# THE ADMISSIBLE DIMENSIONS OF THE CHAN–WEI REFUTATION: CERTIFYING $\Theta_N > 0$, MONOTONICITY BETWEEN RESONANCES, AND THE BLOCK STRUCTURE OF THE ADMISSIBLE SET

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Abstract. Chen, Liu, Wei and Yang (arXiv:2608.10501) disprove the radiality conjecture of Chan and Wei for positive stable solutions of the Lane–Emden equation $-\Delta u = u^p$ in $\mathbb{R}^n$ in every dimension $n \geq 12$ they call *admissible*: with $\ell = 1 + \sqrt{n-2}$, either $\ell$ is an even integer (the resonant case $n = (2m-1)^2 + 2$), or $\Theta_N = 1/\ell + \sigma_\ell - \sigma_{-1/2} > 0$, where $\sigma_\nu$ is an explicit combination of two values of the digamma function. They print the thirteen admissible dimensions below 150 and nothing above. This paper is about the arithmetic condition itself. Evaluating $\sigma_{-1/2}$ from the same Gauss series as $\sigma_\ell$ makes $\gamma_E$, $\log 2$ and $\pi$ cancel and leaves

$$\Theta_N \;=\; -4 + \frac{1}{1+\ell} + \sum_{k\geq 1} \frac{(2\ell+1)^2}{(4k+1)(2k-\ell)(2k+1+\ell)},$$

a series with $O(k^{-3})$ terms and no transcendental constant. We prove, without derivatives, that $\Theta$ is strictly increasing in $\ell$ on every interval $(2m-2, 2m)$ between consecutive even integers, $m \geq 2$; hence the admissible dimensions of each window $((2m-3)^2+2,\ (2m-1)^2+2]$ form a block ending at the resonant dimension, and two sign evaluations settle a window. A verified rational enclosure of the series then certifies the printed list, extends it to $n \leq 400$, and determines the admissible set for every $12 \leq n \leq 14163 = 119^2 + 2$: 58 blocks, 940 dimensions in all, with block lengths $2, 3, 4, 4, 5, 5, \ldots, 28, 28, 28, 29$. The block-length law $\lceil (m+2)/2 \rceil$, which the first eight blocks suggest, is false: it fails first at $m = 25$, where the block has 13 elements and not 14, and at 31 of the 57 windows $4 \leq m \leq 60$. Every statement is verified in Lean 4 against *Mathlib*; the digamma enters only through its Gauss series, and the identification of that series with $\Gamma'/\Gamma$ is the one classical fact used but not formalized.

## 1. Introduction

**1.1. The source and its arithmetic condition.** Chen, Liu, Wei and Yang [1] study positive stable solutions of the Lane–Emden equation

$$-\Delta u = u^p \quad \text{in } \mathbb{R}^n$$

(their equation (1.1)), a solution being *stable* when $\int_{\mathbb{R}^n} \left(|\nabla\varphi|^2 - p\, u^{p-1}\varphi^2\right) dx \geq 0$ for every $\varphi \in C_c^\infty(\mathbb{R}^n)$. In the supercritical range the stability of the positive radial entire solutions is governed by the Joseph–Lundgren exponent [3], which [1] writes as $p_{\mathrm{JL}}(n) = +\infty$ for $n \leq 10$ and, for $n > 10$,

$$p_{\mathrm{JL}}(n) = 1 + \frac{4}{n-4-2\sqrt{n-1}} = \frac{(n-2)^2 - 4n + 8\sqrt{n-1}}{(n-2)(n-10)} \qquad \text{(their (1.2))}.$$

On the interval $p_{\mathrm{JL}}(n) < p < p_{\mathrm{JL}}(n-1)$ — "the first Joseph–Lundgren window" of [1], their (1.3) — Chan and Wei formulated a radiality conjecture, which [1] cites as [2, Conjecture 11.1] and quotes as follows:

> "If $p_{\mathrm{JL}}(n) \leq p < p_{\mathrm{JL}}(n-1)$, then every positive stable solution of the Lane–Emden equation is radially symmetric about some point."

Liu, Wang, Wei and Wu [4] restated the conjecture for the open interval and proved partial results under an additional reflection symmetry and a stronger restriction on $p$. (We have not consulted [2] or [3] directly; both are cited here as [1] cites them, and the quotation is the one printed in [1]. [4] was read in full.)

The main theorem of [1] refutes the conjecture in every dimension satisfying an *arithmetic* condition. With $N = n - 1$, $\ell = 1 + \sqrt{N-1} = 1 + \sqrt{n-2}$, $\gamma_E$ Euler's constant and $\psi = \Gamma'/\Gamma$ the digamma function, [1] sets, for $\nu \notin \{0, 2, 4, \dots\} \cup \{-1, -3, -5, \dots\}$,

$$\sigma_\nu = -\Big[\log 2 + \gamma_E + \tfrac{1}{2}\psi\big(-\tfrac{\nu}{2}\big) + \tfrac{1}{2}\psi\big(\tfrac{1+\nu}{2}\big)\Big] \tag{1}$$

(this constant comes out of the large-$s$ expansion of the even solution of $F'' + \nu(\nu+1)\operatorname{sech}^2 s\, F = 0$, their Proposition 3.7), puts, for $\ell \notin 2\mathbb{N}$,

$$\Theta_N = \frac{1}{\ell} + \sigma_\ell - \sigma_{-1/2}, \tag{2}$$

adds "Note that explicitly, $\sigma_{-1/2} = \pi/2 + 2\log 2$", and calls $n$ admissible if

$$\ell \notin 2\mathbb{N} \text{ and } \Theta_N > 0, \qquad \text{or} \qquad \ell \in 2\mathbb{N} \tag{3}$$

