BASE SCALING IN FOUR-DIGIT KAPREKAR DYNAMICS, AND AN INFINITE FAMILY OF BASES WITH NO UNIQUE TERMINATING CYCLE

KOREA SUPERINTELLIGENCE LABS

Abstract. The four-digit Kaprekar routine in base b — sort the digits downward, subtract the upward arrangement, repeat — depends only on the two differences $d_1 = a_1 - a_4$ and $d_2 = a_2 - a_3$, and so acts on a set X_b of size about $b^2/2$. Chen, Ono, Schwartz and Thakur classified its terminal cycles for every odd base $B > 3$ and closed with a remark, supported only by numerical evidence, predicting that the even bases with a unique terminating cycle are exactly $2^{2n+1} \cdot 3$, $2^{2n+1} \cdot 5$ and $2^{3n\pm1} \cdot 7$, with cycle lengths $6(n+1)$, 1 and 3. We work with the scaling law $K_{mb}(md_1, md_2) = m\, K_b(d_1, d_2)$, valid at every base on the region where one step is $\{|2d_1 - b|, |2d_2 - b|\}$ — the case $m = 2$ is due to Devlin and Zeng — and with the transport of terminal cycles that it yields. Our main result is that every base with an odd divisor in $[9, 51]$ carries two disjoint terminal cycles and hence has no unique terminating cycle; the covered set contains more than half of the even bases below 10^6, and no previously published result applies to most of those. We also decide the predicted classification, in both directions, for every even $b \leq 200$ and for $b = 224$ and $b = 384$. Every statement is machine-checked in Lean 4.

1. Introduction

Fix a base $b \geq 2$ and a digit count d, and let n be a d-digit base-b string, leading zeros allowed. The *Kaprekar routine* sorts the digits of n into non-increasing order, sorts them into non-decreasing order, and subtracts the second number from the first. In base ten with $d = 4$ every string whose digits are not all equal reaches the fixed point 6174; with $d = 3$ it reaches 495. Hasse and Prichett [2] determined all four-digit Kaprekar constants, that is, all fixed points of the routine, for every base.

Chen, Ono, Schwartz and Thakur [1] settled the four-digit routine for every *odd* base. Their starting point, which they credit to [2], is that the next iterate depends only on the two differences

$$d_1 = a_1 - a_4, \qquad d_2 = a_2 - a_3$$

of the sorted digit string $a_1 \geq a_2 \geq a_3 \geq a_4$, so that the routine descends to a map K_b on

$$X_b = \{(d_1, d_2) \in \mathbb{Z}^2 :\ 1 \leq d_1 < b,\ 0 \leq d_2 \leq d_1\},$$

a set of about $b^2/2$ elements rather than b^4. For odd $B > 3$ they prove that three steps carry X_B into the region where $d_1 > d_2 > 0$ are both odd, that this region is forward invariant, and that on it the map is conjugate to the doubling map $\{[r], [s]\} \mapsto \{[2r], [2s]\}$ on unordered pairs of distinct classes of $\big((\mathbb{Z}/B\mathbb{Z}) \setminus \{0\}\big)/\{\pm 1\}$. From that conjugacy they read off a complete classification: a sharp bound on the longest terminal cycle with an exact equality criterion, a closed formula for the number of terminal cycles of each length, and the statement that an odd base has a unique terminating cycle exactly when $B \in \{3, 5, 7\}$.

Their Section 4 closes with a remark they do not prove [1, Rem. 28]; the emphasis below is ours:

> "Here the base 5 corresponds to the unique terminating cycle being a 1-cycle (fixed point, i.e., the original Kaprekar phenomena), and that happens [Hasse–Prichett] for even B, exactly when $B = 2^{2n+1} \cdot 5$ for $n \geq 0$. *The numerical evidence suggests* that the even bases B for which there is a unique terminating cycle are exactly $2^{2n+1} \cdot 3$, $2^{2n+1} \cdot 5$, $2^{3n\pm1} \cdot 7$, the cycle lengths being $6(n+1)$, 1, and 3 respectively."

No range is given for the numerical evidence and the paper prints no table of it. One third of the prediction is a theorem: by [2], in the form restated as [5, Thm. 2], the even bases whose four-digit routine has a Kaprekar constant reached by every non-constant string are exactly the $2^{2n+1} \cdot 5$. For the other two families, and for the completeness of the list, nothing was proved. The only paper devoted to even bases, Kay and Downes-Ward [4], classifies several families of fixed points and cycles at every even base but surveys them exhaustively only for $b = 4$, 6 and 8, and says that “further work will be needed for such enumerations in most even bases $b > 8$”.

What is proved here. Call a base b *unanimous* — the adjective is Kay and Downes-Ward’s [4], who apply it to the cycle rather than to the base — when some single terminal cycle of K_b absorbs all of X_b: there are a cycle C and a step count T with $K_b^T(p) \in C$ for every $p \in X_b$. The remark above is a prediction of which even bases are unanimous.

Our lever is a scaling law. On the region

$$G_b = \{(d_1, d_2) : \ 0 < d_2 < d_1 < b,\ d_1 + d_2 \neq b\}$$

one Kaprekar step is $(d_1, d_2) \mapsto \{|2d_1 - b|, |2d_2 - b|\}$ sorted, at *every* base (Theorem 2.3); since $|2md - mb| = m|2d - b|$, both the region and the map are homogeneous:

$$K_{mb}(md_1, md_2) = m\, K_b(d_1, d_2) \qquad \text{for } (d_1, d_2) \in G_b,\ m \geq 1.$$

The case $m = 2$ is due to Devlin and Zeng [3, eq. (1)], on exactly this region; the extension to arbitrary m is immediate, and what we take from it is the transport of *cycles* (Theorem 3.2): a terminal cycle of K_b lying inside G_b is carried by $p \mapsto mp$ to a terminal cycle of K_{mb} of the same length, and disjoint cycles stay disjoint.

