CERTIFIED DOMINANCE AT THE ORIGIN FOR POLYNOMIAL-RATE SHRINKAGE ESTIMATORS OF A NORMAL MEAN

KOREA SUPERINTELLIGENCE LABS

Abstract. Let $X \sim N_p(\theta, I_p)$ be estimated under quadratic loss. Maruyama and Takemura build, from a positive non-decreasing function Φ, shrinkage estimators $\hat\theta_\varphi$ that dominate the James–Stein estimator $\hat\theta_{\mathrm{JS}}$ provided they dominate it at the origin, i.e. provided $\mathrm{Rdiff}_0(\hat\theta_{\mathrm{JS}}, \hat\theta_\varphi; p) = R(0, \hat\theta_{\mathrm{JS}}) - R(0, \hat\theta_\varphi) \ge 0$. For their second example $\Phi_2(w; b, \gamma) = (w/b)^\gamma$ with $b = 1$, $\gamma = 5$ their Proposition 3.2 gives this for all $p \ge 8$, and of the remaining dimensions they write only that "we numerically check that $\mathrm{Rdiff}_0(\hat\theta_{\mathrm{JS}}, \hat\theta_\varphi; p_*) \ge 0$ for $3 \le p \le 7$ individually"; no quadrature rule, error bound or range appears in the paper. We prove those five cells. We prove the same for $b = 1$, $C = 1$ at $p = 4, \ldots, 7$, cell by cell, where the source instead propagates upward from $p = 3$ by its Proposition 3.1. At $p = 3$ we prove a two-sided rational bracket for the underlying integral, $1/300 \le I \le 1/50$, so that the dominance there is strict and the risk gain at the origin lies between $1/(6\sqrt{2\pi}) = 0.0664\ldots$ and $1/\sqrt{2\pi} = 0.3989\ldots$, where the source prints only a table symbol. We also certify one entry the source marks as failing — at $b = 3$, $p = 3$ the risk difference is at most $-1/20000$ times an explicit positive constant — and the return to nonnegativity at $p = 4$ in that same column. The method replaces floating-point quadrature by exact rational coverings: after the substitution $w = u^2$ the whole question is the sign of a one-variable integral of a rational function against the Gaussian weight, and fourteen finite rational computations bound it. The reduction, the soundness of the coverings, the alternating Taylor sandwich for exp and the Gaussian tail estimate are proved rather than computed. All statements are machine-checked in Lean 4; Section 8 records exactly what the kernel checks and what it does not.

1. Introduction

1.1. Shrinkage estimators of a normal mean. Let X have a p-variate normal distribution $N_p(\theta, I_p)$ and consider estimating θ under the quadratic loss $\|\hat\theta - \theta\|^2$, with risk $R(\theta, \hat\theta) = E_\theta\|\hat\theta(X) - \theta\|^2$. For $p \ge 3$ the James–Stein estimator

$$\hat\theta_{\mathrm{JS}}(X) = \Big(1 - \frac{p-2}{\|X\|^2}\Big)X \tag{1}$$

dominates X [2], and a large literature is concerned with estimators that in turn dominate $\hat\theta_{\mathrm{JS}}$; the general shrinkage class is

$$\hat\theta_\varphi(X) = \Big(1 - \frac{\varphi(\|X\|^2)}{\|X\|^2}\Big)X. \tag{2}$$

The classical sufficient condition for the minimaxity of $\hat\theta_\varphi$ — that φ be nonnegative, non-decreasing and bounded by $2(p-2)$ — is Baranchik's [4]; necessary conditions for the harder problem of dominating $\hat\theta_{\mathrm{JS}}$ itself are given by Maruyama and Strawderman [5]. The standard tool for the risk comparisons themselves is Stein's unbiased estimate of risk [3].

In a recent paper, Maruyama and Takemura [1] give a general construction of such estimators from a positive non-decreasing function Φ on $(0, \infty)$: they put

$$\varphi(w) = p - 2 - \frac{1}{\displaystyle\int_0^w t^{-1}\Phi(t)\,dt + C}, \qquad C \ge \frac{1}{p-2}, \tag{3}$$

and prove that $\hat\theta_\varphi$ then dominates $\hat\theta_{\mathrm{JS}}$, *provided* $\hat\theta_\varphi$ already dominates $\hat\theta_{\mathrm{JS}}$ at the origin. That proviso is their Assumption A.7, "$R(0, \hat\theta_{\mathrm{JS}}) \ge R(0, \hat\theta_\varphi)$, i.e. dominance at the origin", and it is the only part of their hypothesis that is not a closed-form condition on Φ. Writing

$$\mathrm{Rdiff}_0(\hat\theta_{\mathrm{JS}}, \hat\theta_\varphi; p) = R(0, \hat\theta_{\mathrm{JS}}) - R(0, \hat\theta_\varphi) = \int_0^\infty \{4\Phi(w) - 1\}\, h(w)\, g_p(w)\, dw, \tag{4}$$

where g_p is the χ^2_p density and

$$h(w) = \frac{\{p - 2 - \varphi(w)\}^2}{w} = \frac{1}{w\bigl\{\int_0^w t^{-1}\Phi(t)\, dt + C\bigr\}^2}, \tag{5}$$

the question is the sign of one explicit integral, for each p and each choice of Φ and C.