(their (1.5)). The second alternative is the *resonant* case; [1] notes that it is equivalent to $n = (2m-1)^2 + 2$, $m \geq 3$, so that the first resonant dimensions are $27, 51, 83, \dots$. Their Theorem 1.1 then reads, in paraphrase: let $n \geq 12$ be admissible; with $\alpha_\varepsilon = \frac{N-4}{2} - \sqrt{N-1-\varepsilon^2}$ and $p_\varepsilon = 1 + 2/\alpha_\varepsilon$, there is $\varepsilon_0 > 0$ depending only on $n$ such that for every $0 < \varepsilon < \varepsilon_0$ the spherical equation $\Delta_{S^{n-1}}\Phi - \alpha_\varepsilon(n-2-\alpha_\varepsilon)\Phi + \Phi^{p_\varepsilon} = 0$ has a positive nonconstant axially symmetric solution $\Phi_\varepsilon \in C^\infty(S^N)$ whose linearized operator has a positive lowest eigenvalue, namely $\lambda_0(\varepsilon) = \frac{2}{\pi}\Theta_N\varepsilon^3 + O_N(\varepsilon^4|\log\varepsilon|^3)$ when $\ell \notin 2\mathbb{N}$ and $3\varepsilon^2 + \frac{16}{\pi}\Xi_\ell\varepsilon^3 + O_N(\varepsilon^4|\log\varepsilon|^3)$ when $\ell \in 2\mathbb{N}$; and from $\Phi_\varepsilon$ one obtains a smooth positive nonradial stable solution $u_\varepsilon \in C^\infty(\mathbb{R}^n)$ of $-\Delta u_\varepsilon = u_\varepsilon^{p_\varepsilon}$. The paragraph after the theorem is, verbatim,

> "This theorem disproves the Chan–Wei conjecture in every admissible dimension. Numerical evidence indicates that it should be false for all dimension $n > 10$. The admissable dimensions $n$ with $n < 150$ are: 26, 27, 49, 50, 51, 80, 81, 82, 83, 120, 121, 122, 123."

("admissable" is the source's spelling.) A page later [1] adds that when $\ell \notin 2\mathbb{N}$ and $\Theta_N \leq 0$ "the construction of the solution and the eigenvalue expansion are still valid, but they do not give a positive shifted eigenvalue", positivity being what passes from the singular cone to a smooth stable entire solution. Its Remark 4.12 shows that there are infinitely many non-resonant admissible dimensions: at the single dimension $n = (2m-1)^2+1$ one has $\ell = 2m - \frac{1}{4m} + O(m^{-2})$, and $\sigma_{2m-\delta} = 1/\delta + O(\log(m+1))$ uniformly for $0 < \delta < 1/2$ makes $\Theta_N > 0$ "for all sufficiently large $m$", with no bound on $m$.

1.2. **What the source leaves open.** The printed list is given with no method and no error bound; the source prints no admissible dimension above 150, and states no property of the admissible set beyond the resonant subsequence and the existence result of Remark 4.12. Since $\Theta_N$ is an explicit function of $n$ with no PDE content, four questions are purely arithmetic: is the printed list correct, and can that be certified rather than recomputed; what is the admissible set further out; does it have structure; and how thin is it? Every question here is about the condition (3). Nothing is claimed about the equation, and Theorem 1.1 of [1] is not re-proved.

1.3. **What is settled here.** Throughout, $\psi$ is taken as its Gauss series (Definition 2.1); see Remark 2.5 for exactly what that entails.

*No transcendental constant.* Both digamma arguments in $\sigma_{-1/2}$ equal $\frac{1}{4}$. Evaluating $\sigma_{-1/2}$ from the same series as $\sigma_\ell$, instead of from the closed form $\pi/2 + 2\log 2$, cancels $\log 2$ between the two $\sigma$'s, while $\gamma_E$ cancels inside each; combining the three series termwise gives (Proposition 3.1)

$$\Theta_N = -4 + \frac{1}{1+\ell} + \sum_{k\geq 1} u_k, \qquad u_k = \frac{(2\ell+1)^2}{(4k+1)(2k-\ell)(2k+1+\ell)} = O(k^{-3}),$$

and with $s = \sqrt{n-2}$ each term is a single quotient $(4n+1+12s)/((4k+1)(E_k - 3s))$, $E_k = 4k^2+2k-n \in \mathbb{Z}$, linear in $s$ above and below. The source does not make this observation; it is what makes everything else here elementary.

*Monotonicity between resonances* (Theorem 4.1). $\Theta$ is strictly increasing in $\ell$ on every interval $(2m-2, 2m)$ with $m \geq 2$, and on the part $\ell > \frac{1}{2}$ of $(0, 2)$. The proof needs no derivative: the difference $\Theta(L_2) - \Theta(L_1)$ is itself a termwise positive series. Consequently (Theorem 4.3) in every window $((2m-3)^2+2, (2m-1)^2+2]$ the admissible dimensions are a *suffix* — a block ending at the resonant dimension — and two sign evaluations, $\Theta(n_0) > 0 > \Theta(n_0-1)$, determine the whole window.

*The certified admissible set.* A verified rational enclosure of the series (Proposition 5.2) certifies the printed list for $12 \le n < 150$ (Theorem 6.1), extends it to $12 \le n \le 400$ (Proposition 6.3), and, with 116 sign certificates and Theorem 4.3, determines the admissible set for every $12 \le n \le 14163 = 119^2 + 2$ as 58 blocks, 940 dimensions in all (Theorem 6.4, Corollary 6.5).

*A block-length law refuted* (Theorem 6.6). The block lengths $b_m$, $m = 3, \dots, 60$, begin 2, 3, 4, 4, 5, 5, 6, 6 and so on, and the law $b_m = \lceil (m+2)/2 \rceil$ fits every window with $4 \le m \le 24$. It is false: it fails first at $m = 25$ — the window $(2211, 2403]$ has admissible set $\{2391, \dots, 2403\}$, of size 13 where the law predicts 14 — and it is wrong at 31 of the 57 windows $4 \le m \le 60$, including every $m \ge 36$. The failure is not a numerical near-miss: $\Theta(2390) < -0.014$, while the law needs $\Theta(2390) > 0$.

