CONVEX POLYOMINOES BY AREA AND DEGREE OF CONVEXITY

KOREA SUPERINTELLIGENCE LABS

Abstract. The *degree of convexity* $\mathrm{dc}(P)$ of a convex polyomino P is the least k such that any two cells of P are joined by a monotone path inside P with at most k changes of direction; the k-convex polyominoes are those with $\mathrm{dc}(P) \leq k$. Two 2026 papers grade this class along adjacent axes and neither crosses them: Castelli and Massazza count by area but only at $k = 2$, and Conway and Guttmann count by semi-perimeter — explicit series for $k = 3, \ldots, 7$ and a conjectured generating function for every k. We compute the underlying object both papers summarise — the joint distribution $T(n, d)$ of convex polyominoes of area n by exact degree of convexity d — for all $n \leq 19$ and all d, and read off the cumulative columns for every $k \leq 17$; by area only $k = 1$ and $k = 2$ were previously available. Three structural facts come out of the triangle: the largest degree of convexity available at area n is exactly $n - 2$; degree d first occurs at area $d + 2$, and is then attained by exactly four polyominoes; and each near-extremal diagonal $n \mapsto T(n, n-2-j)$ agrees with a polynomial of degree j in n with leading coefficient $4/j!$, for $j \leq 5$ over the computed range. We record the same three phenomena for the semi-perimeter grading, with $n - 2$ replaced by $s - 3$ and $4/j!$ by $4 \cdot 2^j/j!$. We also correct one entry of the published $k = 2$ column: the true value of $|\mathrm{ZConv}(29)|$ is 194304762, not the printed 94304762. The enumerations are calibrated against eleven independently published sequences — three by area, eight by semi-perimeter — and every theorem below is machine-checked in Lean 4; values quoted from a separate implementation in C are marked as such.

1. Introduction

A *polyomino* is a finite edge-connected set of unit cells of $\mathbb{Z}\times\mathbb{Z}$, taken up to translation; it is *convex* when its intersection with every horizontal and every vertical line is a segment. Inside the convex polyominoes there is a grading by how twisty a shape is, and both of the sources this paper starts from state it in one sentence:

> "The degree of convexity of a convex polyomino P is the smallest integer k such that any two cells of P can be joined by a monotone path inside P with at most k changes of direction." [1], abstract

> "A *k-convex polyomino* [...] is a convex polyomino with the additional constraint that any two cells of the polyomino can be joined by a directed *internal* NE, NW, SE or SW path with at most k right-angle bends. The path must be parallel to the axes of the polyomino." [2], §1

The two conventions agree, and their three load-bearing details are worth stating explicitly because a different reading changes every number below. Convexity is *required*: the grading refines the convex class rather than relaxing it, and [2] carries a footnote warning that the "m-convex" polyominoes of Enting, Guttmann, Richmond and Wormald are a relaxation and a different object. The bound is *at most* k, so "k-convex" names a cumulative class. And counting is up to translation only — "fixed" polyominoes, with no identification under rotation or reflection — as in the OEIS entries both papers point at. We write $\mathcal{C}$ for the set of convex polyominoes up to translation, $|P|$ for the area of P, and $\mathrm{dc}(P)$ for its degree of convexity. Then

$$(1)\qquad T(n,d) = \#\{P \in \mathcal{C} : |P| = n,\ \mathrm{dc}(P) = d\}, \qquad C_k(n) = \sum_{d \leq k} T(n,d), \qquad c(n) = \sum_{d \geq 0} T(n,d),$$

so $C_k(n)$ is the number of k-convex polyominoes of area n and $c(n)$ the number of convex ones. Degree 0 is a bar, degree ≤ 1 is the L-convex class, degree ≤ 2 is the class [1] calls ZConv.

The two sources, and what they leave uncrossed. Castelli and Massazza [1] count *by area*, but only at $k = 2$: their Table 1 prints $|\mathrm{ZConv}(n)| = C_2(n)$ for $1 \le n \le 75$, obtained from a set of recurrences implemented in C++. The paper prints no generating function and no closed form, so the table cannot be checked against anything internal to it; the authors record that an earlier publication of this sequence was wrong, "due to a bug in the prototype program used to generate it", and that the sequence of their Table 1 does not appear in the OEIS. Conway and Guttmann [2] count *by semi-perimeter*: explicit series and generating functions for $k = 3, \dots, 7$, the five of them in the OEIS as A398024–A398028, each linking that paper, and then a conjectured closed form $Q_k(x)$ covering every k at once (their §4.2), so in that grading no k is out of reach. The by-area columns for $k \ge 3$ appear in neither paper, and in no OEIS entry. That this is the state of the art and not an accident of our reading is confirmed by a third, independent paper in the same corner, which describes the by-area situation as "Enumeration with respect to the area has been carried out for L-convex polyominoes [...], while for $k \ge 2$ the asymptotic behaviour of k-convex polyominoes has been analyzed" [3] — that is, two and a half months before [1] was posted, exact by-area enumeration was available for $k = 1$ and asymptotics only for $k \ge 2$.

Neither paper states the object of which its own columns are summaries. Both count cumulative classes; the underlying datum is the joint distribution $T(n, d)$ of (1), and it is printed nowhere. Its first columns can be recovered from the published ones by subtraction, since $T(\cdot, d) = C_d - C_{d-1}$: by area that gives $d \le 2$, and by semi-perimeter $d \le 7$ from the printed series, or every d if one is willing to expand the conjecture of [2]. By area, beyond $d = 2$, the distribution requires the enumeration afresh.

What this paper settles.

(1) **The distribution itself** (Theorem 5.1): $T(n, d)$ for every $n \le 19$ and every d, with nothing lumped. All the columns below are its partial sums.
(2) **The by-area columns** (Theorems 5.3 and 5.4): $C_k(n)$ for $3 \le k \le 17$ and $n \le 19$, and for $3 \le k \le 6$ to $n = 23$. By area only $k = 1$ and $k = 2$ were previously available; $k = 17$ is the convex total.
(3) **The maximum** (Theorem 6.1): the largest degree of convexity available at area n is exactly $n - 2$, for every $n \le 19$. This is an easy fact, and we record it as one; its interest is that it says where the cumulative columns stop.
(4) **The onset** (Theorem 6.2): degree d first occurs at area $d + 2$, and there are then exactly four polyominoes of that degree.
(5) **The near-extremal diagonals** (Theorem 7.1): for $j = 0, \dots, 5$ the diagonal $n \mapsto T(n, n-2-j)$ agrees, over the whole computed range from its onset, with a polynomial of degree j in n whose leading coefficient is $4/j!$.
(6) **The semi-perimeter twin** (Theorems 3.5 and 8.1): the same three phenomena by semi-perimeter, the maximum becoming $s - 3$ and the diagonal leading coefficient $4 \cdot 2^j/j!$; and, on the way, the eight published columns of that grading, reproduced at every term to $s = 15$.
(7) **A correction** (Theorem 4.1): Table 1 of [1] prints $|\mathrm{ZConv}(29)| = 94304762$; the value is 194304762, the printed entry with its leading digit lost.