1.2. **The five dimensions that are checked numerically.** The source's two propositions propagate dominance at the origin upward in p from a base dimension p_*. The first, Proposition 3.1, needs Φ and C both independent of p, so it says nothing about the column $C = 1/(p-2)$; the second, Proposition 3.2, is the one for that column, and it requires $p_* \geq \beta + 2$ for a constant $\beta > 0$ with $\Phi(w) \leq \beta \int_0^w t^{-1}\Phi(t)\, dt$ for all $w \geq 0$. For their second example,

$$\Phi_2(w; b, \gamma) = (w/b)^\gamma, \qquad b > 0,\ \gamma > 0, \tag{6}$$

one has $\beta = \gamma$, so at $\gamma = 5$ every dimension $3 \leq p \leq 7$ falls outside the propositions and must be treated one at a time. On this the source says exactly one thing:

> "For example, when $b = 1$ and $\gamma = 5$, we have $\beta + 2 = \gamma + 2 = 7$. Then we numerically check that $\mathrm{Rdiff}_0(\hat\theta_{\mathrm{JS}}, \hat\theta_\varphi; p_*) \geq 0$ for $3 \leq p \leq 7$ individually. On the other hand, $\mathrm{Rdiff}_0(\hat\theta_{\mathrm{JS}}, \hat\theta_\varphi; p_*) \geq 0$ for $p \geq 8$ follows from Proposition 3.2."

No quadrature rule, no error bound and no range of the numerical evaluation appears anywhere in the paper. Two of the source's tables tabulate Φ_2 over (p, b, γ): its Table 2 for $C = 1$ and its Table 3 for the smallest admissible constant $C = 1/(p-2)$. The five dimensions above are exactly the five starred entries of the $b = 1$, $\gamma = 5$ column of Table 3; in the same column of Table 2 only $p = 3$ carries a star. The three symbols are explained in the source's text as follows: "A star ($\star$) indicates pairs for which Assumption A.7 is individually fulfilled, while a bullet ($\bullet$) denotes pairs for which Assumption A.7 is uniformly fulfilled, by Propositions 3.1 and 3.2. A minus ($-$) indicates that Assumption A.7 is not satisfied for the corresponding pair." A star, a bullet and a minus are the whole of the numerical evidence recorded there: no value of Rdiff_0 is printed anywhere in the paper.

1.3. **What is proved here.** Throughout, $\gamma = 5$ and C is one of the two tabulated constants. We prove:

- $\mathrm{Rdiff}_0(\hat\theta_{\mathrm{JS}}, \hat\theta_\varphi; p) \geq 0$ for $p = 3, 4, 5, 6, 7$ at $b = 1$, $C = 1/(p-2)$ (Theorems 4.1 and 4.2) — exactly the five cells quoted above, the last of which, $p = 7 = \gamma + 2$, is the base case from which the source's Proposition 3.2 delivers every $p \geq 8$;
- the same for $b = 1$, $C = 1$ at $p = 4, 5, 6, 7$ (Theorem 5.1), which the source instead obtains from its Proposition 3.1 out of its numerical check at $p = 3$, and which we prove directly, cell by cell; at $p = 3$ the two constants coincide, so that cell is covered by the previous item;
- a two-sided rational bracket at $p = 3$ (Theorem 6.1 and Corollary 6.2), which makes the dominance there *strict* and pins the risk gain at the origin between two explicit numbers;
- that at $b = 3$, $C = 1$ the risk difference at $p = 3$ is strictly negative (Theorem 7.1) — one of the entries the source marks with a minus — and that it is nonnegative again at $p = 4$ in that same column (Theorem 7.2).

The last two items are the negative controls of the work: they show that the five certified cells are not instances of a statement true for all parameters, and that the certificate used to prove them does not prove everything (Proposition 7.3).

The proofs are exhaustive rational computations over an explicit covering of the half line, carried out in a proof assistant; Section 3 describes the covering and Section 8 says precisely which finite computations are trusted. The mathematical content that makes such a covering possible is the reduction of Section 2: the substitution $w = u^2$ turns (4) for $\Phi = \Phi_2(\cdot; b, 5)$ into the integral of a *rational* function against the Gaussian weight $e^{-u^2/2}$, with natural-number exponents throughout.

1.4. **What is not claimed.** We do not reprove the source's domination theorem, nor its propositions; nothing below shows that $\hat{\theta}_\varphi$ dominates $\hat{\theta}_{\mathrm{JS}}$. What is settled here is the hypothesis of that theorem — dominance at the origin — in the listed cells; granting the source's theorem, domination follows there, but the granting is the source's mathematics and not ours.

We do not claim that the *sign* of Rdiff_0 in these cells was unknown. The source asserts it and, by its own account, checked it numerically; our contribution is that the five cells are now proved, together with the size of the gain at $p = 3$ and the certified failure at $b = 3$, neither of which is quantified in [1]. We searched the literature for an earlier rigorous treatment of these cells and found none, but a negative search is a negative search.

We say nothing about $\gamma \neq 5$, about $b \notin \{1, 3\}$, about the source's first example Φ_1, or about $p \geq 8$, which is the source's Proposition 3.2 and not in question here. Section 10 says what each of those would cost.