1.4. **What is not claimed.** The admissible set is not shown to have density zero; that expectation rests on a floating-point probe reported, and labelled as such, in Section 7. The closed form $\sigma_{-1/2} = \pi/2 + 2\log 2$ is neither proved nor used by any statement below; it is invoked once, in the discarded alternative route of Section 7, which is not a theorem of this paper. The identification of the Gauss series with $\Gamma'/\Gamma$ is classical and is not formalized here (Remark 2.5). Nothing is claimed for $n > 14163$ beyond what Theorem 4.1 says for every window. Nothing is claimed about the expectation "false for all dimension $n > 10$" of [1]: Section 7 says only what the certified data mean for the mechanism of its Theorem 1.1. Nothing is claimed about the partial differential equation.

1.5. **Where this was looked for.** Before anything was claimed (7 September 2026) the following were searched. A full-text corpus of about 890 000 arXiv e-prints through early September 2026 contains 169 papers discussing the Joseph–Lundgren exponent, of which four mention the digamma function at all and none carries an admissibility criterion of the form (3); the phrase "radiality conjecture" occurs only in [1]; no paper other than [1] prints two of the four digit blocks $26, 27$ / $49, 50, 51$ / $80, \dots, 83$ / $120, \dots, 123$; and nothing in the corpus cites [1]. OpenAlex records 27 works citing [2], of which exactly one, [4], concerns this conjecture, and it has no recorded citers; [4] contains no occurrence of "digamma", "admissible", "Θ" or any dimension list. Semantic Scholar indexes [1] with no citations. The OEIS returns nothing for the thirteen-term list, for the list extended to 171, or for the block-length sequence; the resonant dimensions $27, 51, 83, 123, \dots$ are the terms from $n = 2$ on of A164897 [6], $a(n) = 4n(n+1)+3 = (2n+1)^2+2$, whose entry concerns something unrelated. Read these negatives as "not found", not as "new": the numbers of Proposition 6.3 take a second for anyone with the definition and a computer algebra system. Theorems 4.1, 4.3 and 6.6 are the statements we did not find anywhere.

## 2. Preliminaries and conventions

**Definition 2.1** (the Gauss series)**.** For $x \in \mathbb{R}$ put

$$\psi_{\mathrm{G}}(x) = \sum_{k=0}^{\infty} \Big( \frac{1}{k+1} - \frac{1}{k+x} \Big),$$

with the convention $1/0 = 0$ for the finitely many $k$ with $k + x = 0$.

The $k$-th term equals $(x-1)/((k+1)(k+x)) = O(k^{-2})$ once $k+x \ge 1$, so the series converges for *every* real $x$ (Lean: `summable_gaussTerm`); this is why no hypothesis appears in Lemma 2.2(ii). Classically, for $x \notin \{0, -1, -2, \dots\}$, $\psi_{\mathrm{G}}(x) = \psi(x) + \gamma_E$ [5, eq. 5.7.6].

**Lemma 2.2** (control on the definition)**.** (i) $\psi_{\mathrm{G}}(1) = 0$*, i.e.* $\psi(1) = -\gamma_E$*;*
(ii) $\psi_{\mathrm{G}}(x+1) = \psi_{\mathrm{G}}(x) + 1/x$ *for every* $x \in \mathbb{R}$ *(with* $1/0 = 0$*);*
(iii) $\psi_{\mathrm{G}}(2) = 1$*;*
(iv) $\psi_{\mathrm{G}}(m+1) = H_m = \sum_{k=0}^{m-1} \frac{1}{k+1}$ *for every* $m \in \mathbb{N}$.

*Proof.* (i) Every term vanishes. (ii) Termwise, $\left(\frac{1}{k+1}-\frac{1}{k+x+1}\right)-\left(\frac{1}{k+1}-\frac{1}{k+x}\right)=\frac{1}{k+x}-\frac{1}{k+x+1}$, whose partial sums telescope to $\frac{1}{x}-\frac{1}{m+x}$; both series converge, so the difference of their sums is the limit $1/x$. (iii) and (iv) follow from (i) and (ii) by induction. □

These are the textbook facts $\psi(1)=-\gamma_E$ and $\psi(x+1)=\psi(x)+1/x$ [5, eqs. 5.4.12, 5.5.2]; *Mathlib* states them over $\mathbb{C}$ for its `Complex.digamma` = `logDeriv` $\Gamma$ (`digamma_one`, `digamma_apply_add_one`, the latter under the hypothesis $s\neq -m$ for all $m\in\mathbb{N}$). The lemma is recorded not as new but as the control pinning $\psi_{\mathrm{G}}$ down: it is the digamma on $1+\mathbb{N}$ with the additive normalisation $\psi+\gamma_E$.

**Definition 2.3.** For $\nu\in\mathbb{R}$ and $n\in\mathbb{N}$ put

$$\sigma(\nu)=-\log 2-\tfrac{1}{2}\psi_{\mathrm{G}}\left(-\tfrac{\nu}{2}\right)-\tfrac{1}{2}\psi_{\mathrm{G}}\left(\tfrac{1+\nu}{2}\right),\qquad \ell(n)=1+\sqrt{n-2},\qquad \Theta(n)=\frac{1}{\ell(n)}+\sigma(\ell(n))-\sigma(-\tfrac{1}{2}),$$

and, for a real parameter $L$, $\Theta_{\mathbb{R}}(L)=1/L+\sigma(L)-\sigma(-\frac{1}{2})$, so that $\Theta(n)=\Theta_{\mathbb{R}}(\ell(n))$ by definition.

For $\nu\notin\{0,2,4,\dots\}\cup\{-1,-3,-5,\dots\}$ the two $\gamma_E$'s hidden in (1) cancel against the $\frac{1}{2}+\frac{1}{2}$, so $\sigma(\nu)$ is the $\sigma_\nu$ of [1] and $\Theta(n)$ is its $\Theta_N$, $N=n-1$, under the identification $\psi_{\mathrm{G}}=\psi+\gamma_E$.

**Definition 2.4** (resonant, admissible)**.** $n\in\mathbb{N}$ is *resonant* if $n=r^2+2$ for some odd $r\in\mathbb{N}$, and *admissible* if it is resonant, or it is not resonant and $\Theta(n)>0$.

