduce γ modulo $w^2 + \frac{5}{4}$: using $w^2 = -\frac{5}{4}$,

$$\gamma(w) = \tfrac{25}{2}w - 10,$$

which vanishes only at $w = \frac{4}{5}$, and $(\frac{4}{5})^2 = \frac{16}{25} \ne -\frac{5}{4}$. So γ is nonzero at all six roots, and Proposition 2.1 turns the six roots into six fiber points. Four of them are the points of Theorem 6.3, whose w_1-coordinates are the four rational roots; the other two lie over the roots $\pm\frac{\sqrt{-5}}{2}$ of the quadratic factor, so the bijection of Proposition 2.1, being given by formulas over $\mathbb{Q}$ and applied to a rational target, puts their coordinates in $\mathbb{Q}(\sqrt{-5})$; the nontrivial automorphism of $\mathbb{Q}(\sqrt{-5})$ over $\mathbb{Q}$ then interchanges the two. □

Remark 6.5. The two non-rational preimages can be written out over $\mathbb{Q}(\sqrt{-5})$; for the sibling target $(1, -\frac{33}{8}, 0, 0, 6)$, which has the same four rational roots, we computed them in exact $\mathbb{Q}(\sqrt{-5})$ arithmetic, the first coordinate being $-\frac{32}{945} \mp \frac{4}{189}\sqrt{-5}$. That computation lies outside the formal development and no statement above uses it.

Theorems 6.1, 6.3 and 6.4 together give a complete picture of one target of F_6 and a uniform constraint on all of them: the fiber over v^* has exactly six points with the maximal rational part four, and no target anywhere has six real points.

7. The published data, reproduced

None of the results above would mean anything if the maps we work with were not the maps of [5]. We therefore rebuilt F_4 and F_5 inside a proof assistant from the construction data of Section 2 — surfaces, direction fields and stage constants — using a sparse multivariate polynomial layer over $\mathbb{Q}$, and re-derived the component polynomials rather than transcribing them. The following two propositions are assertions of [5], verified here as polynomial identities in four indeterminates rather than at sample points; they are not new results.

Proposition 7.1. *Let $C = \gamma x$ and let S_1, S_2, S_3 be the sweep components of Section 2. Then, as identities in $\mathbb{Q}[x, y, z, t]$,*

$$C \mid S_1, \quad C^2 \mid S_2, \quad C \mid S_3 \quad \textit{for } F_4, \qquad C \mid S_1, \quad C^2 \mid S_2, \quad C^3 \mid S_3 \quad \textit{for } F_5,$$

with quotients of degrees $11, 12, 21$ and $12, 14, 16$ and of $32, 39, 114$ and $24, 34, 54$ terms respectively. In particular F_4 and F_5 are polynomial maps with component degrees $4, 11, 12, 21$ and $3, 12, 14, 16$.

Proposition 7.2. *The 4×4 determinants of the formal Jacobian matrices of the component tuples pinned by Proposition 7.1 are the constant polynomials*

$$\det J(F_4) \equiv -\tfrac{44}{9}, \qquad \det J(F_5) \equiv \tfrac{160}{29}.$$

In [5] these are asserted with the remark that all polynomial identities "were verified by exact rational arithmetic; the verification scripts are available from the author", the constancy of $\det J(F_4)$ being additionally "confirmed at random rational points" and that of $\det J(F_5)$ "confirmed by exact evaluation at random rational points". Proposition 7.1 is what makes the components trustworthy: they are pinned by the construction rather than transcribed, and Proposition 7.2 is then a symbolic determinant of the pinned data. The same route does not reach F_6 and F_7: the five components of F_6 have $6, 342, 421, 507, 904$ terms of degree up to 44 in five variables, so a Laplace expansion of $\det J(F_6)$ requires on the order of 10^{10} monomial products, and the expansion of the components of F_7 — whose last four [5] reports to have "between 14,000 and 26,000 terms" — did not complete within 1800 seconds in our implementation. We record instead that the affordable substitutes are the chain factors $\det J(S) = \gamma$ and $\det(\text{stage}) \in \{-290, 119377\}$, both 4×4 determinants in four variables, which are what the proof in [5] multiplies together.

For F_7 we verified the closed-form solution of the transverse system, which [5] checked at sample points.