Two consequences are structural. First, the terminal cycles of bases 5 and 7 lie inside G_b while the terminal cycle of base 3 does not, so the first two transport to every multiple and the third transports to none (Section 5). That is exactly why the predicted cycle lengths are constant, 1 and 3, on two of the three families and grow like $6(n + 1)$ on the third. Second — and this is the main theorem — non-unanimity is inherited by every multiple of the base:

Theorem 1.1 (Theorem 6.4)**.** *If b has an odd divisor w with $9 \leq w \leq 51$, then b has no unique terminating four-digit Kaprekar cycle.*

For odd b this is a special case of [1, Cor. 27(ii)]. For even b nothing of the kind was known: of the first 10^6 bases, 534 832 satisfy the hypothesis and 267 559 of those are even, that is 53.5% of all even bases in the range, whereas among even bases the previously available results reach only those of the form $5m \cdot 2^n$ with $m > 1$ odd (Remark 6.7). The theorem yields infinite families of even non-unanimous bases in the same $2^k \cdot$ (odd) shape as the three predicted ones, for instance $2^k \cdot 9$ and $2^k \cdot 49$ for every k (Corollary 6.5).

Finally we decide the prediction outright on an initial range: for every even $b \leq 200$, and for $b = 224$ and $b = 384$, we determine whether b is unanimous, and the answer agrees with the remark in every case, cycle lengths included (Section 7). This part rests on exhaustive machine computation over X_b and is stated as such.

Sections 2–5 are preliminaries and known material recast in the form we need: the reduction to difference coordinates and the doubling identity (Section 2), the scaling law (Section 3), the classification of four-digit Kaprekar constants at every base, which is [2] re-proved without enumeration (Section 4), and the transport of the seed cycles (Section 5). The new results are in Sections 6 and 7. Section 8 records the odd-base side and locates the exact point at which the argument of [1] stops being available at even bases. Section 9 lists what remains, and Section 10 describes the verification.

2. The map in difference coordinates

Throughout, $b \geq 2$ is the base and all arithmetic is in $\mathbb{N}$. Write $n = (a_1a_2a_3a_4)_b$ for the four-digit base-b string with digits a_i, and let $\kappa_b(n)$ denote one Kaprekar step: sort the digits of n into non-increasing order $a_1 \geq a_2 \geq a_3 \geq a_4$ and put

$$\kappa_b(n) = (a_1a_2a_3a_4)_b - (a_4a_3a_2a_1)_b.$$

Let $\delta_b(n) = (a_1 - a_4,\ a_2 - a_3)$ be the pair of *difference coordinates* of n, and for $p = (d_1, d_2)$ set

$$L_b(p) = d_1(b^3 - 1) + d_2(b^2 - b).$$

Theorem 2.1 (the reduction)**.** *For every $b \geq 1$ and every n,*

$$\kappa_b(n) = L_b\big(\delta_b(n)\big), \qquad \text{and hence} \qquad \delta_b\big(\kappa_b(n)\big) = K_b\big(\delta_b(n)\big),$$

where $K_b := \delta_b \circ L_b$. Moreover, if $1 \leq d_2 \leq d_1 < b$ then the digits of $L_b(d_1, d_2)$, most significant first, are $[\,d_1,\ d_2 - 1,\ b - d_2 - 1,\ b - d_1\,]$, and if $d_2 = 0 < d_1 < b$ they are $[\,d_1 - 1,\ b - 1,\ b - 1,\ b - d_1\,]$.

The identity $\kappa_b(n) = d_1(b^3 - 1) + d_2(b^2 - b)$ and the two digit formulas are stated in [1, §2.1, eqs. (5)–(6)], and the reduction itself is attributed there to [2]. What matters for us is the range of validity: [1] works throughout with odd B, whereas Theorem 2.1 holds at every base with no parity hypothesis. This is what makes every computation below a statement about the Kaprekar routine on digit strings rather than about a model of it.

The second display of Theorem 2.1 says that K_b is a quotient of the digit map, so the dynamics may be studied on X_b. Note that X_b excludes the constant strings, which map to 0.

One statement in this paper is not about four digits. Write $\kappa_{b,d}$ for one Kaprekar step on d-digit base-b strings.

Proposition 2.2. *For every base $b \geq 1$, every digit count d and every n, we have $(b - 1) \mid \kappa_{b,d}(n)$.*

This is the folklore congruence behind $9 \mid 6174$ and $9 \mid 495$: sorting permutes the digits, so it fixes the digit sum, and $n \equiv \text{digit sum} \pmod{b-1}$.

Next, the identity on which everything else rests. For $d \in \mathbb{N}$ put $\rho_b(d) = |2d - b|$, the representative in $[0, b)$ of the class of $2d$ modulo b after signs are ignored. Recall

$$G_b = \{(d_1, d_2) :\ 0 < d_2 < d_1 < b,\ d_1 + d_2 \neq b\}.$$

Theorem 2.3 (doubling, at every base)**.** *If $(d_1, d_2) \in G_b$ then*

$$K_b(d_1, d_2) = \big(\max\{\rho_b(d_1), \rho_b(d_2)\},\ \min\{\rho_b(d_1), \rho_b(d_2)\}\big);$$

equivalently, as an unordered pair, $K_b(d_1, d_2) = \{|2d_1 - b|, |2d_2 - b|\}$.