Resonance is the source's $\ell\in 2\mathbb{N}$ in the form the source itself gives it: $\ell(n)=2m$ iff $\sqrt{n-2}=2m-1$ iff $n=(2m-1)^2+2$. (That one-line equivalence is not part of the formal development, which takes Definition 2.4 as the definition.) At a resonant $n$ the term of $\psi_{\mathrm{G}}(-\ell/2)$ with $k=\ell/2$ is $1/0=0$ under our convention, so $\Theta(n)$ is a real number with no meaning; the definition of admissibility never consults it, and the certificate of Section 5 refuses to evaluate it (Remark 5.5).

*Remark* 2.5 (the one classical fact not formalized). Every theorem below is a theorem about $\psi_{\mathrm{G}}$ of Definition 2.1. The equivalence of the Gauss series with $\Gamma'/\Gamma$ (through the Weierstrass product) is classical and is *not* formalized here, so the results concern the source's $\Theta_N$ under the standard identification of $\psi$ with its Gauss series, not under a machine-checked proof of that identification. Three things bear on the gap. First, Lemma 2.2 pins $\psi_{\mathrm{G}}$ to the values and the recurrence any digamma must have. Second, outside Lean, $\Theta(n)$ was computed by a second, independent route — $\psi$ by the recurrence shifted to a large argument and the Stirling-type asymptotic series of $\Gamma'/\Gamma$, together with the reflection formula for the negative argument $-\ell/2$ — and agreed with the series form to within $8\times 10^{-12}$ at $n=12,26,30,49,80,120,149,170,300$. Third, the value $\sigma_{-1/2}=\pi/2+2\log 2$ printed by [1] is consistent with our normalisation: $\sigma(-\frac{1}{2})=-\log 2-\psi_{\mathrm{G}}(\frac{1}{4})$, and the classical $\psi(\frac{1}{4})=-\gamma_E-\frac{\pi}{2}-3\log 2$ gives exactly $\pi/2+2\log 2$. That value is not proved here and nothing depends on it, because $\sigma(-\frac{1}{2})$ is only ever evaluated from the same series as $\sigma(\ell)$.

## 3. The closed series form

**Proposition 3.1.** *For every $n\in\mathbb{N}$, writing $\ell=\ell(n)$,*

$$\Theta(n)=-4+\frac{1}{1+\ell}+\sum_{k=1}^{\infty}w_k,\qquad w_k=\frac{1}{2k-\ell}+\frac{1}{2k+1+\ell}-\frac{4}{4k+1},\tag{4}$$

*the series converging absolutely. For every $k$ with $2k\neq\ell$,*

$$w_k=u_k:=\frac{(2\ell+1)^2}{(4k+1)(2k-\ell)(2k+1+\ell)},\tag{5}$$

*and for $n\geq 2$, with $s=\sqrt{n-2}$ and $E_k=4k^2+2k-n$,*

$$u_k=\frac{4n+1+12s}{(4k+1)\,(E_k-3s)}.\tag{6}$$

*The same identity (4) holds for $\Theta_{\mathbb{R}}(L)$ with $\ell$ replaced by any real $L$.*

*Proof.* Both arguments of $\psi_{\mathrm{G}}$ in $\sigma(-\frac{1}{2})$ equal $\frac{1}{4}$, so $\Theta(n) = \frac{1}{\ell} + \psi_{\mathrm{G}}(\frac{1}{4}) - \frac{1}{2}\psi_{\mathrm{G}}(-\frac{\ell}{2}) - \frac{1}{2}\psi_{\mathrm{G}}(\frac{1+\ell}{2})$. All three Gauss series converge, so they may be combined termwise. In the $k$-th term the $\frac{1}{k+1}$ parts cancel (coefficient $1 - \frac{1}{2} - \frac{1}{2}$), and $-\frac{1}{k+1/4} = -\frac{4}{4k+1}$, $\frac{1}{2} \cdot \frac{1}{k-\ell/2} = \frac{1}{2k-\ell}$, $\frac{1}{2} \cdot \frac{1}{k+(1+\ell)/2} = \frac{1}{2k+1+\ell}$. Hence $\Theta(n) = \frac{1}{\ell} + \sum_{k\geq 0} w_k$, and $w_0 = -\frac{1}{\ell} + \frac{1}{1+\ell} - 4$ gives (4). For (5), $\frac{1}{2k-\ell} + \frac{1}{2k+1+\ell} = \frac{4k+1}{(2k-\ell)(2k+1+\ell)}$ and $(4k+1)^2 - 4(2k-\ell)(2k+1+\ell) = (2\ell+1)^2$. For (6) substitute $\ell = 1+s$ and $s^2 = n-2$: $(2\ell+1)^2 = 4n+1+12s$ and $(2k-\ell)(2k+1+\ell) = 4k^2+2k-\ell-\ell^2 = E_k - 3s$. □

*Remark* 3.2. No $\pi$, no $\log 2$, no $\gamma_E$ survives. The $\pi/2$ of the source's $\sigma_{-1/2} = \pi/2 + 2\log 2$ never appears because $\sigma_{-1/2}$ is never evaluated in closed form. For $2k > \ell$ every $u_k$ is positive and $O(k^{-3})$; the finitely many terms with $2k < \ell$ are negative, and the one nearest the pole is what drives $\Theta$ to $-\infty$ as $\ell \downarrow 2m-2$, while $\Theta \to +\infty$ as $\ell \uparrow 2m$. Two features of (6) are used below: numerator and denominator are linear in $s$, so a rational bracket $a \leq s \leq b$ encloses $u_k$ with one interval division; and the dependence on $\ell$ in (4) is simple enough to compare two values of $\ell$ termwise.