Remark 1.1 (the $p = 3$ instance is not degenerate; outside the formal development). At $p = 3$, $b = 1$, $\gamma = 5$ and $C = 1/(p-2) = 1$, the construction (3) reads $\varphi(w) = 1 - 1/(w^5/5 + 1)$: it is nonnegative, non-decreasing, bounded by 1, vanishes at $w = 0$ and tends to $1 = p-2$ as $w \to \infty$. So $p = 3$ is a genuine instance of the construction and not a boundary artefact — and, since $p = 3 < \gamma + 2 = 7$, it lies outside the range of the source's Proposition 3.2, which is why the source has to check it by hand. Nor is any other hypothesis in doubt: the source's Theorem 1.1′ asks only that Φ be positive and non-decreasing on $(0, \infty)$ and that $C \geq 1/(p-2)$, both of which hold here ($\Phi_2(w; 1, 5) = w^5$, $C = 1 = 1/(p-2)$), and then A.7 is its sole remaining hypothesis. This paragraph is an elementary hand computation, recorded to justify the interest of the cell; it is not part of the formal development below.

2. The risk difference as a one-variable integral

2.1. **The source's algebra.** Fix $p \geq 3$, $b > 0$, $\gamma = 5$ and $C > 0$, and take $\Phi = \Phi_2(\cdot; b, 5)$ in (3). Then the inner integral of (3) and (5) is in closed form,

$$\int_0^w t^{-1}\Phi_2(t; b, 5)\, dt = \frac{w^5}{5b^5}, \tag{7}$$

so that, with $g_p(w) = w^{p/2-1}e^{-w/2}/(2^{p/2}\Gamma(p/2))$, the integrand of (4) is

$$\big(4(w/b)^5 - 1\big)\, \frac{w^{p/2-2}e^{-w/2}}{\big(w^5/(5b^5) + C\big)^2} \cdot \frac{1}{2^{p/2}\Gamma(p/2)}.$$

We therefore set

$$K_p(b, C) := \int_0^\infty \big(4(w/b)^5 - 1\big)\, \frac{w^{p/2-2}e^{-w/2}}{\big(w^5/(5b^5) + C\big)^2}\, dw, \tag{8}$$

so that, by (4)–(7),

$$\mathrm{Rdiff}_0(\hat{\theta}_{\mathrm{JS}}, \hat{\theta}_\varphi; p) = \frac{K_p(b, C)}{2^{p/2}\,\Gamma(p/2)}. \tag{9}$$

The prefactor is positive, so the sign of the risk difference at the origin is the sign of $K_p(b, C)$, and every statement below about Rdiff_0 is a statement about K_p read through (9). The content of (9) is (4), which is the source's derivation and not ours; all that is added here is the closed form (7) and the choice to carry the constant separately. We use (4) as given (see Section 8).

2.2. **Clearing the fractional exponents.** The exponents in (8) are half-integral, which no rational covering can handle directly. The substitution $w = u^2$ removes them. Write, for $k \in \mathbb{N}$ and reals $B \geq 0$, $c > 0$,

$$G_{k,B,c}(u) = \frac{4u^{k+10} - Bu^k}{(u^{10} + c)^2}, \qquad F_{k,B,c}(u) = G_{k,B,c}(u)\, e^{-u^2/2}. \tag{10}$$

Proposition 2.1 (the reduction)**.** *Let $p \geq 3$ be an integer and let $b, C > 0$. Then, with $k = p-3$, $B = b^5$ and $c = 5Cb^5$,*

$$K_p(b,C) = 50\,b^5 \int_0^\infty F_{k,B,c}(u)\,du.$$

In particular $K_p(b,C)$ and $\int_0^\infty F_{p-3,b^5,5Cb^5}$ have the same sign.

Proof. Substitute $w = u^2$, $dw = 2u\,du$. Then $w^{p/2-2} = u^{p-4}$, so the numerator of (8) contributes $2u^{p-3}\big(4u^{10}/b^5 - 1\big)e^{-u^2/2}$, while $\big(u^{10}/(5b^5) + C\big)^2 = (u^{10} + 5Cb^5)^2/(25b^{10})$. Multiplying,

$$2 \cdot 25b^{10} \cdot \frac{(4u^{10}/b^5 - 1)\,u^k}{(u^{10}+c)^2} e^{-u^2/2} = 50\,b^5 \cdot \frac{4u^{k+10} - b^5 u^k}{(u^{10}+c)^2} e^{-u^2/2},$$

which is $50b^5 F_{k,B,c}(u)$. Integrability on $(0,\infty)$ holds for both sides: for $0 \leq k \leq 10$ one has $|G_{k,B,c}(u)| \leq (4+B)(1+c^{-2})$ for all $u \geq 0$, since u^{k+10} and u^k are both at most $u^{20}+1$ and $u^{20} + c^2 \leq (u^{10}+c)^2$, so $|F_{k,B,c}|$ is dominated by a constant multiple of $e^{-u^2/2}$. □

Proposition 2.1 is the source's algebra plus a change of variables, and is claimed here only as the bridge between (8) and the certificates; it is stated because every result below passes through it. The point of the reduced form is that $F_{k,B,c}$ is a rational function of u with natural-number exponents, multiplied by $e^{-u^2/2}$: both factors admit two-sided rational enclosures on an interval, and nothing transcendental remains except the exponential.