Proposition 7.3. *With $H_3 = w_2 + w_1^3 + w_1^2 w_3$ and $H_4 = w_3 + w_1^4$ and with p_1, p_2 as in (4), the identities*

$$H_3\big(w_1, p_1, p_2\big) = Y_3 - 2Y_2 w_1 + Y_1 w_1^2, \qquad H_4\big(w_1, p_1, p_2\big) = Y_4 - 3Y_2 w_1^2 + 2Y_1 w_1^3$$

hold in $\mathbb{Q}[Y_1, Y_2, Y_3, Y_4, w_1]$; and the specialisation R_0 of (4) at $Y = 0$ factors as

$$R_0(w_1) = w_1^5(w_1 - 1)\big(w_1^6 + w_1^5 + w_1^4 - w_1^3 - w_1^2 - w_1 + 1\big).$$

The first two identities are the case $n = 5$ of the univariate fiber theory of [5]; there the corresponding identity is reported as "checked at 150 random (Y, w_1)". The second is the factorisation of the fiber polynomial of F_7 over the axis.

Beyond these, we reproduced from the construction data of [5], in exact rational arithmetic: the component degrees and term counts of all four maps; the tangency quintic (1) and its printed expansion; all eleven coefficient polynomials of the degree-ten resultant (2), coefficient by coefficient, after the rescaling used there; the threefold $X_1, \ldots, X_4$ of F_6 recovered from its potentials; the fiber sextic (3) and the special factorisation $R_0 = w_1^2(w_1 - 1)(2w_1^3 - 1)$; and the degree-twelve identity (4) as an identity in (Y, w_1). Nothing failed to reproduce.

8. Verification

The certificates on which the statements above rest have been formally verified in Lean 4 against *Mathlib*, in three modules of a repository that is private at the time of writing; repository reference to be added at submission. The development contains no `sorry`. What is checked in the kernel, with no axioms beyond `propext`, `Classical.choice` and `Quot.sound`, is: the five exact rational evaluations of Theorem 4.1, carried out by `norm_num` on the division-free form of Remark 2.2, including the degree-21 component; the Bézout identity (6) and the factorisation (7), both stated over an arbitrary field of characteristic zero, together with the implication that constitutes Theorem 5.4; the sum-of-squares argument of Theorem 5.1 over $\mathbb{R}$; the Newton identities and power-sum estimates of Theorem 6.1, again over $\mathbb{R}$, with a separate polynomial identity exhibiting the Vieta expansion that its three hypotheses encode; and the polynomial content of Theorem 6.4 — the factorisation of R, the distinctness of its six roots, the four evaluations of γ, the reduction $\gamma \equiv \frac{25}{2}w - 10$ modulo the quadratic factor, and the conclusion that γ vanishes at no root. What is delegated to compiled evaluation, through Lean's `native_decide`, is exactly two things: the twenty rational evaluations of Theorem 6.3, whose arguments have height up to $1.4 \cdot 10^{25}$; and the polynomial-identity checks of Propositions 7.1 and 7.2, which run inside a sparse multivariate polynomial layer over $\mathbb{Q}$ written for the purpose (exponent vectors packed in base 128, like-term collection through a hash map, formal partial derivatives, Laplace expansion along the first row), a module that checks in 11 seconds. What is *not* formalised, and is done by hand here, is the passage between those certificates and the geometry: Proposition 2.1, which we quote from [5]; the reading of Vieta off the printed coefficients in Lemma 3.1, which supplies the hypotheses of the two reality theorems; the coefficient comparisons of Corollary 5.2 and Remark 5.3; the deduction of an empty fiber and of Corollary 5.6 from the implication of Theorem 5.4; and Corollaries 4.2 and 6.2 together with the last sentence of Theorem 6.4, each of which is Proposition 2.1 applied to a kernel-checked root statement. Negative controls accompany each group of statements, since a fiber certificate that proves nothing looks much like one that proves something: displacing one coordinate of a preimage in Theorem 4.1 or Theorem 6.3 breaks the equation $C(p) = 1$; γ is not identically zero for F_5 ($\Gamma(0,0) = 23$) and the two equations of Theorem 5.4 are not vacuous ($\mathcal{A}(0,0) \neq 0$), so that theorem is neither about an empty hypothesis set nor about an identically vanishing conclusion; γ and R are not constant along the w_1-line in Theorem 6.4 ($\gamma(1) = -\frac{7}{8}$, $R(1) = -\frac{9}{4}$), two controls which caught two mistaken hand-written constants on their first run; the determinants of Proposition 7.2 are checked to be *not* equal to nearby constants, so that F_4 and F_5 are genuinely different maps and the constancy test is not returning "true" blindly; and the polynomial layer itself is controlled by the facts that its formal derivative is not identically zero and that a Jacobian matrix with a repeated row has determinant zero. Finally, every numerical value quoted above was first produced by an independent implementation in Python and the formal statement was then written against it, and the reproduction of the published data described in Section 7 was run before any of the new statements was formulated.

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