## 4. Monotonicity between consecutive resonances

**Theorem 4.1** ($\Theta$ is strictly increasing between resonances)**.** *Let $m \in \mathbb{N}$ and let $L_1 < L_2$ be real numbers with $2m-2 < L_1 < L_2 < 2m$ and $L_1 > \frac{1}{2}$. Then $\Theta_{\mathbb{R}}(L_1) < \Theta_{\mathbb{R}}(L_2)$.*

*Proof.* Put $d = L_2 - L_1 > 0$ and, for $k \geq 1$,

$$P_k = \frac{d}{(2k-L_2)(2k-L_1)}, \qquad S_k = \frac{d}{(2k-1+L_2)(2k-1+L_1)}.$$

By Proposition 3.1 for $\Theta_{\mathbb{R}}$, $\Theta_{\mathbb{R}}(L_2) - \Theta_{\mathbb{R}}(L_1) = \left[\frac{1}{1+L_2} - \frac{1}{1+L_1}\right] + \sum_{k\geq 1}\left[w_k(L_2) - w_k(L_1)\right]$, and termwise $\frac{1}{2k-L_2} - \frac{1}{2k-L_1} = P_k$, $\frac{1}{2k+1+L_2} - \frac{1}{2k+1+L_1} = -S_{k+1}$, while the bracket in front is $-S_1$. So the difference is $\sum_{k\geq 1} P_k - \sum_{k\geq 1} S_k = \sum_{k\geq 1}(P_k - S_k)$, every series involved being absolutely convergent ($P_k, S_k = O(k^{-2})$; summability of $P$ follows from that of the other two). Each term is strictly positive. The product $(2k-L_2)(2k-L_1)$ is positive because both factors have the same sign: for $k < m$ both are negative, for $k \geq m$ both are positive, since $2m-2 < L_i < 2m$. And for $L > \frac{1}{2}$ and $k \geq 1$,

$$|2k - L| < 2k - 1 + L$$

— if $2k \geq L$ this is $1 < 2L$, if $2k < L$ it is $1 < 4k$ — so $0 < (2k-L_2)(2k-L_1) < (2k-1+L_2)(2k-1+L_1)$ and $P_k > S_k$. A convergent series of positive terms is positive. No derivative, no differentiation under the sum, and no uniform convergence is used. □

In terms of the dimension: $\ell(n) \in (2j+2, 2j+4)$ exactly when $(2j+1)^2 + 2 < n < (2j+3)^2 + 2$, and $\ell(n) \geq 1 > \frac{1}{2}$ always. For $m \geq 2$ the hypothesis $L_1 > \frac{1}{2}$ is implied by $L_1 > 2m-2 \geq 2$, so it bites only on the first interval $(0,2)$ — and there it is sharp, not an artefact of the argument: evaluating (4) numerically (a computation, not a theorem) makes $\Theta_{\mathbb{R}}$ strictly *decreasing* on $(0, \frac{1}{2})$, with $\Theta_{\mathbb{R}}(\frac{1}{2}) \approx -3.1416$, so the theorem would be false on the whole of $(0,2)$. No dimension reaches that interval.

**Corollary 4.2.** *For $j, n_1, n_2 \in \mathbb{N}$ with $(2j+1)^2 + 2 < n_1 < n_2 < (2j+3)^2 + 2$ one has $\Theta(n_1) < \Theta(n_2)$.*

Only the top of such a window is resonant: $r^2$ strictly between the consecutive odd squares $(2j+1)^2$ and $(2j+3)^2$ forces $2j+1 < r < 2j+3$ (Lean: `not_resonant_of_window`). Hence:

**Theorem 4.3** (blocks are suffixes)**.** *Let $j, n_0 \in \mathbb{N}$ with $(2j+1)^2 + 2 < n_0 < (2j+3)^2 + 2$, $\Theta(n_0) > 0$ and $\Theta(n_0 - 1) < 0$. Then for every $n$ with $(2j+1)^2 + 2 < n \leq (2j+3)^2 + 2$,*

$$n \textit{ is admissible} \iff n_0 \leq n.$$

*Proof.* The top $(2j+3)^2 + 2$ is resonant, hence admissible, and lies above $n_0$. An interior $n$ is not resonant, so it is admissible iff $\Theta(n) > 0$; by Corollary 4.2, $\Theta(n) \leq \Theta(n_0 - 1) < 0$ for $n \leq n_0 - 1$ and $\Theta(n) \geq \Theta(n_0) > 0$ for $n \geq n_0$. □

With $m = j + 2$ the window is $((2m-3)^2 + 2, (2m-1)^2 + 2]$, and its admissible set is the block $\{n_0, \dots, (2m-1)^2 + 2\}$ of length $b_m = (2m-1)^2 + 2 - n_0 + 1$. Determining the admissible set on a window therefore costs two sign evaluations instead of one per dimension — the reason $n \le 14163$ is affordable in Section 6. Remark 4.12 of [1] proves $\Theta > 0$ at the single dimension $(2m-1)^2 + 1$, one below the resonance, for all sufficiently large $m$; Theorem 4.3 says that whenever the block has length at least 2 it contains that dimension, and Section 6 certifies that it does for every $3 \le m \le 60$.

## 5. A verified rational enclosure of $\Theta$

Fix $n \ge 2$, $s = \sqrt{n-2}$, and rationals $0 \le a \le s \le b$. The certificate uses

$$a = \frac{\lfloor \sqrt{(n-2) \cdot 10^{30}} \rfloor}{10^{15}}, \qquad b = a + 10^{-15},$$

where $a^2 \le n - 2 \le b^2$ follows from the defining inequalities of the integer square root, for every $n$ at once.

*One term.* For $1 \le k \le K$ the guard "$E_k - 3b > 0$ or $E_k - 3a < 0$" says that the rational interval $[E_k - 3b,\, E_k - 3a]$ enclosing the denominator of (6) misses 0. When it holds, a single interval division encloses $u_k$: with $N_- = 4n + 1 + 12a \le 4n + 1 + 12s \le 4n + 1 + 12b = N_+$ and $0 \le N_-$,

$$\frac{N_-}{(4k+1)(E_k - 3a)} \le u_k \le \frac{N_+}{(4k+1)(E_k - 3b)} \quad \text{(denominator positive)},$$

and the two bounds swap roles when the denominator is negative. The guard also implies $2k \ne \ell$, so (5) applies to every guarded term.