What is claimed, and what is not. The columns of items (1), (2), (4), (5) and (6) were searched for in the OEIS and in the literature and not found; the searches are described in Section 9. "Not found" is not "new", and here it is weaker than usual on two counts. The cumulative columns $\widetilde{C}_8, \widetilde{C}_9, \widetilde{C}_{10}$ of item (6) are not new at all: they follow by expansion from the conjectured closed form of [2], as Section 8 sets out, and we report them because we compute them, not because they were missing. And for the by-area columns one sentence of [2] weakens the claim deliberately and in the authors' own words: "We have developed an algorithm to enumerate k-convex polyominoes by both semi-perimeter and area. [...] However, in this article we are only subsequently discussing the analysis of the semi-perimeter data." So the by-area data is within reach of an algorithm that already exists; what is absent is the printed data and the structure read off it. Two of our findings — the onset law and the diagonal polynomials — are statements about the distribution rather than about any one column, and neither source computes the distribution.

Everything below is a finite computation, and Section 3 says exactly which finite computation. The maximum-degree law, the onset law and the diagonal identities are verified over the stated finite ranges, not proved for all n; where an elementary argument for the general case exists we give it and say that it has not been formalised.

2. The objects, in coordinates

Two encodings of a convex polyomino are used, and they are genuinely different: the results about the $k=2$ column come from one, the distribution from the other, and the agreement of the two on their overlap is part of the calibration.

Definition 2.1 (column intervals)**.** A convex polyomino with a tight bounding box of width w is the sequence of its column intervals $I_j=[a_j,b_j]$, $1\le j\le w$, taken up to vertical translation. Convexity is exactly: each I_j is non-empty, consecutive columns meet ($I_j\cap I_{j+1}\neq\emptyset$), the sequence (a_j) of bottoms is a *valley* (non-increasing, then non-decreasing) and the sequence (b_j) of tops is a *mountain* (non-decreasing, then non-increasing). The degree of convexity is read on *cells*: $\mathrm{dc}(P)\le k$ iff any two cells of P are joined by a monotone lattice path inside P with at most k right-angle bends.

Definition 2.2 (row intervals and the bounding box)**.** Dually, a convex polyomino with a tight $h\times w$ bounding box is the sequence of its row intervals $[L_i,R_i]$, $0\le i<h$, with L a valley, R a mountain, $L_i\le R_i$, consecutive rows overlapping, $\min L=0$ and $\max R=w-1$. Because a convex polyomino has the perimeter of its bounding box, its semi-perimeter is $h+w$; we write

$$\widetilde{T}(s,d)=\#\{P\in\mathcal{C}:\ P \text{ has semi-perimeter } s,\ \mathrm{dc}(P)=d\},\qquad \widetilde{C}_k(s)=\sum_{d\le k}\widetilde{T}(s,d)\tag{2}$$

for the same distribution graded by $s=h+w$ instead of by area.

Figure 1 shows the parameter on two shapes that recur below. On the left is the Z pentomino, the shape the class ZConv is named after: its two extreme cells are joined by a path with two bends and by none with fewer, so its degree of convexity is exactly 2. On the right is the *thin staircase* of area 6: its rows are intervals of length at most two, each meeting the row below in a single column, and its two extreme cells are joined by exactly one monotone path, which turns at every interior step. Its degree of convexity is $4=6-2$, the largest area 6 allows (Theorem 6.1).

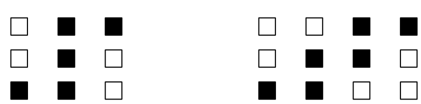

FIGURE 1. Left: the Z pentomino, of degree of convexity 2. Right: the thin staircase of area 6, of degree of convexity 4.

For $k=2$ there is a criterion on columns that avoids paths altogether. [1] quotes it from [4] in the form: $P\in$ ZConv if and only if for any two disjoint columns I_i and I_j of P there is a column I_m with $i<m<j$ and $I_i\subsetneq I_m$, $I_j\subsetneq I_m$. The betweenness clause can be dropped.

Lemma 2.3. *Let P be a convex polyomino with column intervals $I_1,\dots,I_w$ and let $i<j$ with $I_i\cap I_j=\emptyset$. If some column of P contains $I_i\cup I_j$, then some such column I_m has $i<m<j$.*

Proof. Say $b_i<a_j$; the other case is the reflection $(a,b)\mapsto(-b,-a)$, which exchanges valley and mountain and preserves the column order. Let q be an index at which the mountain b peaks and p one at which the valley a bottoms out. Suppose $m<i$. From $I_i\cup I_j\subseteq I_m$ we get $b_m\ge b_j>b_i$. If $i\le q$ then b is non-decreasing on $[1,q]$, so $b_m\le b_i$, a contradiction; hence $i>q$, and then b is non-increasing on $[q,w]$, so $b_j\le b_i$, again a contradiction. Hence $m>i$. Suppose $m>j$. From $a_i\le b_i<a_j$ we get $a_i<a_j$ with $i<j$, so $j>p$ (otherwise a would be non-increasing on $[1,p]$ and give $a_j\le a_i$); then a is non-decreasing on $[p,w]$, so $a_m\ge a_j>a_i$, contradicting $a_m\le a_i$. Hence $m<j$. Finally $m=i$ and $m=j$ are impossible because I_i and I_j are disjoint. □

Two consequences carry the computation of Section 4. First, the criterion is *order-free*: membership in ZConv is a condition on the *set* of column intervals, with no indices in it. Second, ZConv is *prefix-closed* in the column order — an initial segment of the columns of a Z-convex polyomino is again convex, and a covering column for two of its disjoint columns lies strictly between them and hence inside the segment. Prefix closure is what turns an infeasible filter-after-generate into a search whose node count is the answer itself.

Proposition 2.4. *Over every convex polyomino of area at most* 15*, the order-free column criterion — for any two disjoint columns there is a column containing both — agrees with the cell-level definition "*$\mathrm{dc}(P) \leq 2$*" of Definition 2.1. Over every convex polyomino of area at most* 13 *it also agrees with the criterion of* [4] *as* [1] *quotes it, betweenness clause included, and* ZConv *is prefix-closed in the column order. The agreement is not vacuous: at area* 13 *the criterion rejects* 103438 *of the* 138536 *convex polyominoes.*

3. The two enumerations, and what calibrates them

3.1. **Column intervals: an exact enumeration.** Let $E(n)$ be the list of convex polyominoes of area n generated by appending column intervals left to right, normalised so that the leftmost column starts in row n. The four local constraints of the generator are Definition 2.1 read left to right, with two further bounds that prune on remaining area.

Proposition 3.1. *For every n: $E(n)$ contains no repetitions, every member of $E(n)$ is a convex polyomino of area n in normal form, and every convex polyomino of area n has exactly one translate in $E(n)$. Consequently the counts obtained by filtering $E(n)$ are exact cardinalities of sets of translation classes, not lower bounds.*

The two pruning bounds are the delicate part: they are not constraints but consequences, and a generator that tightened either by one would silently lose shapes while remaining sound. Both implications are part of the formal proof.