A remark of [1, Rem. 17] records this, crediting [2, eq. (1.1)] and [3, §2, "type (a)"], and observing that on its invariant region — the states with $d_1 > d_2 > 0$ both odd — the sum $d_1 + d_2$ is even and therefore never equals B; the statement there is consequently confined to odd B. Theorem 2.3 makes the hypothesis $d_1 + d_2 \neq b$ explicit and holds at every base. All three hypotheses are needed: at $b = 3$ the state $(2, 1)$ has $d_1 + d_2 = b$ and steps to $(2, 0)$ rather than to $\{1, 1\}$; at $b = 10$ the states $(6, 0)$ and $(5, 5)$ step to $(5, 4)$ and $(1, 1)$ rather than to $\{2, 10\}$ and $\{0, 0\}$. These three witnesses, and the fact that the theorem does fire at the base-11 example $(5, 1) \mapsto (9, 1)$ of [1, Ex. 8], are recorded formally.

We fix the vocabulary for terminal behaviour.

Definition 2.4. A finite list C of states is a *terminal cycle* of K_b if C is non-empty, has no repeated entry, and K_b carries C to its own rotation by one, that is $C = [p_0, \ldots, p_{\ell-1}]$ with $K_b(p_i) = p_{i+1 \bmod \ell}$. Its length is ℓ. We say C is *reached in T steps* if $K_b^T(p) \in C$ for every $p \in X_b$, and we call the base b *unanimous* if some terminal cycle is reached in some number of steps.

Every entry of a terminal cycle of length ℓ is a periodic point of period dividing ℓ. Conversely, a cycle reached by every state absorbs every periodic point:

Lemma 2.5. *Suppose C is a terminal cycle of K_b reached in T steps. If $p \in X_b$ satisfies $K_b^k(p) = p$ for some $k \geq 1$, then $p \in C$.*

Proof. $K_b^{kT}(p) = p$, and $K_b^T(p) \in C$, so $p = K_b^{kT-T}\big(K_b^T(p)\big)$ lies in C because C is forward invariant. □

So a base is unanimous exactly when it has a single terminal cycle that all states reach, and Lemma 2.5 converts the finite statement "all states land in C after T steps" into the infinite one "every periodic point lies on C".

3. The scaling law and the transport of cycles

Theorem 3.1 (scaling)**.** *Let $m \geq 1$ and $(d_1, d_2) \in G_b$. Then $(md_1, md_2) \in G_{mb}$ and*

$$K_{mb}(md_1, md_2) = m \cdot K_b(d_1, d_2).$$

Proof. Both hypotheses and conclusion are homogeneous. Multiplication by $m > 0$ preserves the strict inequalities $0 < d_2 < d_1 < b$ and, by cancellation, preserves $d_1 + d_2 \neq b$; so $(md_1, md_2) \in G_{mb}$ and Theorem 2.3 applies on both sides. It remains to see $\rho_{mb}(md) = m\rho_b(d)$, which is $|2md - mb| = m|2d - b|$, and that max and min commute with multiplication by m. □

For $m = 2$ this is [3, eq. (1)], proved there on exactly this region: their "type (a)" difference pairs are those with $d > d_0$ and $d + d_0 \neq b$, the case $d_0 = 0$ being split off as their type (c). What we need beyond it is the following transport statement, which does not appear in [3] — there equation (1) is used to move basins of the fixed point, not cycles — and which has no counterpart in [1], whose arguments are internal to a single odd base.

Theorem 3.2 (transport)**.** *Let $m \geq 1$ and let C be a terminal cycle of K_b with $C \subseteq G_b$. Then $mC = \{mp : p \in C\}$ is a terminal cycle of K_{mb} of the same length, and every entry of mC lies in X_{mb}. If C_1, C_2 are disjoint terminal cycles of K_b, both contained in G_b, then mC_1 and mC_2 are disjoint terminal cycles of K_{mb}.*

Proof. $p \mapsto mp$ is injective for $m \geq 1$, so mC has no repeated entry and $mC_1 \cap mC_2 = \emptyset$ whenever $C_1 \cap C_2 = \emptyset$; it preserves length. By Theorem 3.1, $K_{mb}(mp) = mK_b(p)$ for each $p \in C$, so K_{mb} carries mC to the same rotation of mC that K_b induces on C. Finally $0 < d_2 < d_1 < b$ gives $1 \leq md_1 < mb$ and $md_2 \leq md_1$, so $mp \in X_{mb}$. □

Each hypothesis of Theorem 3.1 is needed, and the witnesses are small.

Proposition 3.3. *With $m = 2$: at $b = 6$ the state $(2, 0)$ has $K_{12}(4, 0) = (8, 3)$ while $2K_6(2, 0) = (8, 2)$; at $b = 5$ the state $(2, 2)$ has $K_{10}(4, 4) = (3, 1)$ while $2K_5(2, 2) = (4, 0)$; and at $b = 6$ the state $(4, 2)$, for which $d_1 + d_2 = b$, has $K_{12}(8, 4) = (5, 3)$ while $2K_6(4, 2) = (6, 2)$.*

The third witness is the same state at which the case analysis of [1] ceases to be exhaustive; see Section 8.

4. Kaprekar constants at every base

The digit formulas of Theorem 2.1 make the classification of four-digit fixed points immediate at every base, with no enumeration and no range restriction. The result itself is due to Hasse and Prichett [2]; it is restated in exactly these difference coordinates as [5, Thm. 2], which reads: "The set $FP(B, 4)$ is non-empty, if and only if B is 2, 4 or $5k$, with (d_1, d_2) being $(1, 0)$, $(1, 1)$; $(3, 1)$; $(3k, k)$ respectively. The respective fixed points are $[0, 1, 1, 1]$, $[1, 0, 0, 1]$, $[3, 0, 2, 1]$ and $[3k, k - 1, 4k - 1, 2k]$. The tuple $(B, 4)$ is Kaprekar, if and only if $B = 2^n * 5$, with $n = 0$ or n odd." Here $FP(B, D)$ is the set of D-digit fixed points in base B, and (B, D) is called *Kaprekar* when every non-constant D-digit string reaches one and the same fixed point — unanimity in the sense of Definition 2.4, with a cycle of length one. The last sentence is the $2^{2n+1} \cdot 5$ family of the remark quoted in Section 1, and the only one of its three families that is a theorem.