*The partial sum.* The term bounds are accumulated with outward rounding at a fixed denominator $D$: $\Sigma_0^- = \Sigma_0^+ = 0$, $\Sigma_k^- = \lfloor D(\Sigma_{k-1}^- + \text{lower bound of } u_k) \rfloor / D$, $\Sigma_k^+ = \lceil D(\Sigma_{k-1}^+ + \text{upper bound of } u_k) \rceil / D$, so the rationals stay of bounded size however large $K$ is, and $\Sigma_K^- \le \sum_{k=1}^{K} u_k \le \Sigma_K^+$.

*The tail.* Put $\lambda = 1 - \frac{1+b}{2(K+1)}$ and assume $\lambda > 0$ (this forces $\ell < 2(K+1)$, so every tail term has $2k > \ell$).

**Lemma 5.1.** *For $K \ge 1$: $\sum_{k>K} k^{-3} \le \frac{1}{2K^2}$; for $k \ge K+1$: $0 \le u_k \le \frac{(2b+3)^2}{16\lambda k^3}$; hence*

$$0 \le \sum_{k>K} u_k \le T_K(b) := \frac{(2b+3)^2}{32\,\lambda\,K^2}.$$

*Proof.* $k^{-3} \le \frac{1}{2(k-1)^2} - \frac{1}{2k^2}$ for $k \ge 2$ (it reduces to $2 \le 3k$), and the right side telescopes. For $k \ge K+1$: $4k+1 \ge 4k$, $2k + 1 + \ell \ge 2k$, and $2k - \ell \ge 2k\big(1 - \frac{\ell}{2(K+1)}\big) \ge 2k\lambda$ because $\ell \le 1 + b$; so the denominator of (5) is at least $16\lambda k^3$, while $(2\ell+1)^2 \le (2b+3)^2$. ☐

**Proposition 5.2** (enclosure)**.** *Let $n \ge 2$, $K \ge 1$, $D \ge 1$, $0 \le a \le \sqrt{n-2} \le b$ rational, suppose the guard holds for every $1 \le k \le K$ and $\lambda > 0$. Then*

$$-4 + \frac{1}{2+b} + \Sigma_K^- \;\le\; \Theta(n) \;\le\; -4 + \frac{1}{2+a} + \Sigma_K^+ + T_K(b).$$

*Proof.* Split (4) at $K$; the head is enclosed by $\Sigma_K^{\pm}$, the tail by Lemma 5.1, and $2 + a \le 1 + \ell \le 2 + b$ encloses the $\frac{1}{1+\ell}$ term. ☐

**Definition 5.3** (the certificates)**.** $\texttt{certPos}(n, K, D)$ is the Boolean "the guard holds for $1 \le k \le K$, $\lambda > 0$, and $-4 + \frac{1}{2+b} + \Sigma_K^- > 0$"; $\texttt{certNeg}(n, K, D)$ is "the guard holds for $1 \le k \le K$, $\lambda > 0$, and $-4 + \frac{1}{2+a} + \Sigma_K^+ + T_K(b) < 0$", with $a, b$ as above.

**Lemma 5.4.** *Let $n \ge 2$, $K \ge 1$, $D \ge 1$. If $\texttt{certPos}(n, K, D)$ is $\texttt{true}$ then $\Theta(n) > 0$; if $\texttt{certNeg}(n, K, D)$ is $\texttt{true}$ then $\Theta(n) < 0$.*

Everything in this section up to and including Lemma 5.4 is proved by ordinary reasoning, for all $n$, $K$, $D$ at once; the only computation is the evaluation of the two Booleans at particular arguments, in Section 6.

*Remark* 5.5 (what the certificate refuses and discriminates). At a resonant $n$, $\ell = 2m$, the $m$-th denominator $E_m - 3s = (2m - \ell)(2m + 1 + \ell)$ vanishes and the guard fails, so neither certificate can be true: the certificate refuses the pole rather than mis-signing it. It also discriminates, in that it does not certify $\Theta > 0$ at the non-admissible 25 nor $\Theta < 0$ at the admissible 26. The formal controls, at $K = 120$, $D = 10^{20}$, are

$$\texttt{certPos}(27, K, D) = \texttt{false}, \qquad \texttt{certNeg}(27, K, D) = \texttt{false},$$
$$\texttt{certPos}(25, K, D) = \texttt{false}, \qquad \texttt{certNeg}(26, K, D) = \texttt{false}.$$

## 6. The admissible set

Every statement in this section is the conjunction of Section 5 with a finite evaluation, and we say for each which evaluation was run.

### 6.1. **The printed list, certified.**

**Theorem 6.1** ($12 \le n < 150$)**.** *For every $n$ with $12 \le n < 150$, $n$ is admissible in the sense of Definition 2.4 if and only if*

$$n \in \{26, 27, 49, 50, 51, 80, 81, 82, 83, 120, 121, 122, 123\}.$$

*Proof.* For each of the 138 integers $12 \le n \le 149$ the following Boolean was evaluated with $K = 120$ and $D = 10^{20}$: if $n$ is in the list, "$n$ resonant or $\texttt{certPos}(n, K, D)$"; if not, "$n$ not resonant and $\texttt{certNeg}(n, K, D)$". All 138 are `true` (one compiled evaluation). By Lemma 5.4 the first alternative gives admissibility and the second, since $\Theta(n) < 0$, its failure. □

This is the sentence of [1] quoted in Section 1.1, character for character, now with a proof from the source's own definition. The values are not new; the certificate is, in that the source gives no method and no error bound.

**Proposition 6.2** (controls)**.** *(a) 26 is not resonant and is admissible; (b) $27 = 5^2 + 2$ is resonant and admissible; (c) 25 is not admissible; (d) 79 is not admissible; (e) 123 is admissible; (f)* $\dfrac{357}{10000} < \Theta(80) < \dfrac{368}{10000}$.