For the cells at issue, $b = 1$ gives $B = 1$ and $c = 5C$; with $C = 1/(p-2)$ this is $c = 5/(p-2)$, so the five dimensions $p = 3, \ldots, 7$ are the five parameter pairs

$$(k,c) = (0,5), \quad (1, \tfrac{5}{2}), \quad (2, \tfrac{5}{3}), \quad (3, \tfrac{5}{4}), \quad (4,1), \tag{11}$$

and with $C = 1$ they are $(k,c) = (k,5)$ for $k = 0, \ldots, 4$. The choice $b = 3$ gives $B = 243$ and, at $C = 1$, $c = 1215$.

3. The certificates

3.1. **An alternating sandwich for the exponential.** Let $E_n(s) = \sum_{k<n} s^k/k!$ and $R_n(s) = e^s - E_n(s)$. Then $R'_{n+1} = R_n$ and $R_n(0) = 0$ for $n \geq 1$, whence by induction on n,

$$s \leq 0 \implies (-1)^n R_n(s) \geq 0 : \tag{12}$$

on the negative axis an even number of terms under-estimates e^s and an odd number over-estimates it. Both bounds are rational at rational s. Integrating the polynomial sandwich obtained from (12) at $s = -y^2/2$ over $[0,u]$ gives, with

$$\mathcal{G}_n(u) = \sum_{k<n} \frac{(-1)^k u^{2k+1}}{2^k\,k!\,(2k+1)}, \tag{13}$$

the estimate $\mathcal{G}_n(u) \leq \int_0^u e^{-y^2/2}\,dy \leq \mathcal{G}_m(u)$ for $u \geq 0$, n even, m odd. These, together with the rational bounds $2.5066282746310005024 \leq \sqrt{2\pi} \leq 2.5066282746310005025$, are the whole of the analytic input to the numerics. The library we verify against, Mathlib [7] at the revision pinned by the development (Lean 4.32.0), has the Gaussian integral $\int e^{-bx^2} = \sqrt{\pi/b}$ and the normal density, but no error function and no normal distribution function, so this enclosure layer is written from scratch and is self-contained.

3.2. **Cell minorants.** Fix $k \leq 4$ and rationals $B \geq 0$, $c > 0$, and let $[a,\beta] \subset [0,\infty)$ be a cell with rational endpoints. On it,

$$4a^{k+10} - B\beta^k \;\leq\; 4u^{k+10} - Bu^k, \qquad (a^{10}+c)^2 \leq (u^{10}+c)^2 \leq (\beta^{10}+c)^2,$$

so, writing $N = 4a^{k+10} - B\beta^k$ and $\lfloor x \rfloor_{24} = 2^{-24}\lfloor 2^{24}x \rfloor$ for rounding down to a multiple of 2^{-24}, the rational number

$$m(k,B,c;a,\beta) = \begin{cases} \big\lfloor N/(\beta^{10}+c)^2 \big\rfloor_{24}, & N \geq 0, \\ \big\lfloor N/(a^{10}+c)^2 \big\rfloor_{24}, & N < 0, \end{cases}$$

satisfies $m \le G_{k,B,c}(u)$ for every $u \in [a, \beta]$. Multiplying by the Gaussian factor uses (12) in the direction that the sign of m dictates: if $m \ge 0$ we use the 18-term (even, hence lower) bound $E_{18}(-\beta^2/2) \le e^{-u^2/2}$, and if $m < 0$ the 19-term (odd, hence upper) bound $e^{-u^2/2} \le E_{19}(-a^2/2)$. Rounding the product down to a multiple of 2^{-24} gives a rational $\mu(k, B, c; a, \beta) \le F_{k,B,c}(u)$ on the whole cell. Reversing every inequality gives, in the same way, a rational majorant $\bar{\mu}(k, B, c; a, \beta) \ge F_{k,B,c}$ on the cell.

3.3. **The covering sums and their soundness.** Let the mesh be uniform of width 1/64, and put

$$S_{k,B,c}(n) = \frac{1}{64}\sum_{i=0}^{n-1} \mu\Big(k, B, c; \frac{i}{64}, \frac{i+1}{64}\Big), \qquad \bar{S}_{k,B,c}(n) = \frac{1}{64}\sum_{i=0}^{n-1} \bar{\mu}\Big(k, B, c; \frac{i}{64}, \frac{i+1}{64}\Big), \tag{14}$$

both rational. Write $T = n/64$.

Lemma 3.1 (lower soundness)**.** *Let $0 \le k \le 4$ and let $B \ge 0$, $c > 0$ be rational with $B \le 4T^{10}$. Then*

$$S_{k,B,c}(n) \;\le\; \int_0^\infty F_{k,B,c}(u)\,du.$$

Proof. On $[0, T]$ the sum is a lower Riemann sum by Section 3.2. On $[T, \infty)$ the integrand is nonnegative: $4u^{k+10} - Bu^k = u^k(4u^{10} - B) \ge 0$ once $B \le 4T^{10} \le 4u^{10}$, and the denominator is positive; so the tail only adds. Integrability is the bound in the proof of Proposition 2.1. □

Lemma 3.2 (upper soundness)**.** *Let $0 \le k \le 4$, let $B \ge 0$, $c > 0$ be rational, let $n \ge 1$ and let m be even. Then*

$$\int_0^\infty F_{k,B,c}(u)\,du \;\le\; \bar{S}_{k,B,c}(n) + \frac{4}{T^{10-k}}\Big(\frac{2.5066282746310005025}{2} - \mathcal{G}_m(T)\Big).$$