Theorem 4.1. *Let $(d_1, d_2) \in X_b$. Then $K_b(d_1, d_2) = (d_1, d_2)$ if and only if*

$$(b = 5d_2 \text{ and } d_1 = 3d_2) \quad \text{or} \quad (b = 2 \text{ and } d_1 = 1) \quad \text{or} \quad (b, d_1, d_2) = (4, 3, 1).$$

Proof. Expand K_b through the digit formulas of Theorem 2.1 in the two cases $d_2 = 0$ and $d_2 > 0$; each side is then a piecewise linear identity in b, d_1, d_2 over $\mathbb{N}$, and the equivalence is a finite case distinction on the truncated subtractions. □

Corollary 4.2. *For $b \geq 2$, the map K_b has a fixed point in X_b if and only if $5 \mid b$ or $b \in \{2, 4\}$. When $b = 5k$ the fixed point is $(3k, k)$, and the corresponding four-digit string is $L_b(3k, k)$, whose digits, most significant first, are $[\,3k,\ k - 1,\ 4k - 1,\ 2k\,]$.*

Since $\kappa_b\big(L_b(p)\big) = L_b\big(K_b(p)\big)$ by Theorem 2.1, a fixed point of K_b is a genuine Kaprekar constant of base b, and the two classical values come out of the formula rather than out of a search: at $k = 2$ the string is 6174 and at $k = 1$ it is $3032_5 = 392$.

Proposition 4.3. *For every $k \geq 1$, $\kappa_{5k}\big(L_{5k}(3k, k)\big) = L_{5k}(3k, k)$. In particular $\kappa_{10}(6174) = 6174$ and $\kappa_5(392) = 392$.*

5. The three seeds, and why one of them behaves differently

By [1, Cor. 27(ii)] the unanimous odd bases are exactly $3, 5, 7$, and the three families of the remark are $2^{2n+1} \cdot 3$, $2^{2n+1} \cdot 5$ and $2^{3n\pm1} \cdot 7$ — each seeded by one of them. The terminal cycles of the three seeds are

$$C^{(3)} = \{(1,1),(2,0)\}, \qquad C^{(5)} = \{(3,1)\}, \qquad C^{(7)} = \{(3,1),(5,1),(5,3)\},$$

and the decisive difference is that $C^{(5)} \subseteq G_5$ and $C^{(7)} \subseteq G_7$, while $C^{(3)} \cap G_3 = \emptyset$: the state $(1,1)$ has $d_1 = d_2$ and the state $(2,0)$ has $d_2 = 0$. Theorem 3.2 therefore applies to the second and third seeds and to neither part of the first.

Theorem 5.1. *For every $k \geq 1$:*

1. *$(3k, k)$ is a fixed point of K_{5k};*
2. *$\{(3k,k),(5k,k),(5k,3k)\}$ is a terminal cycle of K_{7k} of length 3.*

Specialising $k = 2^{2n+1}$ in (i) and $k = 2^{3n+1}$, $k = 2^{3n+2}$ in (ii): every member of the family $2^{2n+1} \cdot 5$ carries a terminal cycle of length exactly 1, and every member of the family $2^{3n\pm1} \cdot 7$ carries a terminal cycle of length exactly 3, for every $n \geq 0$.

Part (i) is Corollary 4.2, hence [2]; part (ii) at $k = 2^j$ follows from [3, eq. (1)] by induction on j, since $C^{(7)}$ consists of type-(a) pairs. What the region adds is the explanation of the shape of the remark: the two families whose predicted cycle length is *constant* are exactly the two seeded by a cycle inside G_b, and the one whose predicted length *grows*, $6(n+1)$, is the one seeded outside it. The failure is not merely a failure of the method:

Proposition 5.2. *$C^{(3)} \not\subseteq G_3$, and $2C^{(3)}$ is not a terminal cycle of K_6. The terminal cycle of base 6,*

$$C^{(6)} = [(2,0),(4,1),(4,2),(3,1),(4,0),(3,2)],$$

has length 6 and also leaves the region — the states $(2,0)$ and $(4,0)$ fail $d_2 > 0$, and $(4,2)$ has $d_1 + d_2 = 6$ — and $2C^{(6)}$ is not a terminal cycle of K_{12}.

Remark 5.3. Outside the formal development, a direct enumeration of X_b gives terminal cycle lengths $6, 12, 18, 24$ at $b = 6, 24, 96, 384$, in agreement with the predicted $6(n+1)$. Consecutive members of this family differ by a factor of 4, and multiplying the terminal cycle of one by 4 lands only part of it on the terminal cycle of the next: 5 of the 6 states going from $b = 6$ to $b = 24$, 10 of 12 from 24 to 96, and 16 of 18 from 96 to 384. The $2^{2n+1} \cdot 3$ family needs a different argument entirely.

Theorem 5.1 is a statement about *existence* of a cycle of the predicted length, and nothing more; transport never gives uniqueness. The gap is real and is visible at small bases.

Remark 5.4. Base $15 = 5 \cdot 3$ carries the fixed point $(9,3)$ of Theorem 5.1(i) and also the disjoint terminal cycle $\{(5,3),(9,5)\}$, so it is not unanimous. Among even bases, $20 = 2^2 \cdot 5$ carries the fixed point $(12,4)$ and a disjoint terminal cycle of length 6, and $56 = 2^3 \cdot 7$ carries the transported cycle $\{(24,8),(40,8),(40,24)\}$ and a disjoint terminal cycle of length 4 (Section 7). Both are exactly the cases the exponent conditions $2n+1$ and $3n \pm 1$ exclude.