(a)–(e) are the negative controls on Theorem 6.1: a list one element too large (adding 25, or extending the block 80–83 down to 79) is refuted, a list one element too small (dropping 123, or 26) is refuted, and both branches of Definition 2.4 are non-vacuous. (f) is two-sided: from Proposition 5.2 at $K = 120$, $D = 10^{20}$ both bounds are evaluated, a window of width $1.1 \times 10^{-3}$ around the smallest positive value of $\Theta$ on $n \le 400$, showing the enclosure is genuinely tight and not a sign scraped off a loose bound.

### 6.2. **To** $n = 400$**.**

**Proposition 6.3** ($12 \le n \le 400$)**.** *For every $n$ with $12 \le n \le 400$, $n$ is admissible if and only if $n$ is in the list of Theorem 6.1 or in one of the four blocks*

$$167\text{–}171, \qquad 223\text{–}227, \qquad 286\text{–}291, \qquad 358\text{–}363.$$

*In particular no $n$ with $150 \le n \le 166$ is admissible.*

*Proof.* The same sweep as in Theorem 6.1, over the 389 integers $12 \le n \le 400$, at $K = 120$, $D = 10^{20}$, against the 35-element list; the last sentence is read off the list. □

The source prints nothing above 150. These numbers are routine — a reader with (1)–(3) and a computer algebra system gets them in about a second — and what is added is that they now carry a proof, with no floating point anywhere in the decision, together with the emptiness of $[150, 166]$ as the check that the four new blocks are not an artefact of a mis-set range. The four blocks end at the resonant dimensions $171 = 13^2 + 2$, $227 = 15^2 + 2$, $291 = 17^2 + 2$, $363 = 19^2 + 2$, as Theorem 4.3 says they must.

6.3. **Fifty-eight blocks.** For $t = 0, 1, \ldots, 57$ let $m = t+3$ and let $\beta_t$ be the $t$-th entry of the following table, the bottom of the $m$-th block; the window is $((2m-3)^2+2,\ (2m-1)^2+2] = ((2t+3)^2+2,\ (2t+5)^2+2]$.

| $m$ | $\beta$ | top | $b_m$ | $m$ | $\beta$ | top | $b_m$ | $m$ | $\beta$ | top | $b_m$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 26 | 27 | 2 | 23 | 2015 | 2027 | 13 | 43 | 7206 | 7227 | 22 |
| 4 | 49 | 51 | 3 | 24 | 2199 | 2211 | 13 | 44 | 7550 | 7571 | 22 |
| 5 | 80 | 83 | 4 | 25 | 2391 | 2403 | 13 | 45 | 7902 | 7923 | 22 |
| 6 | 120 | 123 | 4 | 26 | 2590 | 2603 | 14 | 46 | 8261 | 8283 | 23 |
| 7 | 167 | 171 | 5 | 27 | 2798 | 2811 | 14 | 47 | 8629 | 8651 | 23 |
| 8 | 223 | 227 | 5 | 28 | 3013 | 3027 | 15 | 48 | 9004 | 9027 | 24 |
| 9 | 286 | 291 | 6 | 29 | 3237 | 3251 | 15 | 49 | 9388 | 9411 | 24 |
| 10 | 358 | 363 | 6 | 30 | 3468 | 3483 | 16 | 50 | 9779 | 9803 | 25 |
| 11 | 437 | 443 | 7 | 31 | 3708 | 3723 | 16 | 51 | 10179 | 10203 | 25 |
| 12 | 525 | 531 | 7 | 32 | 3955 | 3971 | 17 | 52 | 10587 | 10611 | 25 |
| 13 | 620 | 627 | 8 | 33 | 4211 | 4227 | 17 | 53 | 11002 | 11027 | 26 |
| 14 | 724 | 731 | 8 | 34 | 4474 | 4491 | 18 | 54 | 11426 | 11451 | 26 |
| 15 | 835 | 843 | 9 | 35 | 4746 | 4763 | 18 | 55 | 11857 | 11883 | 27 |
| 16 | 955 | 963 | 9 | 36 | 5026 | 5043 | 18 | 56 | 12297 | 12323 | 27 |
| 17 | 1082 | 1091 | 10 | 37 | 5313 | 5331 | 19 | 57 | 12744 | 12771 | 28 |
| 18 | 1218 | 1227 | 10 | 38 | 5609 | 5627 | 19 | 58 | 13200 | 13227 | 28 |
| 19 | 1361 | 1371 | 11 | 39 | 5912 | 5931 | 20 | 59 | 13664 | 13691 | 28 |
| 20 | 1513 | 1523 | 11 | 40 | 6224 | 6243 | 20 | 60 | 14135 | 14163 | 29 |
| 21 | 1672 | 1683 | 12 | 41 | 6543 | 6563 | 21 | | | | |
| 22 | 1840 | 1851 | 12 | 42 | 6871 | 6891 | 21 | | | | |

Here "top" is the resonant dimension $(2m-1)^2+2$ and $b_m = (2m-1)^2+2-\beta+1$ is the block length; the columns $\beta$ and $b_m$ are the data of the theorem, the others are arithmetic.

**Theorem 6.4** ($12 \le n \le 14163$)**.** *For every $t < 58$ and every $n$ with $(2t+3)^2+2 < n \le (2t+5)^2+2$,*

$$n \text{ is admissible} \iff \beta_t \le n.$$

*Consequently the admissible dimensions with $12 \le n \le 14163$ are exactly the 58 blocks $\{\beta_t, \ldots, (2t+5)^2+2\}$ of the table.*

*Proof.* For each $t < 58$ the Boolean "$(2t+3)^2+2 < \beta_t$ and $\beta_t < (2t+5)^2+2$ and $\texttt{certPos}(\beta_t, K, D)$ and $\texttt{certNeg}(\beta_t-1, K, D)$" was evaluated at $K = 2000$, $D = 10^{20}$; all 58 are `true` (one compiled evaluation, 116 sign certificates). Lemma 5.4 turns the two certificates into $\Theta(\beta_t) > 0 > \Theta(\beta_t - 1)$, and Theorem 4.3 with $j = t+1$ does the rest. The windows for $t = 0, \ldots, 57$ partition $(11, 14163]$. ☐

The proof is where the monotonicity theorem earns its place: 116 evaluations certify 116 signs, and without Theorem 4.3 they would say nothing about the 14 000 dimensions between them.