3.2. **Row intervals and a bitboard wavefront: the distribution.** The distribution $T(n,d)$ needs the degree of *every* convex polyomino, not a Boolean, and is computed from the row-interval encoding of Definition 2.2. Cells are packed one per bit; for a source cell a the cells reachable from a by a monotone NE path with at most t bends split into those whose last step was horizontal (H_t) and vertical (V_t), with H_0 the maximal east run from a, V_0 the maximal north run, and $H_{t+1} = H_t \cup \mathrm{eastfill}(V_t)$, $V_{t+1} = V_t \cup \mathrm{northfill}(H_t)$. Each fill is a Kogge–Stone doubling. The degree of P is the largest t at which $H_t \cup V_t$ still grows, maximised over a and over two quadrant families rather than four: reversing a monotone path keeps it monotone in the opposite quadrant and changes no bend, so the minimum bend count is symmetric in its two endpoints, and every unordered pair of cells is in NE/SW or in NW/SE relation. Only boxes with $h \leq w$ are walked, the $h < w$ ones counted twice, transposition being an area- and degree-preserving bijection.

Three things about this walk are checked rather than assumed.

Proposition 3.2. (1) *Over every cell subset of every box that can hold a polyomino of area at most* 9*, the row-interval walk generates exactly the convex polyominoes, with the same distribution by area and by degree, where convexity is decided directly on the cell set (every row a non-empty segment, every column a non-empty segment, and a* 4*-adjacency flood fill reaching everything). The two enumerations share no code below the degree routine.*

(2) *Over every convex polyomino of area at most* 8*, the wavefront degree equals the minimum number of right-angle bends over all monotone lattice paths, computed by an unmemoised depth-first search — the definition transcribed. Since that search returns a sentinel when no monotone path joins some pair of cells, the agreement also certifies that in each of those polyominoes every pair of cells is so joined.*

(3) *The predicate is neither vacuous nor over-permissive: the two diagonal cells of a 2×2 box have segment rows and segment columns and are still rejected, which is what the flood-fill clause is for; the U pentomino is rejected for a non-segment row; the T tetromino is accepted. On each rejected*

shape the path definition returns its sentinel, while the wavefront returns a small finite number, so a finite wavefront value is not *evidence of convexity.*

Item (1) is verified to area 9 *and item (2) to area* 8*; neither is proved for general* n*.*

Cells are packed into one 64-bit word in the first implementation and into two in the second, split at row 4 so that no row straddles the boundary; the ceilings of both layouts are theorems in both directions — one word holds every box a convex polyomino of area ≤ 15 can have and fails at area 16 (the 8×9 box needs 72 bits), two words hold every box up to area 19 and fail at 20 (the 4×17 box needs 68 bits in the low word).

Proposition 3.3. *The two-word walk agrees with the one-word walk cell for cell on the whole overlap: the distributions of area* ≤ 15 *by degree,* 15 *areas by* 13 *degree buckets, are equal. Folding degrees* ≥ 12 *of the uncapped table of area* ≤ 19 *back into one bucket returns the capped table entry for entry, and the top bucket of the uncapped table is empty — so the uncapped table is the distribution and not a longer truncation of it. Both statements are kernel reductions of two literals, not second evaluations.*

The point of Proposition 3.3 is transfer: the two walks differ in the cell packing, the number of doublings, the guarded shifts and the north fill, so everything Proposition 3.2 establishes about the narrow walk carries to the wide one, and everything the wide one establishes carries to the uncapped one.

3.3. **Calibration against published data.** The enumerator is not trusted; it is calibrated, and at exactly the k where its output is new.

Proposition 3.4. *The by-area total* $c(n)$ *is OEIS A067675 at every* $n \leq 19$*, the column* $C_1(n)$ *is OEIS A126764 at every* $n \leq 19$*, and the column* $C_2(n)$ *is the first* 19 *entries of Table 1 of* [1]*. The same three columns come out of the independent column-interval enumeration of Proposition 3.1 — the total and* C_2 *for* $n \leq 16$*, and* C_1 *for* $n \leq 12$ *— and there they are exact cardinalities of sets of translation classes rather than counts over an enumerated family.*

Those three gates pin the walk for $k \leq 2$ and for the total, and say nothing about how degrees $3, 4, 5, 6, \ldots$ split among themselves — which is where the new columns live. That hole is closed by changing the grading and nothing else.

Theorem 3.5. *Grading the same walk of the same row intervals by* $h + w$ *instead of by area reproduces eight independently published sequences at every one of the fourteen terms* $s = 2, \ldots, 15$*:*

$$\widetilde{C}_1 = A003480, \quad \widetilde{C}_2 = A128611, \quad \widetilde{C}_3 = A398024, \quad \widetilde{C}_4 = A398025,$$

$$\widetilde{C}_5 = A398026, \quad \widetilde{C}_6 = A398027, \quad \widetilde{C}_7 = A398028, \quad \textstyle\sum_d \widetilde{T}(\cdot, d) = A005436.$$

The eight columns are pairwise distinct at $s = 15$*, so this is fourteen matches against eight different sequences and not one match restated. The distribution* $\widetilde{T}(s, d)$ *they are cumulative sums of is verified as a single literal and printed in Tables 7 and 8.*

published sequence	grading	is our column	terms reproduced
A067675	area	$c(n)$, all convex	19 verified, 23 in C
A126764	area	C_1, L-convex	19 verified, 23 in C
Table 1 of [1]	area	C_2, Z-convex	19 verified; 37 in C, one corrected
A005436	semi-perimeter	all convex	14 verified, 15 in C
A003480	semi-perimeter	$\widetilde{C}_1$	14 verified, 15 in C
A128611	semi-perimeter	$\widetilde{C}_2$	14 verified, 15 in C
A398024–A398028	semi-perimeter	$\widetilde{C}_3, \ldots, \widetilde{C}_7$	14 verified each, 15 in C

TABLE 1. The calibration. Eleven sequences computed by other people, by other methods — three by area, eight by semi-perimeter — come out of the two enumerations of Section 3; every value agrees at every term computed, the single exception being the entry corrected by Theorem 4.1.

Five of those eight are the series [2] itself published, covering every k for which those authors print an explicit series. So the degree routine is checked against published exact data at exactly $k = 3, 4, 5, 6, 7$: the range in which the by-area columns of Theorem 5.3 are new, and the range at which those of Theorem 5.4 begin. Only the grading differs between the two runs. The values at $s = 13, 14, 15$ are printed in Table 9.

4. The $k = 2$ column, and a correction to Table 1 of [1]

Prefix closure (Section 2) licenses a depth-first search that appends columns left to right and abandons a prefix the moment it stops being Z-convex. Such a search visits exactly one node per Z-convex polyomino of area at most n — about 5×10^8 at $n = 29$, against roughly 10^{11} convex polyominoes of that area that a filter-after-generate would have to touch. Getting from area 16 to area 37 was a change of algorithm and not a change of budget.