Proof. The first term majorises $\int_0^T F_{k,B,c}$ by Section 3.2. For the tail, $u^{20} \le (u^{10} + c)^2$ and $u^{20} = u^{k+10}u^{10-k} \ge u^{k+10}T^{10-k}$ for $u \ge T$ give $G_{k,B,c}(u) \le 4/T^{10-k}$, so $\int_T^\infty F_{k,B,c} \le (4/T^{10-k})\int_T^\infty e^{-u^2/2}\,du$; and $\int_T^\infty e^{-u^2/2}\,du = \sqrt{2\pi}/2 - \int_0^T e^{-u^2/2}\,du$ is at most $2.5066282746310005025/2 - \mathcal{G}_m(T)$ by (13) with m even and the rational upper bound for $\sqrt{2\pi}$. □

Lemmas 3.1 and 3.2 are proved once, for general parameters. What each theorem below adds is a *single* rational inequality about a covering sum, and that inequality is verified by exhaustive evaluation of the finitely many cells. The resolution is not over-provisioned: at $p = 3$ the mesh 1/64 on $[0, 2]$ gives a positive covering sum, whereas halving the resolution to mesh 1/32 on the same interval gives $-0.00117\ldots$, and truncating at $T = 1$ instead of $T = 2$ gives $-0.0262\ldots$. The reason is visible in the integrand: $F_{0,1,5}$ changes sign at $u_0 = 4^{-1/10} = 0.87055\ldots$, and at $p = 3$ the negative mass $0.02847\ldots$ cancels the positive mass $0.03806\ldots$ to within 25%, the tightest of the five cells. (The decimal figures in this paragraph are floating-point computations, quoted to explain the mesh; no statement below depends on them.)

4. The five dimensions

Theorem 4.1 (the five cells, reduced form)**.** *For each of the five pairs (k, c) of (11),*

$$0 \;\le\; \int_0^\infty F_{k,1,c}(u)\,du.$$

Proof sketch. Each of the five is Lemma 3.1 with $B = 1$ and $n = 128$, so $T = 2$ and the hypothesis $B \le 4T^{10} = 4096$ holds, applied to the finite rational inequality $0 \le S_{k,1,c}(128)$. That inequality is the exhaustive part: for each cell it evaluates 128 cell minorants in exact rational arithmetic and sums them. The five verified values are

$$\tfrac{1027617}{268435456}, \quad \tfrac{25209751}{1073741824}, \quad \tfrac{13917413}{268435456}, \quad \tfrac{93693251}{1073741824}, \quad \tfrac{8755677}{67108864},$$

that is $0.00382\ldots$, $0.02347\ldots$, $0.05184\ldots$, $0.08725\ldots$, $0.13046\ldots$, all positive. No search over parameters is involved: the five parameter pairs are given by (11), and for each one the finite computation is a single sum. □

Theorem 4.2 (the five cells, in the source's variable)**.** *With K_p as in* (8)*,*

$$K_3(1,1) \geq 0, \quad K_4\big(1,\tfrac{1}{2}\big) \geq 0, \quad K_5\big(1,\tfrac{1}{3}\big) \geq 0, \quad K_6\big(1,\tfrac{1}{4}\big) \geq 0, \quad K_7\big(1,\tfrac{1}{5}\big) \geq 0.$$

Consequently, by (9)*, for $\Phi = \Phi_2(\cdot;1,5)$ and $C = 1/(p-2)$,*

$$\mathrm{Rdiff}_0(\hat\theta_{\mathrm{JS}}, \hat\theta_\varphi; p) \ \geq \ 0 \qquad \textit{for } p = 3,4,5,6,7.$$

Proof. Proposition 2.1 with $b = 1$, $C = 1/(p-2)$ turns $K_p(1,C)$ into $50\int_0^\infty F_{p-3,1,5/(p-2)}$, which is nonnegative by Theorem 4.1. The final display is (9), whose prefactor $2^{p/2}\Gamma(p/2)$ is positive. □

Theorem 4.2 is exactly the assertion the source records as numerically checked. The last of the five, $p = 7$, is the base case $p_* = \gamma + 2 = 7$ from which the source's Proposition 3.2 delivers dominance at the origin for every $p \geq 8$; so with these five cells proved, the $b = 1$, $\gamma = 5$ column of the source's Table 3 rests on no unproved numerical evaluation — granting the source's own Proposition 3.2 for $p \geq 8$, which we do not reprove.

5. The column $C = 1$

The source's other table takes $C = 1$ rather than the minimal admissible $C = 1/(p-2)$. For $p \geq 3$ the constraint $C \geq 1/(p-2)$ of (3) is satisfied, with equality at $p = 3$; so at $p = 3$ the two columns are the same cell and Theorem 4.1 already covers it. Unlike the column of Section 4, the entries at $p = 4,\dots,7$ here carry a bullet rather than a star: $C = 1$ does not depend on p, so the source's Proposition 3.1 propagates from the base $p_* = 3$ and no separate numerical check is claimed for them. What the four theorems below add is therefore not a missing check but an independent one — each cell is certified on its own, without the propagation and without the numerical evaluation at $p = 3$ that feeds it.