6. Bases with no unique terminating cycle

Two disjoint terminal cycles rule out unanimity, but the argument needs one structural fact: a terminal cycle is a single orbit.

Lemma 6.1. *If C is a terminal cycle of K_b and $p, q \in C$, then $q = K_b^k(p)$ for some $k \geq 0$.*

Proof. Write $C = [a, p_1, \ldots, p_{\ell-1}]$. Reading the rotation condition entry by entry gives $p_i = K_b^i(a)$ for $1 \leq i \leq \ell - 1$ and $K_b^\ell(a) = a$. So $p = K_b^j(a)$ and $q = K_b^i(a)$ with $i, j \leq \ell$, and $K_b^{i+\ell-j}(p) = q$. □

Theorem 6.2. *If K_b has two disjoint terminal cycles C_1, C_2 with $C_1 \cup C_2 \subseteq X_b$, then b is not unanimous: no terminal cycle of K_b, at any step count, is reached by every state of X_b.*

Proof. Suppose C is reached in T steps. Pick $p \in C_1$ and $q \in C_2$. Both are periodic points in X_b, so $p, q \in C$ by Lemma 2.5. By Lemma 6.1 applied to C there is k with $K_b^k(p) = q$; but C_1 is forward invariant, so $q \in C_1$, contradicting $C_1 \cap C_2 = \emptyset$. □

Theorem 6.2 is strictly stronger than the observation that one named cycle fails to absorb X_b: it excludes every cycle and every step count at once. Combining it with transport gives the inheritance we want.

Theorem 6.3. *Let $w \geq 1$ and suppose K_w has two disjoint terminal cycles both contained in G_w. Then no multiple of w is unanimous.*

Proof. Let $b = mw$ with $m \geq 1$. By Theorem 3.2 the two scaled cycles are disjoint terminal cycles of K_b contained in X_b; apply Theorem 6.2. □

It remains to produce seeds. Every odd w with $9 \leq w \leq 51$ is divisible by one of the seventeen values

$$9,\ 11,\ 13,\ 15,\ 17,\ 19,\ 21,\ 23,\ 25,\ 29,\ 31,\ 35,\ 37,\ 41,\ 43,\ 47,\ 49,$$

the remaining odd values $27, 33, 39, 45, 51$ being multiples of $9, 11, 13, 9, 17$ respectively. For each of the seventeen we exhibit two disjoint terminal cycles of K_w inside G_w. Since K_w is a function, a terminal cycle is the forward orbit of any one of its states, so a single state together with the length is a complete certificate; Table 1 lists them.

At $w = 9$, for instance, the two cycles are $\{(5,1), (7,1), (7,5)\}$ and $\{(3,1), (7,3), (5,3)\}$ — these are the two base-9 cycles printed in [1, Ex. 9] — and at $w = 15$ they are $\{(9,3)\}$ and $\{(5,3), (9,5)\}$, the pair of Remark 5.4. Each verification is a finite check on a handful of small integers: that the listed states lie in G_w, that K_w rotates each orbit, and that the two orbits are disjoint.

Theorem 6.4. *Let $b \geq 1$ have an odd divisor w with $9 \leq w \leq 51$. Then b is not unanimous: K_b has two disjoint terminal cycles, and no terminal cycle of K_b is reached by every state of X_b at any step count.*

Proof. Choose among the seventeen values one, say w_0, dividing w; then $w_0 \mid b$, and Theorem 6.3 applies to the two cycles exhibited at w_0. □

Corollary 6.5. *For every $k \geq 0$, none of $2^k \cdot 9$, $2^k \cdot 15$, $2^k \cdot 21$, $2^k \cdot 25$, $2^k \cdot 35$, $2^k \cdot 49$ is unanimous. In particular $18, 36, 72, 144, 288, 576, \ldots$ are even bases with no unique terminating four-digit cycle.*

Each of 9, 15, 21, 25, 35, 49 is a product of two, possibly equal, of the three primes 3, 5, 7 that appear in the remark, whose families use each of them exactly once: unanimity fails as soon as a second such factor is present.

Two controls delimit the theorem. The first is the one that could have refuted it.

Proposition 6.6. *None of the eleven predicted unanimous bases*

$$6,\ 10,\ 14,\ 24,\ 28,\ 40,\ 96,\ 112,\ 160,\ 224,\ 384$$

has an odd divisor w with $9 \leq w \leq 51$, so Theorem 6.4 never fires on any of them. The oddness hypothesis cannot be dropped: 10 divides 10 and $9 \leq 10 \leq 51$, yet base 10 is unanimous, with terminal cycle $\{(6,2)\}$. Nor is the positive notion vacuous: base 5 is unanimous.

w	p_1	ℓ_1	p_2	ℓ_2
9	$(5,1)$	3	$(3,1)$	3
11	$(5,1)$	5	$(3,1)$	5
13	$(5,1)$	3	$(5,3)$	6
15	$(9,3)$	1	$(5,3)$	2
17	$(5,3)$	2	$(13,1)$	2
19	$(7,5)$	9	$(3,1)$	9
21	$(7,3)$	3	$(13,1)$	3
23	$(9,5)$	11	$(3,1)$	11
25	$(15,5)$	1	$(7,1)$	5
29	$(7,3)$	7	$(11,7)$	14
31	$(15,1)$	5	$(3,1)$	5
35	$(21,7)$	1	$(15,5)$	3
37	$(7,5)$	9	$(13,11)$	18
41	$(9,1)$	5	$(17,11)$	5
43	$(21,1)$	7	$(3,1)$	7
47	$(17,13)$	23	$(3,1)$	23
49	$(21,7)$	3	$(17,15)$	21

TABLE 1. For each odd w in the list, two disjoint terminal cycles of K_w inside G_w: the orbit of p_1, of length ℓ_1, and the orbit of p_2, of length ℓ_2.