The test at a node is incremental and costs $O(1)$ after one scan of the prefix. Appending a column $[a, b]$ to a Z-convex prefix $I_1, \ldots, I_m$, only the pairs involving the new column are new, and by Lemma 2.3 a covering column may be sought among $I_1, \ldots, I_m$. Write $S_1 = \{c \le m : b_c < a\}$ and $S_2 = \{c \le m : a_c > b\}$ for the prefix columns lying wholly below and wholly above the new one. Then the new column keeps the prefix Z-convex exactly when

$$S_1 \neq \emptyset \implies \max\{b_c : a_c \le \alpha\} \ge b, \qquad \alpha = \min\{a_c : c \in S_1\},$$
$$S_2 \neq \emptyset \implies \min\{a_c : b_c \ge \beta\} \le a, \qquad \beta = \max\{b_c : c \in S_2\},$$

because the two profile functions $\alpha \mapsto \max\{b_c : a_c \le \alpha\}$ and $\beta \mapsto \min\{a_c : b_c \ge \beta\}$ are monotone, so only the extreme member of S_1 and of S_2 can bind. Run to $n = 37$, the search gives

1, 2, 6, 19, 55, 148, 370, 874, 1966, 4242, 8838, 17851, 35098, 67356, 126518,
233033, 421696, 750780, 1316916, 2278259, 3891347, 6567788, 10962524, 18108061,
29619788, 48004616, 77126190, 122896541, **194304762**, 304931206, 475173306,
735490162, 1131122763, 1728912988, 2627129510, 3969544022, 5965539010.

Thirty-six of these thirty-seven values agree with Table 1 of [1] exactly, including all twelve entries $n = 17, \ldots, 28$ and all eight entries $n = 30, \ldots, 37$ on the far side of the one disagreement.

Theorem 4.1. *Of the* 74 *steps of Table 1 of* [1] *exactly one fails to increase, namely* $122896541 \mapsto 94304762$ *at* $n = 28 \mapsto 29$*; and replacing the entry at* $n = 29$ *by* 194304762 *makes the whole table strictly increasing. The computed value of* $|\mathrm{ZConv}(29)|$ *is* $194304762 = 100000000 + 94304762$.

The table was read from the arXiv LaTeX source, and the defect is not an artefact of that copy: the version of record — EPTCS 445, pp. 57–65 — prints the same 94304762 at $n = 29$, and its 75-entry table has the same single descent and no other.

The two halves of Theorem 4.1 have different standing and it matters. The statements about the printed table — that it has exactly one descent, where it is, and that the proposed value removes it — are arithmetic about the transcribed numbers and are proved outright, depending on no axioms at all. That 194304762 is the true value rests on the computation above, which is an independent implementation and not a re-run of the authors' recurrences. Since every neighbouring entry on both sides is reproduced exactly, the defect is in the typesetting of the table and not in the authors' program: a wrong program does not agree with a published table at thirty-six places and disagree at one, and a lost leading digit is exactly what the disagreement is. Context that raises the prior rather than establishing anything: [1] records that this sequence was published wrong once before, and the paper contains no closed form against which the table could have been checked internally.

Proposition 2.4 matters here. What the search counts is the criterion's class; what [1] defines is the class of convex polyominoes of degree of convexity at most 2. Combining Propositions 3.1 and 2.4 identifies the two for $n \le 15$ as an equality of cardinalities of sets defined by monotone paths between cells, with no criterion in the statement: there are exactly 35098, 67356 and 126518 convex polyominoes of area 13, 14 and 15 with $\mathrm{dc}(P) \le 2$, up to translation, which are entries 13, 14 and 15 of Table 1. For larger n the identification is [4] as cited, checked independently over all 234145 convex polyominoes of area at most 13.

Proposition 4.2. *The class counted in Theorem 4.1 is a proper non-empty subclass, and the hypotheses of the column criterion are all exercised. Each headline count is refuted one above and one below. Convexity is not a redundant hypothesis: the column list* $[(0,5),(0,0),(5,5)]$ *satisfies the criterion and is not a convex polyomino, its tops forming a valley rather than a mountain. The inner quantifier "for any two disjoint columns" goes all three ways — a Z pentomino has disjoint columns and they are covered, a* 2×2 *square has none, and a three-step staircase has disjoint columns that are not covered — and the threshold* 2 *is real at both ends, the Z pentomino having degree exactly* 2 *and the three-step staircase degree* 3.

Remark 4.3. The natural worry is that the defective entry propagated into the asymptotics [1] reports; it cannot have changed a published number, because no number is published. What [1] says is that "[b]y using the new sequence, the conjecture for $|\mathrm{ZConv}(n)|$" of [5] "is corrected in" the shape $C\exp\left(\pi\sqrt{17n/6}\right)/n^{3/2}$, "for a suitable constant C", and its acknowledgements thank "Anthony J. Guttmann for updating the conjecture on $|\mathrm{ZConv}(n)|$ based on the new sequence". What the new sequence changed is thus the *form* — the exponential factor and the power of n — while C is left unevaluated, in the arXiv source and in the EPTCS version alike. There is no published fit for C to examine, and we make no claim about which value at $n = 29$ entered the work behind that form; the erratum above is independent of it either way.

5. The distribution by degree of convexity

Theorem 5.1. *For* $1 \le n \le 19$ *and* $0 \le d \le 17$ *the values of* $T(n,d)$ *are those of Tables 2 and 3;* $T(n,d) = 0$ *for* $d \ge 18$ *and* $n \le 19$.

$n\backslash d$	0	1	2	3	4	5	6	7	8
1	1	0	0	0	0	0	0	0	0
2	2	0	0	0	0	0	0	0	0
3	2	4	0	0	0	0	0	0	0
4	2	13	4	0	0	0	0	0	0
5	2	33	20	4	0	0	0	0	0
6	2	74	72	24	4	0	0	0	0
7	2	154	214	100	28	4	0	0	0
8	2	308	564	332	132	32	4	0	0
9	2	588	1376	968	488	168	36	4	0
10	2	1096	3144	2570	1564	684	208	40	4
11	2	1982	6854	6354	4516	2388	924	252	44
12	2	3513	14336	14860	12064	7452	3488	1212	300
13	2	6092	29004	33192	30298	21380	11692	4916	1552
14	2	10396	56958	71358	72358	57372	35836	17608	6728
15	2	17432	109084	148420	165676	145796	102276	57392	25632
16	2	28835	204196	300036	365908	353970	275392	173504	88492
17	2	47036	374658	591388	783278	826562	706204	493208	282376
18	2	75818	674960	1139834	1631190	1866044	1737100	1331396	844700
19	2	120792	1196122	2152930	3314942	4089660	4121346	3438916	2393336

Table 2. $T(n,d)$, the number of convex polyominoes of area n with degree of convexity exactly d, for $d \le 8$. Column $d = 0$ is the two bars. Full row sums, taken together with Table 3, are OEIS A067675.

The two facts one notices first in Table 2 are the constant column $T(n,0) = 2$ for $n \ge 2$ — the horizontal and the vertical bar — and the leading 4 that walks one step right per row. Both are read off the table; the second is Theorem 6.2.