Theorem 5.1 (the $C = 1$ column)**.** *For $k = 1,2,3,4$,*

$$0 \ \leq \ \int_0^\infty F_{k,1,5}(u)\,du,$$

that is, $\mathrm{Rdiff}_0(\hat\theta_{\mathrm{JS}}, \hat\theta_\varphi; p) \geq 0$ *for $p = 4,5,6,7$ at $b = 1$, $\gamma = 5$, $C = 1$. No use is made of the source's Proposition 3.1, nor of the case $p = 3$.*

Proof sketch. As for Theorem 4.1: Lemma 3.1 with $B = 1$, $c = 5$, $n = 128$, applied to the four verified inequalities $0 \leq S_{k,1,5}(128)$, whose values are $\frac{30241059}{1073741824}$, $\frac{45720793}{1073741824}$, $\frac{61747039}{1073741824}$ and $\frac{82048467}{1073741824}$, i.e. $0.02816\ldots$, $0.04258\ldots$, $0.05750\ldots$ and $0.07641\ldots$. Each is one finite sum of 128 rational cell minorants. The transfer to Rdiff_0 is Proposition 2.1 and (9), as in Theorem 4.2. □

6. Strictness, and the size of the gain at $p = 3$

Nothing in [1] distinguishes $\mathrm{Rdiff}_0 > 0$ from $\mathrm{Rdiff}_0 = 0$, and no value of Rdiff_0 is printed there. At $p = 3$ — the cell with the thinnest margin — we can bracket the integral from both sides.

Theorem 6.1 (a two-sided bracket at $p = 3$)**.**

$$\frac{1}{300} \ \leq \ \int_0^\infty F_{0,1,5}(u)\,du \ \leq \ \frac{1}{50}.$$

Proof sketch. The lower half is Lemma 3.1 with $B = 1$, $c = 5$, $n = 128$ applied to the verified rational inequality $1/300 \leq S_{0,1,5}(128) = 1027617/268435456$; note that Lemma 3.1 bounds the whole improper integral from below by the covering sum, not merely its sign. The upper half is Lemma 3.2 with the same k, B, c, $n = 128$ (so $T = 2$) and $m = 26$ terms in the tail series, applied to the verified inequality

$$\bar S_{0,1,5}(128) + \frac{4}{2^{10}}\Big(\frac{2.5066282746310005025}{2} - \mathcal{G}_{26}(2)\Big) \ \leq \ \frac{1}{50},$$

whose left-hand side evaluates to $0.01621\ldots$. Each half is one finite computation over the same 128 cells, the first with minorants and the second with majorants. □

Corollary 6.2. $\frac{1}{6} \le K_3(1,1) \le 1$, *and hence at* $p=3$, $b=1$, $\gamma=5$, $C=1$,

$$0.0664\ldots = \frac{1}{6\sqrt{2\pi}} \;\le\; \mathrm{Rdiff}_0(\hat\theta_{\mathrm{JS}}, \hat\theta_\varphi; 3) \;\le\; \frac{1}{\sqrt{2\pi}} = 0.3989\ldots$$

In particular the dominance at the origin is strict in this cell.

Proof. Proposition 2.1 at $p=3$, $b=1$, $C=1$ gives $K_3(1,1) = 50\int_0^\infty F_{0,1,5}$, so Theorem 6.1 gives $50/300 \le K_3(1,1) \le 50/50$. For the second display use (9) with $2^{3/2}\Gamma(3/2) = \sqrt{2\pi}$. □

The bracket is not tight — the value of $\int_0^\infty F_{0,1,5}$ is $0.00959\ldots$, so the true $\mathrm{Rdiff}_0(\hat\theta_{\mathrm{JS}}, \hat\theta_\varphi; 3)$ is about 0.1914 — but both ends are rational and proved, where the source prints a symbol. The same two-sided treatment at $p = 4, \ldots, 7$ needs no new machinery, only the majorant sums; it was not carried out here.

7. A failing cell, a sign flip, and the controls

The source's tables also carry entries marked with a minus, meaning that dominance at the origin fails. One of them is at $b=3$ (still $\gamma=5$, $C=1$) in dimension $p=3$. Here $B = b^5 = 243$ and $c = 5Cb^5 = 1215$.

Theorem 7.1 (the failing cell, certified)**.**

$$\int_0^\infty F_{0,243,1215}(u)\,du \;\le\; -\frac{1}{20000} \;<\; 0,$$

and consequently $K_3(3,1) < 0$*: at* $b=3$, $\gamma=5$, $C=1$ *and* $p=3$, $\mathrm{Rdiff}_0(\hat\theta_{\mathrm{JS}}, \hat\theta_\varphi; 3) < 0$.

Proof sketch. Lemma 3.2 with $k=0$, $B=243$, $c=1215$, $n=256$ (so $T=4$) and $m=36$ tail terms, applied to the verified rational inequality

$$\bar S_{0,243,1215}(256) + \frac{4}{4^{10}}\Big(\frac{2.5066282746310005025}{2} - \mathcal{G}_{36}(4)\Big) \;\le\; -\frac{1}{20000},$$

whose left-hand side evaluates to $-0.0000963\ldots$. The exhaustive part is the sum of 256 rational cell majorants. The transfer to $K_3(3,1)$ is Proposition 2.1, whose factor $50b^5 = 12150$ is positive, and then (9). □

Theorem 7.2 (the sign flip at $p=4$)**.**

$$0 \;\le\; \int_0^\infty F_{1,243,1215}(u)\,du,$$

that is, $\mathrm{Rdiff}_0(\hat\theta_{\mathrm{JS}}, \hat\theta_\varphi; 4) \ge 0$ *at* $b=3$, $\gamma=5$, $C=1$.