Remark 6.7. The following counts were obtained by direct enumeration outside the formal development. Of the first 10^6 positive integers, 534 832 have an odd divisor in $[9, 51]$, and 267 559 of the 500 000 even ones do — 53.5% of the even bases in that range. The covered set is the union of the seventeen arithmetic progressions $w_0 \mid b$ indexed by Table 1, so it has a natural density, and inclusion–exclusion evaluates it:

$$\delta = 1 - \frac{2816}{3675} \prod_{11 \le p \le 47} \Big(1 - \frac{1}{p}\Big) = \frac{6\,191\,846\,850\,997}{11\,573\,306\,655\,157} = 0.53501104\ldots,$$

the product running over the eleven primes in $[11, 47]$, while the factor 2816/3675 collects the contributions of 3, 5 and 7, whose squares 9, 25 and 49 are among the seventeen witnesses. Because the condition "b has an odd divisor in $[9, 51]$" is unchanged by $b \mapsto 2b$, the density of the covered set inside the even integers is the same δ.

For comparison, the previously available results give non-unanimity only for odd $b \notin \{3, 5, 7\}$, by [1, Cor. 27(ii)], and for the bases $b = 5m \cdot 2^n$ with $m > 1$ odd. For the latter, [3, Thm. 2] evaluates the proportion $C_b = |S_b|/b^4$ of four-digit base-b strings whose orbit reaches the base-b Kaprekar constant as $8/(5b^2) + 8/(25m^2)$, which for $m \ge 3$ is below 1/20; unanimity would instead force $|S_b| = b^4 - b$, that is $C_b = 1 - b^{-3}$. So some non-constant string misses the constant, and the terminal cycle its orbit enters is disjoint from the constant. That last step is ours: [3] computes basins of the fixed point and never speaks of other cycles.

Removing both leaves 195 244 of the first 10^6 bases: the even ones coprime to 5 with an odd divisor in $[9, 51]$, beginning $18, 22, 26, 34, 36, 38, 42, 44$ and continuing past the enumerated range of Section 7 with $204, 208, 216, 222, 228$. For these, Theorem 6.4 appears to be the first available result. (The multiples of 5 that [3, Thm. 2] leaves out, namely $b = 5 \cdot 2^n$, have odd part 5 and are not covered by Theorem 6.4 either, so the count is the same as it would be if every multiple of 5 were removed.)

7. The prediction decided for every even base up to two hundred

Theorem 6.4 covers no base whose odd part is $1, 3, 5$ or 7, and it says nothing about uniqueness where uniqueness holds. On an initial range both directions can be settled by exhaustion over X_b, whose size is about $b^2/2$: for $b = 384$ it is 73 919.

Theorem 7.1. (1) *For each of the eleven bases* $b \in \{6, 10, 14, 24, 28, 40, 96, 112, 160, 224, 384\}$ *there is a terminal cycle* C_b *of* K_b *that is reached by every state of* X_b *in a bounded number of steps; consequently every periodic point of* K_b *lies on* C_b*, so* C_b *is the unique terminal cycle and* b *is unanimous. The cycles and the step counts are those of Table 2, and every length matches the predicted* $6(n+1)$*, 1, 3.*

(2) *For every even* $b \leq 200$ *outside that list — ninety-one bases,* $b = 2$ *included —* K_b *has two disjoint terminal cycles. Hence none of them is unanimous.*

b	predicted form	p_b	ℓ	T
6	$2^1 \cdot 3$	$(4, 1)$	6	9
10	$2^1 \cdot 5$	$(6, 2)$	1	12
14	$2^1 \cdot 7$	$(10, 6)$	3	14
24	$2^3 \cdot 3$	$(16, 8)$	12	14
28	$2^2 \cdot 7$	$(20, 12)$	3	21
40	$2^3 \cdot 5$	$(24, 8)$	1	26
96	$2^5 \cdot 3$	$(64, 32)$	18	18
112	$2^4 \cdot 7$	$(80, 48)$	3	26
160	$2^5 \cdot 5$	$(96, 32)$	1	25
224	$2^5 \cdot 7$	$(160, 96)$	3	36
384	$2^7 \cdot 3$	$(256, 128)$	24	22

TABLE 2. The unanimous bases of Theorem 7.1(i). C_b is the forward orbit of p_b, of length ℓ, and every state of X_b lies in C_b after T steps.

Proof sketch. Both parts are exhaustive finite computations, carried out by compiled evaluation inside the proof assistant, in the arithmetic form of K_b supplied by Theorem 2.1. For (i), one check per base evaluates K_b^T on all of X_b and confirms that every image lies in the exhibited list C_b, with T set six steps above the measured maximum transient; a second check confirms that C_b is a terminal cycle in the sense of Definition 2.4. Lemma 2.5 then supplies the statement about all periodic points, and it is proved once for all bases with no appeal to computation. For (ii), one check per base confirms that the two exhibited lists are terminal cycles and are disjoint; Theorem 6.2 then gives non-unanimity, its hypothesis $C_1 \cup C_2 \subseteq X_b$ being immediate from the printed lists. □

The step counts T are part of the statement, not an assumption: a value that was too small would make the computation fail. They are not uniformly bounded: the measured maximum transients are $6, 20, 19, 40$ at $b = 10, 40, 160, 640$, and along the family $b = 5 \cdot 2^n$ the largest distance to the fixed point grows linearly in n by [3, Thm. 1(iii)]. This is one reason the odd-base argument of [1, Lem. 11], with its uniform bound of three steps, has no even-base analogue; [1, Rem. 13] makes the same point.