Remark 5.2. Three of the columns of Table 2 were already implicit in the literature. $T(\cdot,0)$ is the two bars, $T(\cdot,1) = C_1 - 2$ is A126764 shifted, and $T(\cdot,2) = C_2 - C_1$ is the difference of Table 1 of [1] and A126764. Everything from $d = 3$ rightwards needs C_3, which is not available anywhere, and so needs the enumeration. The same accounting by semi-perimeter has no boundary at all, since a conjectured closed form covers every k there; see Section 8.

$n\backslash d$	9	10	11	12	13	14	15	16	17
11	4	0	0	0	0	0	0	0	0
12	48	4	0	0	0	0	0	0	0
13	352	52	4	0	0	0	0	0	0
14	1948	408	56	4	0	0	0	0	0
15	8984	2404	468	60	4	0	0	0	0
16	36260	11748	2924	532	64	4	0	0	0
17	132132	50056	15088	3512	600	68	4	0	0
18	443648	191932	67656	19076	4172	672	72	4	0
19	1392640	676200	272212	89772	23788	4908	748	76	4

TABLE 3. $T(n,d)$ for $9 \le d \le 17$; $T(n,d) = 0$ for $n \le 10$ in this range.

Theorem 5.3. *For $k = 3, 4, 5, 6$ and every $n \le 19$, the number $C_k(n)$ of k-convex polyominoes of area n is the corresponding partial sum of Table 2; the values are printed in Table 4.*

n	C_1 (A126764)	C_2 ([1])	C_3	C_4	C_5	C_6	$c(n)$ (A067675)
1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2
3	6	6	6	6	6	6	6
4	15	19	19	19	19	19	19
5	35	55	59	59	59	59	59
6	76	148	172	176	176	176	176
7	156	370	470	498	502	502	502
8	310	874	1206	1338	1370	1374	1374
9	590	1966	2934	3422	3590	3626	3630
10	1098	4242	6812	8376	9060	9268	9312
11	1984	8838	15192	19708	22096	23020	23320
12	3515	17851	32711	44775	52227	55715	57279
13	6094	35098	68290	98588	119968	131660	138536
14	10398	67356	138714	211072	268444	304280	331032
15	17434	126518	274938	440614	586410	688686	783630
16	28837	233033	533069	898977	1252947	1528339	1841867
17	47038	421696	1013084	1796362	2622924	3329128	4306172
18	75820	750780	1890614	3521804	5387848	7124948	10028276
19	120794	1316916	3469846	6784788	10874448	14995794	23288394
beyond the machine-checked range, from the C implementation only:							
20	190479	2278259	6271189	12861515	21591561	31064599	53974959
21	297365	3891347	11174263	24018731	42219667	63389851	124925967
22	460056	6567788	19650132	44234872	81380756	127514712	288878550
23	705576	10962524	34133258	80415936	154769488	253044326	667602492

TABLE 4. k-convex polyominoes of area n, up to translation. The columns C_1, C_2 and c are published; the four between them are not. Rows 20–23 are a separate computation in C and are not part of the machine-checked development; on those rows the three published columns also agree at every entry.

Theorem 5.4. *For $7 \le k \le 17$ and every $n \le 19$, $C_k(n)$ is the corresponding partial sum of Tables 2 and 3; the values at $n = 13, \ldots, 19$ are printed in Table 5. In particular*

$$C_7 = 1,\ 2,\ 6,\ 19,\ 59,\ 176,\ 502,\ 1374,\ 3630,\ 9308,\ 23272,\ 56927,\ 136576,$$
$$321888,\ 746078,\ 1701843,\ 3822336,\ 8456344,\ 18434710,$$

and $C_{17}(n) = c(n)$ for $n \le 19$, the cumulative columns stopping there for the reason given by Theorem 6.1.

Two remarks on Table 4. The column C_6 is not the convex total truncated: the two separate already at $n = 9$, 3626 against 3630. And none of the four inner columns has collapsed onto its neighbour at the

$k\backslash n$	13	14	15	16	17	18	19
7	136576	321888	746078	1701843	3822336	8456344	18434710
8	138128	328616	771710	1790335	4104712	9301044	20828046
9	138480	330564	780694	1826595	4236844	9744692	22220686
10	138532	330972	783098	1838343	4286900	9936624	22896886
11	138536	331028	783566	1841267	4301988	10004280	23169098
12	138536	331032	783626	1841799	4305500	10023356	23258870
13	138536	331032	783630	1841863	4306100	10027528	23282658
14	138536	331032	783630	1841867	4306168	10028200	23287566
15	138536	331032	783630	1841867	4306172	10028272	23288314
16	138536	331032	783630	1841867	4306172	10028276	23288390
17	138536	331032	783630	1841867	4306172	10028276	23288394

TABLE 5. The tail cumulative columns $C_k(n)$. The entries omitted on the left are partial sums of Table 2; by Theorem 6.1, $C_k(n) = c(n)$ exactly when $n \le k+2$. Here $k = 7$ is the largest k for which an explicit series has been published for this object, and only by semi-perimeter.

right-hand end of the computed range: at $n = 23$ the six columns $C_1, \ldots, C_6$ are 0.0011, 0.0164, 0.0511, 0.1205, 0.2318 and 0.3790 of the convex total.

6. The maximum degree, and the onset

Theorem 6.1. *For every n with $2 \le n \le 19$,*

$$\max\{\mathrm{dc}(P) : P \in \mathcal{C},\ |P| = n\} = n - 2.$$

Both halves hold separately: no convex polyomino of area n has degree of convexity $\ge n-1$, and degree exactly $n-2$ occurs at every area $3 \le n \le 19$.

Theorem 6.1 is easy, and we claim nothing more for it than that it has been checked. The upper bound is one line: a monotone path inside P never revisits a cell, so it has at most n cells, at most $n-1$ steps and at most $n-2$ changes of direction. The lower bound is the *thin staircase* of area n — row i occupying columns $[i, i+1]$, with a final row of a single cell when n is odd — whose two extreme cells are joined by exactly one monotone path, and that path turns at every interior step. Neither argument is formalised here; both would need the fact that the wavefront's optimal path visits distinct cells, which in the present setting is a statement about a loop with an early exit. The identity was confirmed at areas 20 and 21 by the separate C implementation. Its value is that it explains where the cumulative columns of Table 5 stop: $C_k(n) = c(n)$ as soon as $k \ge n-2$, and not one step earlier.

Theorem 6.2. *For every d with $1 \le d \le 17$,*

$$\min\{\,|P| : P \in \mathcal{C},\ \mathrm{dc}(P) = d\,\} = d + 2, \qquad \textit{and} \qquad T(d+2, d) = 4.$$

Equivalently: a convex polyomino of degree of convexity d has area at least $d+2$, the bound is sharp, and it is attained by exactly four polyominoes.