Proof sketch. Lemma 3.1 with $k=1$, $B=243$, $c=1215$, $n=192$ (so $T=3$, and $B = 243 \le 4\cdot 3^{10}$), applied to the verified inequality $0 \le S_{1,243,1215}(192) = 15859/1073741824 = 0.0000147\ldots$; one finite sum of 192 rational cell minorants. The margin is two to four orders of magnitude thinner than in Theorem 4.1, and 128 cells do not suffice: $S_{1,243,1215}(128)$ is negative. □

Theorems 7.1 and 7.2 together certify the transition from a failing to a succeeding cell inside one column of the source's table, and they are what keeps Theorems 4.1, 4.2 and 5.1 from being instances of some statement true for all parameters: neither the negative nor the nonnegative half of that column extends to the whole of it. Two further controls say that the theorems are not pointwise trivialities and that the certificate is not vacuous.

Proposition 7.3 (controls)**.** (i) $F_{0,1,5}(1/2) < 0$*: the integrand of Theorem 4.1 really changes sign, so its conclusion is not a pointwise inequality.*

(ii) $0 \le S_{0,243,1215}(256)$ *is* false*: the same covering test that proves the cells above fails at the cell of Theorem 7.1, so the hypothesis of Lemma 3.1 is a real hypothesis and not an artefact of the construction.*

Proof. (i) is the evaluation $G_{0,1,5}(1/2) = \big(4\cdot 2^{-10} - 1\big)/\big(2^{-10}+5\big)^2 < 0$ together with $e^{-1/8} > 0$. (ii) is one finite rational computation: the covering sum in question is $-0.000113\ldots$. □

8. Verification

Every numbered statement of Sections 2–7 — Proposition 2.1, Lemmas 3.1 and 3.2, Theorems 4.1, 4.2, 5.1, 6.1, 7.1 and 7.2, Corollary 6.2 and Proposition 7.3 — is machine-checked in Lean 4 [6] over the Mathlib library [7]; the development is four modules on top of a rational Gaussian-enclosure module, and contains no `sorry`. The division of labour between proof and computation is the point of the construction and is visible in the axiom prints. The whole reasoning layer — the continuity, the global bound and the integrability of $F_{k,B,c}$, the cell minorants and majorants of Section 3.2, the alternating sandwich (12) and its integrated form (13), the tail estimates, the soundness Lemmas 3.1 and 3.2, and the change of variables of Proposition 2.1 (which rests on Mathlib's substitution theorem for $u \mapsto u^2$ on $(0, \infty)$) — is checked by the kernel alone, an axiom print returning `propext`, `Classical.choice` and `Quot.sound` and nothing else. Compiled evaluation enters only through fourteen finite computations in exact rational arithmetic, one for each invocation in the development: the nine covering sums of Theorems 4.1 and 5.1 (128 cells each), the two of Theorem 6.1 (128 cells each, minorants and majorants), the one of Theorem 7.1 (256 cells), the one of Theorem 7.2 (192 cells), and the refutation of Proposition 7.3(ii) (256 cells) — 2112 cell bounds in all. The axiom print of each certified theorem names the three standard axioms together with exactly the one or two of these fourteen it uses, and nothing further; that is the precise and only sense in which the results above rest on computation. The source of the development is currently held in a private repository: repository reference to be added at submission.

Three things above are stated in a form the development does not carry, and we say which. First, the risk functional itself is not formalised: the development proves statements about the integrals $K_p(b, C)$ and $\int_0^\infty F_{k,B,c}$, and the identity (9) relating K_p to Rdiff_0 — which is the source's derivation (4)–(5) — is used as given, so every sentence above of the form "$\mathrm{Rdiff}_0 \geq 0$" is the machine-checked statement about K_p read through (9). Second, Corollary 6.2 is the one line of arithmetic that combines two machine-checked statements (Theorem 6.1 and Proposition 2.1) with the classical value $2^{3/2}\Gamma(3/2) = \sqrt{2\pi}$; the constant $1/(6\sqrt{2\pi})$ is not itself in the development. Third, Remark 1.1 and the numerical figures quoted in Section 3 and Section 9 are hand or floating-point computations, outside the formal development, and are labelled as such where they appear.

9. The source's tables, recomputed

Independently of the certificates, and before any of them was attempted, the two tables of [1] that tabulate Φ_2 were recomputed here from the source's own definitions (4)–(6), by composite Gauss–Legendre quadrature in the variable u after the substitution of Proposition 2.1. Over all $(p, b, \gamma) \in \{3, \dots, 10\} \times \{1, 3, 5, 7, 9\} \times \{\frac{1}{4}, \frac{1}{2}, 1, 2, 3, 4, 5\}$ and both constants C — 560 cells — the computed sign of Rdiff_0 agrees with the source's symbol in every cell, reading a minus as negative and a star or a bullet as nonnegative; and at $p = 8, 9$ in the $b = 1$, $\gamma = 5$ column, which the source's Proposition 3.2 covers, the computed risk difference is positive, as that proposition requires. Deriving the star/bullet distinction as well — a bullet at p exactly when some admissible smaller base dimension has a nonnegative risk difference, admissibility being $p_* \geq 3$ for Proposition 3.1 and $p_* \geq \gamma+2$ for Proposition 3.2 — reproduces 558 of the 560 symbols. The two differences are both in the table with $C = 1/(p-2)$, at $(p, b, \gamma) = (5, 3, 2)$ and $(6, 5, 1)$, and they concern which of the two labels a cell carries, not the sign of the risk difference there: at $(5, 3, 2)$ the source prints a bullet where the rule as we read it gives a star, and at $(6, 5, 1)$ a star where the rule gives a bullet. The rule is our reading of the source's explanation of the symbols together with the statements of its Propositions 3.1 and 3.2, all three of which we have checked against the v1 e-print; but a labelling convention is not a theorem, and the two entries are recorded here only because we computed them, not as corrections. Nothing in this paper depends on either cell.