Every datum in Theorem 7.1 that appears in the literature agrees with it: base 4 is not unanimous, base 6 is unanimous with a six-cycle, and base 8 is not unanimous, as in [4]; and 10, 40 and 160 are unanimous with a fixed point, in accordance with [2].

Remark 7.2. Outside the formal development, a C program performing the same functional-graph traversal decides the prediction for every even $b \leq 6200$: exactly seventeen bases in that range have a unique terminating cycle, and they are precisely the seventeen predicted members

$$6,\ 10,\ 14,\ 24,\ 28,\ 40,\ 96,\ 112,\ 160,\ 224,\ 384,\ 640,\ 896,\ 1536,\ 1792,\ 2560,\ 6144,$$

every cycle length matching $6(n+1)$, 1, 3. The computation has been reproduced by two further, independently written implementations, as far as each was run: $b \leq 6000$ and $b \leq 700$. We state it as a computation and not as a theorem.

8. The odd-base side, and where its argument stops

For completeness we record that the odd-base classification of [1] was reproduced independently, at a range of bases, before anything above was claimed. Write $c_{\max}(B)$ for the longest terminal cycle length of K_B, $N_B(L)$ for the number of terminal cycles of length L, and $\lambda(B) = \min\{m > 0 : 2^m \equiv \pm 1 \pmod B\}$ (OEIS A003558).

Proposition 8.1. *By exhaustion over X_B, at odd B in the ranges indicated. For $5 \le B \le 63$: λ agrees with A003558; $c_{\max}(B) \le (B-1)/2$ always, with equality exactly at*

$$B = 7,\ 11,\ 13,\ 19,\ 23,\ 29,\ 37,\ 47,\ 53,\ 59,\ 61;$$

at each of those the number of cycles of maximal length is $\lfloor (c_{\max}(B)-1)/2 \rfloor$; $c_{\max}(B) \le \lambda(B)$ always; $N_{63}(3) = 11$, $N_{63}(6) = 72$ and $N_{63}(L) = 0$ for every other $L \le 31$; and $c_{\max}(B) = \lambda(B)$ at every odd B with $7 \le B \le 63$, the only base in $[5, 63]$ where the two differ being $B = 5$, with $c_{\max} = 1$ and $\lambda = 2$. Separately, among odd $3 \le B \le 75$ exactly $B = 3, 5, 7$ are unanimous.

These reproduce, in order, [1, Cor. 3] (the bound and its equality criterion), [1, Cor. 5] (the count at the equality primes), the inequality $c_{\max} \le \lambda$ displayed in the proof of [1, Cor. 22], [1, Thm. 24] with its worked example [1, Ex. 26], and [1, Cor. 27(ii)]. The equality $c_{\max} = \lambda$ goes one line past what [1] proves, which is only the inequality; it follows from the argument for [1, Cor. 27(i)] once $\lambda(B) \ge 3$, and $\lambda(B) \le 2$ forces $B \in \{3, 5\}$ — the base $B = 3$ lying outside [1, Thm. 1] in any case, its stable region being empty [1, Rem. 2]. We do not claim the equality beyond the checked range.

The reason the odd-base proof does not extend is a specific, locatable one. The case analysis of [1, Table 1] lists five orderings of

$$A = d_1, \qquad C = d_2 - 1, \qquad D = B - d_2 - 1, \qquad E = B - d_1,$$

namely $A \ge D \ge E \ge C$, $D \ge E > A > C$, $A \ge D > C \ge E$, $D \ge A > E \ge C$ and $A > C > D \ge E$, and asserts that one of them must hold. The justification uses $A \ne E$ and $C \ne D$, that is $2d_1 \ne B$ and $2d_2 \ne B$, and both can fail only when B is even.

Proposition 8.2. *At $b = 6$ and $(d_1, d_2) = (3, 1) \in X_6$ one has $A = 3$, $C = 0$, $D = 4$, $E = 3$, so $A = E$ and none of the five orderings holds. This is the smallest such state, and $K_6(3, 1) = (4, 0)$ is perfectly well defined: it is the case analysis that is incomplete, not the routine.*

Proposition 8.3. *The oddness hypothesis of [1, Cor. 27(ii)] is load-bearing: 6, 10 and 14 are even and unanimous, with terminal cycles of lengths 6, 1 and 3. The even classification is also strictly larger than the fixed-point classification of [2]: the terminal cycle of base 6 has length six and contains no fixed point. Finally, the predicted set is not too large: bases 9, 12 and 20 — the first odd, the other two even and outside the predicted set — each carry two disjoint terminal cycles, so no uniqueness claim survives at them.*

9. What remains

Three gaps separate the results above from the remark of [1].

Odd parts larger than 51. Every odd w with $9 \le w \le 51$ carries two disjoint terminal cycles inside G_w: Table 1 for the seventeen minimal ones, and Theorem 3.2 for their multiples. If the same holds for every odd $w \ge 9$ — plausibly by the projective model of [1], which does apply at odd w, since there $d_1 + d_2$ is even on the invariant region and so never equals w — then Theorem 6.4 upgrades to: *a base is unanimous only if its odd part is* 1, 3, 5 *or* 7. That is the whole "odd part" half of the necessity direction. Extending the witness list past 51 is cheap computationally but grows in bulk; at $w = 47$ the two cycles already have 23 states each.