Remark 6.3. What the four extremal polyominoes *are* is not part of the verified statement, which counts them. Inspection identifies them as the thin staircase of Figure 1 in its four orientations; the staircase has a diagonal symmetry, so its orbit under the eight symmetries of the square has four elements, and since polyominoes are counted up to translation only, all four are distinct.

Theorem 6.2 is the transpose of Theorem 6.1 and carries the same elementary bound, but the multiplicity 4 is a statement about the extremal shapes and not only about the extremal value: at the onset the staircase is rigid. The two together say that the boundary of the triangle in Tables 2 and 3 is a constant 4. That constant is the $j = 0$ case of the next section.

7. The near-extremal diagonals

Read T along $d = n - 2 - j$ for fixed j: this counts the convex polyominoes of area n whose degree of convexity falls j short of the largest that area allows. Over the whole computed range each such diagonal is a polynomial in n.

Theorem 7.1. *With $D_j(n) = T(n, n-2-j)$, the following identities hold at every integer n in the stated range:*

$$\begin{aligned}
D_0(n) &= 4, && 3 \le n \le 19,\\
D_1(n) &= 4n, && 5 \le n \le 19,\\
D_2(n) &= 2n^2 + 2n - 12, && 6 \le n \le 19,\\
D_3(n) &= \tfrac{1}{3}\big(2n^3 + 6n^2 - 68n + 132\big), && 8 \le n \le 19,\\
D_4(n) &= \tfrac{1}{6}\big(n^4 + 6n^3 - 97n^2 + 354n - 456\big), && 9 \le n \le 19,\\
D_5(n) &= \tfrac{1}{30}\big(n^5 + 10n^4 - 205n^3 + 950n^2 - 1116n - 1800\big), && 11 \le n \le 19.
\end{aligned}$$

Each range is tight at its left end: for $j = 1, \dots, 5$ the identity is verified to fail at the area one step below the range stated for it, and for $j = 0$ the table gives $T(2,0) = 2 \ne 4$. The leading coefficient of D_j is $4/j!$ for $j = 0, \dots, 5$.

The identities are stated and verified multiplied through by the denominator, so that no division occurs and each is an identity between integers.

Table 6 records what the evidence is. A polynomial of degree j is determined by $j+1$ of its values, so the content of each line is the number of *further* terms of the computed range that the fitted polynomial then predicts correctly. For $j \le 5$ that number ranges from 18 down to 5. Carrying the C implementation to area 21 adds two lines: the $j = 6$ fit predicts three further terms and the $j = 7$ fit predicts none, so the latter is an interpolation and is reported only because its leading coefficient came out $1/1260 = 4/7!$ unbidden.

j	$D_j(n) = T(n, n-2-j)$	leading coeff.	verified range	terms beyond the fit
0	4	$4/0!$	$3 \le n \le 19$	18
1	$4n$	$4/1!$	$5 \le n \le 19$	15
2	$2n^2 + 2n - 12$	$4/2!$	$6 \le n \le 19$	13
3	$(2n^3 + 6n^2 - 68n + 132)/3$	$4/3!$	$8 \le n \le 19$	10
4	$(n^4 + 6n^3 - 97n^2 + 354n - 456)/6$	$4/4!$	$9 \le n \le 19$	8
5	$(n^5 + 10n^4 - 205n^3 + 950n^2 - 1116n - 1800)/30$	$4/5!$	$11 \le n \le 19$	5
from the C implementation only, to area 21:				
6	degree 6, leading coefficient $1/180$	$4/6!$	$12 \le n \le 21$	3
7	degree 7, leading coefficient $1/1260$	$4/7!$	$14 \le n \le 21$	0

Table 6. The near-extremal diagonals of the by-area distribution. "Terms beyond the fit" is the number of computed values the fitted polynomial predicts correctly after the $j+1$ it was fitted to. Lines $j \le 5$ are the machine-checked identities of Theorem 7.1.

One further pattern is visible in Table 6 and we have no explanation for it. Write $n_0(j)$ for the smallest area at which D_j agrees with its polynomial; for $j \le 5$ this endpoint is verified tight, and for $j = 6, 7$ it is where the fit begins to hold in the C range. The eight observed values $3, 5, 6, 8, 9, 11, 12, 14$ are exactly

$$n_0(j) = \left\lceil \frac{3(j+2)}{2} \right\rceil, \qquad j = 0, \dots, 7.$$

The semi-perimeter diagonals of Section 8 behave more simply: their four observed left endpoints are $5, 7, 9, 11$, that is $2j + 5$.

Conjecture 7.2. For each $j \geq 0$ there is an $n_0(j)$ such that $n \mapsto T(n, n-2-j)$ agrees with a polynomial of degree j in n for $n \geq n_0(j)$, and

$$T(n, n-2-j) \;\sim\; \frac{4\,n^j}{j!} \qquad (n \to \infty).$$

We offer no proof and none is claimed. The shape $4\binom{n}{j}$ suggested by the leading coefficient is what one would expect if the near-extremal polyominoes were staircases carrying j independent defects, in four rotation classes, and that is the direction in which a proof should be looked for; identifying the defects is precisely what the present method, which is exhaustive enumeration, does not do.

8. The semi-perimeter grading past the published series

The eight published columns of Theorem 3.5 are cumulative sums of the same uncapped rows $\widetilde{T}(s, \cdot)$, printed in Tables 7 and 8. Reading those rows past $k = 7$ gives the same three phenomena as the by-area grading. As in Remark 5.2, the accounting is worth being explicit about, and here it comes out differently from the by-area case, because [2] does not stop at $k = 7$. Its §4.2 — the numbering is the same in both postings — conjectures a closed form for every k at once,

$$Q_k(x) = \frac{-2x^4 + x^2(1-6x+7x^2+4x^3)\,U_k(t) + x^3(1-4x+8x^2)\,U_{k-1}(t)}{(1-4x)^2\,U_k(t)} - \frac{4x^4\,T_k^2\big(1/(2\sqrt{x})\big)}{(1-4x)^{3/2}\,U_k(t)}$$

with $t = (1-2x)/(2x)$ and T_k, U_k the Chebyshev polynomials of the first and second kind. Expanding it is elementary, so by semi-perimeter every cumulative column — and hence, by subtraction, every exact-degree column — was in principle available, conditionally on that conjecture. The expansion does agree with our rows: at $k = 8, 9, 10$ it gives $\widetilde{C}_8$, $\widetilde{C}_9$ and $\widetilde{C}_{10}$ at every term of Tables 7 and 8, and at $k = 2, \ldots, 7$ it gives A128611 and A398024–A398028. That is a series expansion done outside the formal development, and it settles provenance rather than correctness, the columns themselves being Theorem 8.1. One caveat on which copy to expand: the version of record prints $4x^2$ where the display above has $4x^4$, and with $4x^2$ the expansion is not a counting sequence at all; the exponent is corrected in arXiv:2607.12448v2, among the three typographical corrections that posting records. What is *not* available from [2] in either grading is the maximum degree and the diagonals, which are statements about the distribution rather than about any one column.