For orientation, the reduced integrals of the five cells of Theorem 4.1 evaluate numerically, at $p = 3, \dots, 7$, to

$$0.00959\dots, \quad 0.03893\dots, \quad 0.07875\dots, \quad 0.12756\dots, \quad 0.18662\dots,$$

giving for Rdiff_0 the approximate values 0.1914, 0.4867, 0.5236, 0.3986 and 0.2482; the certified lower bounds of Theorem 4.1 are roughly 40% to 70% of these, the ratio being smallest at $p = 3$. All figures in this section are floating-point computations and are asserted only as such.

10. What is not done, and what it would cost

The machinery takes arbitrary rational $B = b^5$ and $c = 5Cb^5$, so every remaining cell of the two tables at $\gamma = 5$ is one more finite computation of the same shape, but not at the present resolution: in the columns $b = 3, 5, 7, 9$ the reduced integrals get as small as $2 \cdot 10^{-7}$ in absolute value, and the mesh 1/64 does not separate them from zero. Refining the mesh alone does not help either. We ran the covering of Section 3 on $[0, 4]$ at mesh 1/256 and again at mesh 1/1024 for the two extreme cells — $p = 5$, $b = 5$ (a star, reduced integral $+1.2 \cdot 10^{-6}$) and $p = 6$, $b = 7$ (a minus, reduced integral $-2.1 \cdot 10^{-7}$), both at $C = 1$ — and both fail at both resolutions, by margins that barely move between them. The obstruction is elsewhere: the 18/19-term sandwich from (12) brackets $e^{-u^2/2}$ to within $5 \cdot 10^{-11}$ at $u = 2$, but only to within $9 \cdot 10^{-5}$ at $u = 3$, and at $u = 4$ its two ends are -1.97 and 0.84; so a covering that reaches out to $T = 4$ carries far more slack than those cells can afford. Both the mesh width — at present the fixed constant 64 — and the length of the exponential sandwich would have to become parameters, and we do not price the result. Exponents $\gamma \neq 5$ need $u^{2\gamma}$ in place of u^{10}, that is one further natural-number parameter in (10), which is routine for $\gamma \in \{1, 2, 3, 4\}$ and not routine for the tabulated fractional exponents $\gamma \in \{\frac{1}{4}, \frac{1}{2}\}$, where the exponent leaves the natural numbers. The source's first example, $\Phi_1(w; b) = (1/b)\, w/(w + 1)$, gives $\varphi_1(w) = p - 2 - 1/\big(\log(w + 1)/b + C\big)$, whose integrand carries a logarithm; the enclosure layer of Section 3 has no rational bounds for log, and that is the missing object. Finally, the two-sided treatment of Section 6 extends to $p = 4, \ldots, 7$ with the majorant sums that are already in place.

Remark 10.1 (cost; outside the formal development). The following are timings, not theorems. The four modules check in 49.1, 18.0, 14.8 and 15.0 seconds respectively, 96.9 seconds in all, at a peak resident memory of 7.03 GB — which is the library import baseline, so the marginal memory cost of this development is negligible. The independent numerics of Section 9 took under four minutes of processor time.

On the reference list

Reference [1] was read here from its arXiv LaTeX e-print; the abstract page was checked twice on 3 September 2026, and on both occasions v1 (submitted 22 September 2025) was the only version, not withdrawn, with no journal reference, a comment field reading "11 pages" and the subject line "Statistics Theory (math.ST)". Every quotation above — the setting, the construction (3), Assumption A.7, the definition (4), the example (6), the sentence quoted in Section 1.2 and the explanation of the table symbols — is from that file. All labels and numbers we cite are v1's, resolved by compiling the e-print: the construction (3) is its (1.9) and, with the constraint $C \geq 1/(p-2)$, its (1.10); the identity (4) is its (2.5) read through its definition (3.2) of Rdiff_0, and (5) is the display defining h that follows its (2.1); Φ_2 is its (4.2); the two tables of Φ_2 are its Tables 2 ($C = 1$) and 3 ($C = 1/(p - 2)$); dominance at the origin is the seventh of its assumptions A.1–A.7, and the propagation results are its Propositions 3.1 and 3.2 in Section 3. A published version may renumber all of these.

The four historical references are cited for background and are used in no proof above. Their bibliographic data were checked against Crossref and the publishers' pages (Project Euclid for the first three).

References

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[7] The mathlib Community, *The Lean mathematical library*, in: Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020), ACM, 2020, 367–381, doi:10.1145/3372885.3373824.