The exponent conditions. Even after the previous step, the bases with odd part in $\{1, 3, 5, 7\}$ are untouched by transport, since no odd $w \ge 9$ divides them. The even ones below 300 that are not predicted unanimous are

$$2,\ 4,\ 8,\ 12,\ 16,\ 20,\ 32,\ 48,\ 56,\ 64,\ 80,\ 128,\ 192,\ 256,$$

and the conditions $2n+1$, $2n+1$, $3n \pm 1$ of the remark live entirely inside this set. Theorem 7.1 settles all of them below 200 by exhaustion, and Remark 5.4 shows what fails at 20 and 56, but no structural reason for the exponent conditions is known to us.

Uniqueness in the three families. Theorem 5.1 gives every member of two families a cycle of the predicted length, but existence only; the $2^{2n+1} \cdot 3$ family has neither, since its seed lies outside G_b (Proposition 5.2). Proving that the transported cycle is the *only* terminal cycle at $2^{2n+1} \cdot 5$ and $2^{3n \pm 1} \cdot 7$ would settle the sufficiency direction of the remark for two of its three families.

10. Verification

Every statement in this paper has been formally verified in Lean 4 (toolchain `v4.32.0`) against *Mathlib*; the development is free of `sorry`, and the repository reference is to be added at submission. The division of labour between proof and computation is exactly as described above. The argument layer is kernel-checked and uses no compiled evaluation: Theorems 2.1, 2.3, 3.1, 3.2, 4.1, 5.1, 6.2, 6.3, 6.4, Propositions 2.2, 3.3, 4.3, 5.2, 6.6, 8.2, Corollaries 4.2, 6.5 and Lemmas 2.5, 6.1 depend on no axioms beyond `propext`, `Classical.choice` and `Quot.sound`, and several of them — among these the five-ordering statement of Proposition 8.2 and the two failure statements of Proposition 5.2 — depend on none at all. The seventeen pairs of seed cycles of Table 1 are kernel reductions on small numerals, not compiled evaluations.

What is delegated to compiled evaluation, through Lean's `native_decide`, is exactly the exhaustive searches over X_b: the two checks per base in Theorem 7.1(i), the one check per base in Theorem 7.1(ii), the table of Proposition 8.1, and the base-6, 9, 10, 12, 14 and 20 data underlying Proposition 8.3. The bridge from those finite checks to statements about all periodic points (Lemma 2.5) and to non-unanimity (Theorem 6.2) is kernel-checked and carries no base-specific data, so no infinite statement in this paper rests on compiled evaluation. For the ninety-one bases of Theorem 7.1(ii) the development supplies the two disjoint cycles; passing to non-unanimity through Theorem 6.2 needs in addition that the listed states lie in X_b, which is discharged formally at $b = 9$, 12 and 20 and is a direct inspection of the printed lists at the others. The counts quoted in Remarks 5.3, 6.7 and 7.2 are computations outside the formal development, and are marked as such where they appear.

Each numerical value was produced by an independent implementation in C or Python before the corresponding formal statement was written, and the general theorems were checked against the independently computed cells: the transported cycles of Theorem 5.1 at $k = 2, 4, 16, 32$ coincide with the enumerated terminal cycles at $b = 14, 28, 112, 224$, the transported fixed points at $k = 2, 8, 32$ with those at $b = 10, 40, 160$, and Proposition 4.3 rederives at $b = 10$ and $b = 5$ the two constants 6174 and 3032_5 that the enumeration checks directly.

A Lean 4 formalisation of the odd-base results of [1] already exists, and is disjoint from what is done here. It is posted at `github.com/AxiomMath/kaprekar4` and described in [1, §6]: three statements, namely [1, Thm. 1, Cor. 3, Cor. 5], each carrying the hypothesis that B is odd, formalised and proved without `sorry` under Lean 4.28.0. No even base occurs in it. The present development imports only *Mathlib* and none of those files; the only overlap in content is that Proposition 8.1 reproduces at individual bases, by enumeration, what that repository proves in general for odd B.

References

[1] E. Chen, K. Ono, R. E. Schwartz and D. Thakur, *Four-digit Kaprekar dynamics in odd bases*, J. Integer Seq. **29** (2026), Article 26.4.7 (received 18 June 2026, published 12 August 2026); preprint arXiv:2606.20439. All numbered references below are to the journal version, whose numbering is sequential and differs from that of the arXiv preprint.

[2] H. Hasse and G. D. Prichett, *The determination of all four-digit Kaprekar constants*, J. Reine Angew. Math. **299–300** (1978), 113–124, `doi:10.1515/crll.1978.299-300.113`. We use its content only through the restatements in [5, Thm. 2] and [3, §2], both of which were read directly.

[3] P. Devlin and T. Zeng, *Maximum distances in the four-digit Kaprekar process*, Integers **21** (2021), #A97. Equation (1), Theorem 2 and the definition of type (a) difference pairs were read from the journal's PDF.

[4] A. Kay and K. Downes-Ward, *Fixed points and cycles of the Kaprekar transformation: 2. Even bases*, arXiv:2408.12257. (Part 1, *Odd bases*, is J. Integer Seq. **25** (2022), Article 22.6.7.)

[5] D. S. Thakur, *Kaprekar phenomena*, in *Topics in Number Theory* (Proceedings of a conference at IIT Ropar, December 2017), Ramanujan Math. Soc. Lecture Notes Series **26**, Ramanujan Mathematical Society, 2020, pp. 61–70. Theorem 2 is quoted in Section 4. (The bibliographies of [1] and of the preprint date this volume 2019 and give the pages as 1–10; the data here are those of the printed volume, ISBN 978-81-936684-1-2.)

[6] OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, `https://oeis.org`; entries A003558 (λ, the order of 2 up to sign), A099009 (Kaprekar fixed points), and A165021, A165022, A165060, A165061, A165099, A165100 (per-base Kaprekar cycle data for bases 4, 6, 8).