Theorem 8.1. *For $3 \leq s \leq 15$ the largest degree of convexity of a convex polyomino of semi-perimeter s is exactly $s-3$, and for $5 \leq s \leq 15$ it is attained by exactly four polyominoes. The exact-degree columns $\widetilde{T}(\cdot, d)$ for $2 \leq d \leq 11$ and the cumulative columns $\widetilde{C}_8, \widetilde{C}_9, \widetilde{C}_{10}$ are the corresponding readings of Tables 7 and 8. With $\widetilde{D}_j(s) = \widetilde{T}(s, s-3-j)$,*

$$\widetilde{D}_0(s) = 4 \;\; (5 \leq s \leq 14), \qquad \widetilde{D}_1(s) = 8s - 16 \;\; (7 \leq s \leq 14), \qquad \widetilde{D}_2(s) = 8s^2 - 36s + 44 \;\; (9 \leq s \leq 14),$$

the ranges for $j = 1, 2$ being verified to fail one step below their left endpoints, and the range for $j = 0$ likewise, since $\widetilde{T}(4, 1) = 5 \neq 4$. The leading coefficient of $\widetilde{D}_j$ is $4 \cdot 2^j/j!$ for $j = 0, 1, 2$.

The leading coefficients are the by-area ones with n replaced by $2s$. We record that coincidence without explaining it: the two gradings are not related by a substitution, since a thin staircase of area n has semi-perimeter $n+1$ — which is why the two maxima agree, $s - 3 = n - 2$ on those shapes — and the diagonals count far more than the staircases. A $j = 3$ fit from the C implementation gives leading coefficient $16/3 = 4 \cdot 2^3/3!$ with two terms to spare; at $j \geq 4$ the fits have no terms left over and are not reported.

Proposition 8.2. *A convex polyomino of semi-perimeter s lives in a box with $h + w = s$, so after the $h \leq w$ reduction the largest box to consider is $\lfloor s/2 \rfloor \times \lceil s/2 \rceil$: at $s = 16$ that is $8 \times 8 = 64$ cells and at $s = 17$ it is $8 \times 9 = 72$. Hence the one-word layout of Section 3 holds every box of semi-perimeter at most 16 and fails at 17; the two-word layout buys nothing below $s = 17$.*

Proposition 8.2 is the reason Table 9 stops where it does: what binds below $s = 17$ is running time, not representation. The row $s = 16$ was computed in C only; its eight cumulative sums are 17651648, 185780308, 261791420, 295481944, 309548722, 314691982, 316282360 and 316786960, and all eight agree with the published values of A003480, A128611, A398024, A398025, A398026, A398027, A398028 and A005436 respectively, at a fifteenth term.

$s\backslash d$	0	1	2	3	4	5	6
2	1	0	0	0	0	0	0
3	2	0	0	0	0	0	0
4	2	5	0	0	0	0	0
5	2	22	4	0	0	0	0
6	2	80	34	4	0	0	0
7	2	278	204	40	4	0	0
8	2	954	1066	270	48	4	0
9	2	3262	5184	1540	368	56	4
10	2	11142	24146	8012	2306	484	64
11	2	38046	109328	39384	12892	3336	616
12	2	129902	485324	186406	67044	20146	4640
13	2	443518	2123540	859016	331896	111628	30188
14	2	1514270	9190704	3881506	1586092	582556	178210
15	2	5170046	39441756	17277836	7383696	2910232	982076

TABLE 7. $\widetilde{T}(s,d)$, convex polyominoes of semi-perimeter s with degree of convexity exactly d, for $d \leq 6$. Full row sums, taken together with Table 8, are OEIS A005436.

$s\backslash d$	7	8	9	10	11	12
10	4	0	0	0	0	0
11	72	4	0	0	0	0
12	764	80	4	0	0	0
13	6248	928	88	4	0	0
14	43612	8192	1108	96	4	0
15	273624	61104	10504	1304	104	4

TABLE 8. $\widetilde{T}(s,d)$ for $7 \leq d \leq 12$; the entries are 0 for $s \leq 9$ in this range.

k	published as	$s = 13$	$s = 14$	$s = 15$	$s = 16$ (C only)
1	A003480	443520	1514272	5170048	17651648
2	A128611	2567060	10704976	44611804	185780308
3	A398024	3426076	14586482	61889640	261791420
4	A398025	3757972	16172574	69273336	295481944
5	A398026	3869600	16755130	72183568	309548722
6	A398027	3899788	16933340	73165644	314691982
7	A398028	3906036	16976952	73439268	316282360
all	A005436	3907056	16986352	73512288	316786960

TABLE 9. The last terms of the semi-perimeter gate. Every entry is an independently published value; the columns $s \leq 15$ are machine-checked, the column $s = 16$ is from the C implementation only.

9. PROVENANCE OF THE NEW COLUMNS

The columns of Theorems 5.3, 5.4, 6.2, 7.1 and 8.1 were searched for before being offered. In the OEIS, each of the by-area columns $C_3, \ldots, C_6$ was queried at 23, 13 and 9 terms and each of $C_7, \ldots, C_{11}$ on its first window that separates it from its neighbours; so were the exact-degree columns $T(\cdot, d)$ for $d = 2, 3, 4, 5$, the diagonals D_2 and D_3, and by semi-perimeter $\widetilde{C}_8$ and $\widetilde{T}(\cdot, d)$ for $d = 2, 3, 4$. All returned nothing. The channel was controlled in both directions on the same session: a positive control returning

A000040 was run before, between and after the queries, and a junk control returning nothing was run in between, so that an empty result is distinguishable from a failed request. Keyword queries returned only the two by-area entries already known to us, A067675 and A126764, and two honeycomb-lattice entries.

On arXiv, a metadata search over titles, abstracts and comments — not full text, which bounds the claim — found the whole niche to be five papers, of which the three relevant ones are [1], [2] and [3]; all three were read from their own LaTeX sources, and the definition of the degree of convexity appears in all three in agreement. The other two are *The number of directed k-convex polyominoes* (arXiv:1501.00872), which counts the *directed* convex polyominoes by width, height, last-column size and number of corners — a different class and different statistics — and *Enumeration of polyominoes defined in terms of pattern avoidance or convexity constraints*. The following channels were *not* searched, and so this is not a clean negative: full-text search over an arXiv corpus, MathSciNet, zbMATH, Google Scholar, and any journal-only publication absent from arXiv and the OEIS.

What weakens the claim most is not any of that but [2], in two places. Its §4.2 conjectures a closed form for every k by semi-perimeter, which is why nothing in Section 8 is offered as a new cumulative column. And one sentence of its §1, quoted in Section 1, says that the authors' own algorithm enumerates by area as well as by semi-perimeter and that they simply do not analyse the area data. The by-area columns are therefore a gap in the printed literature inside a corner that the same authors actively curate — which is what makes the absence worth recording — and quite possibly a short run for them. We record it as such.

10. Verification

Every statement above has been formally verified in Lean 4, in the directories `LeanProblemSpec/KConvexArea/` and `LeanProblemSpec/ZConvex/` of a repository that is private at the time of writing (repository reference to be added at submission); the development contains no `sorry` and no declared axiom. What is proved outright, with no appeal to evaluation, is: Proposition 3.1 in full, for every n — soundness, duplicate-freedom and completeness of the column-interval enumeration, together with the fact that the normal form picks exactly one member of each translation class, which is what turns every $k \leq 2$ count into an exact cardinality; the arithmetic half of Theorem 4.1, which depends on no axioms at all; the packing ceilings, including Proposition 8.2; and the two bridges of Proposition 3.3, which are kernel reductions of table literals rather than second evaluations.

What is delegated to compiled evaluation, through Lean's `native_decide`, is exactly the exhaustive searches, and they are few: one evaluation produces the whole capped distribution to area 19, one more produces the uncapped distribution of Theorem 5.1, and every statement of Sections 5–7 — all the columns, the maximum, the onset and the diagonal identities — is a kernel reduction of one of those two literals, so it introduces no further evaluation and no further trust. By semi-perimeter there are four such evaluations, one per expensive row, assembled by kernel reduction into Theorem 3.5. Propositions 2.4 and 3.2 cost one evaluation per area checked, and the negative controls of Propositions 3.2 and 4.2 one apiece. Every count in Theorems 5.1–7.1 should therefore be read as a count over the enumerated family, with that family verified to be the intended one up to area 9 and the degree routine verified against the sources' path definition up to area 8 (Proposition 3.2), and with the published columns of Proposition 3.4 and Theorem 3.5 agreeing at every computed term.

Two of the finite ranges are worth stating in the negative. Area 20 is not reached: the two-word cell layout provably cannot represent the 4×17 box. Semi-perimeter 16 is representable but was not run in Lean; it is in the tables as a C-only row. Everywhere a value is quoted from the C implementation rather than from the formal development, the tables say so. Finally, every numerical value here was produced twice, by two implementations that share no code below the tally — a bitboard wavefront in Lean and a rolling two-row dynamic program in C — and the two agree at every entry of every table.

For a reader who wants to redo the computation rather than the proof, the costs are these, all measured. The uncapped by-area distribution to area 19 — Theorem 5.1, and with it every column, the maximum, the onset and the diagonals — is a single evaluation of 13 to 17 minutes with a peak resident set of about 1.65 GB; the capped distribution it is bridged to costs 12 minutes. Reading the eleven cumulative and ten exact-degree columns off the capped literal costs 3.3 seconds, since it is kernel reduction and not evaluation. By semi-perimeter, the rows to $s = 14$ are 3 minutes together and the row $s = 15$ is 10 minutes on its own, which is why it is a separate module. The identification of the criterion with the definition at areas 13, 14 and 15 (Proposition 2.4) is $13\frac{1}{2}$ minutes and 3.8 GB, the memory going to the library import rather than to the data. The C implementation reaches area 19 in 75 seconds and area 21 in about 8 minutes; semi-perimeter 15 in about $5\frac{1}{2}$ minutes and 16 in about 31; and the pruned Z-convex search of Section 4 reaches $n = 37$ in 917 seconds on four threads. On this workload compiled Lean runs at 2.3 to 2.5 times the cost of the same walk in C compiled with `gcc -O2`, measured in both directions.

All of it builds under the Lean toolchain `leanprover/lean4:v4.32.0`, with Mathlib pinned at its `v4.32.0` tag.

References

[1] L. Castelli and P. Massazza, *On the Counting Sequence of Z-convex Polyominoes*, in: S. Brlek and L. Ferrari (eds.), *Proceedings of GASCom 2026*, Electronic Proceedings in Theoretical Computer Science 445 (2026), pp. 57–65, `doi:10.4204/EPTCS.445.8`; arXiv:2606.13158 (11 June 2026). Quotations and Table 1 were read from the arXiv LaTeX source and checked against the EPTCS version of record, which prints the same 94304762 at $n = 29$.

[2] A. R. Conway and A. J. Guttmann, *k-Convex Polyominoes by Semi-perimeter*, in: S. Brlek and L. Ferrari (eds.), *Proceedings of GASCom 2026*, Electronic Proceedings in Theoretical Computer Science 445 (2026), pp. 74–86, `doi:10.4204/EPTCS.445.10`; arXiv:2606.13845 (11 June 2026). This is the version of record and the one A398024–A398028 link. The same paper was reposted as arXiv:2607.12448, whose v2 (17 July 2026) is commented “13 pages, 2 figures. Appeared in GASCom26 (3 minor typos corrected)”; the two agree verbatim on the §1 sentences quoted here, and differ in the numerator of $Q_{2,k}$ as noted in Section 8.

[3] N. Beaton and S. Rinaldi, *Ascending Convex Polyominoes*, arXiv:2603.26395 (27 March 2026).

[4] G. Castiglione and P. Massazza, *An Efficient Algorithm for the Generation of Z-Convex Polyominoes*, in: *Combinatorial Image Analysis* (IWCIA 2014), Lecture Notes in Computer Science 8466, Springer, 2014, pp. 51–61, `doi:10.1007/978-3-319-07148-0_6`. This is the reference [1] cites as [CaMa14]. It is used here only through the verbatim quotation of its Theorem 1 in [1], and independently re-verified over all convex polyominoes of area at most 13; the original was not read.

[5] A. J. Guttmann and P. Massazza, *Asymptotics of Z-convex polyominoes*, RAIRO Theoretical Informatics and Applications 58 (2024), `doi:10.1051/ita/2024009`. Cited as [GuMa24] in [1]; not read here.

[6] P. Massazza, *On Counting k-Convex Polyominoes*, in: U. Dal Lago and D. Gorla (eds.), *Proceedings of the 23rd Italian Conference on Theoretical Computer Science (ICTCS 2022)*, Rome, Italy, September 7–9, 2022, CEUR Workshop Proceedings 3284, CEUR-WS.org, 2022, pp. 116–121. Cited by [2] at the definition of a k-convex polyomino; verified against the open-access volume at `https://ceur-ws.org/Vol-3284/`, not read directly.

[7] OEIS Foundation Inc., *The On-Line Encyclopedia of Integer Sequences*, `https://oeis.org`, entries read 2026-08-28. By area: A067675, “Number of fixed convex polyominoes with n cells”, and A126764, “Number of L-convex polyominoes with n cells, that is, convex polyominoes where any two cells can be connected by a path internal to the polyomino and which has at most 1 change of direction”. By semi-perimeter: A398024–A398028, whose names run from “Number of fixed 3-convex polyominoes with perimeter 2*n” to “Number of fixed 7-convex polyominoes with perimeter 2*n”; A005436, “Number of convex polygons of perimeter 2n on square lattice”; A128611, “Number of Z-convex polyominoes with semiperimeter n”; and A003480, whose name is the recurrence “a(0) = 1, a(1) = 2, a(2) = 7, for n > 2, a(n) = 4*a(n-1) - 2*a(n-2)”, its L-convex reading being a comment on the entry rather than its name. Seven of the eight are indexed by the semi-perimeter itself — “perimeter 2n” means index n — and A003480 is the exception: our $\widetilde{C}_1(s)$ is its term $s - 2$.