**Corollary 6.5.** $\sum_{t<58} b_{t+3} = 940$*: there are exactly 940 admissible dimensions $n$ with $12 \le n \le 14163$, i.e. 6.6% of the 14152 integers in that range.*

6.4. **The block-length law is false.** The block lengths $b_m$ for $m = 3, \ldots, 60$ read

$2, 3, 4, 4, 5, 5, 6, 6, 7, 7, 8, 8, 9, 9, 10, 10, 11, 11, 12, 12, 13, 13, \mathbf{13}, 14, 14, 15, 15, 16, 16,$
$17, 17, 18, 18, \mathbf{18}, 19, 19, 20, 20, 21, 21, 22, 22, \mathbf{22}, 23, 23, 24, 24, 25, 25, \mathbf{25}, 26, 26, 27, 27, 28, 28, \mathbf{28}, 29.$

The first eight blocks — all that a computation to $n < 400$ shows — fit the law $b_m = \lceil (m+2)/2 \rceil = 3, 4, 4, 5, 5, 6, 6, \ldots$ for $m \ge 4$ exactly, and so do the next thirteen.

**Theorem 6.6** (the law $\lceil (m+2)/2 \rceil$ fails)**.** (a) *For every $n$ with $2211 < n \le 2403$, $n$ is admissible iff $2391 \le n$; and 2390 is not admissible. So $b_{25} = 13$, whereas $\lceil (25+2)/2 \rceil = 14$.*

(b) *For every $n$ with $2027 < n \le 2211$, $n$ is admissible iff $2199 \le n$; and 2198 is not admissible. So $b_{24} = 13 = \lceil (24+2)/2 \rceil$: the law holds at the window just below the first failure.*

(c) *Among the 57 windows $4 \le m \le 60$, $b_m \ne \lceil (m+2)/2 \rceil$ for exactly 31 values of $m$, and for every $m$ with $36 \le m \le 60$.*

(d) $b_3 = 2$, *whereas* $\lceil (3+2)/2 \rceil = 3$.
(e) $2403 = 49^2 + 2$ *is resonant.*

*Proof.* (a), (b), (e) are instances of Theorem 6.4 at $t = 22$ and $t = 21$, i.e. $m = 25$ and $m = 24$: the windows are $(47^2+2, 49^2+2]$ and $(45^2+2, 47^2+2]$ and the table gives $\beta = 2391$ and 2199. (c) and (d) are the finite check $b_m \neq \lfloor (m+3)/2 \rfloor$ over the table (in the formal statement, with $m = t+3$, the law is $(t+6)/2$ in integer division, which equals $\lceil (m+2)/2 \rceil$); the 31 failures are at $m = 25, 27, 29, 31, 33, 35$ and at every $m$ from 36 to 60, as one reads off the table. □

The failure at $m = 25$ is by a wide margin. The two signs certified in Lean are $\Theta(2390) < 0 < \Theta(2391)$; the enclosures behind them (Section 7) are $\Theta(2390) \in [-0.0163, -0.0148]$ and $\Theta(2391) \in [0.6202, 0.6217]$ at $K = 500$, so the law misses by $1.5 \times 10^{-2}$, more than ten times the width of those enclosures — and, at the $K = 2000$ actually used in Theorem 6.4, where $\Theta(2390) \in [-0.015014, -0.014934]$, nearly two hundred times it. The law was suggested by an unrigorous probe of $n < 400$ made while this work was being planned; it is ours, not the source's, and Theorem 6.6 is its refutation. Part (b) shows that the break at $m = 25$ is a real change of behaviour and not an indexing slip, and (e) that the top of the failing window is resonant, as every top is. The run $13, 13, 13$ at $m = 23, 24, 25$ is the first repetition of length three; the next are $18, 18, 18$ ($m = 34, 35, 36$), $22, 22, 22$ ($m = 43, 44, 45$), $25, 25, 25$ ($m = 50, 51, 52$) and $28, 28, 28$ ($m = 57, 58, 59$), and the growth is visibly slower than $m/2$.

## 7. Outside the formal development

Nothing in this section is a theorem of this paper; each item is a computation or a probe, labelled as such, and none supports any statement of Sections 2–6.

**Margins of the certificates (exact rational arithmetic).** The certificate of Section 5 was also run outside Lean, in exact rational arithmetic with the identical guard, rounding and tail bound, and its enclosures were printed. At $K = 120$ every $12 \leq n \leq 400$ is decided, with maximum enclosure width $4.37 \times 10^{-3}$ and the sweep reproducing the list of Proposition 6.3. At $K = 500$, $\Theta(2390) \in [-0.016201, -0.014866]$ and $\Theta(2391) \in [0.620280, 0.621616]$. At $K = 2000$ all 116 certificates of Theorem 6.4 succeed, the smallest distance of a deciding bound from 0 over the 116 being $1.9 \times 10^{-3}$ (attained at $m = 57$), against a tail bound $T_{2000}(b)$ below $5 \times 10^{-4}$ at $n = 14163$; the same 116 boundaries are already decided by $K = 500$ for every $m \leq 56$, and $K$ must grow roughly like $\ell$ divided by the square root of the required margin. The $10^{-15}$ bracket on $\sqrt{n-2}$ is far tighter than needed — $10^{-4}$ decides all 116.

**A floating-point probe of the block lengths (not a result).** Bisecting for the root $\delta_m^* = 2m - \ell^*$ of $\Theta_{\mathbb{R}}$ in the gap $(2m-2, 2m)$ in double precision gives

| $m$ | $\delta_m^*$ | $1/\delta_m^*$ | $\log m + 3\log 2 + \gamma_E + \pi/2$ |
|---|---|---|---|
| 5 | 0.16926 | 5.9080 | 5.8369 |
| 20 | 0.13661 | 7.3199 | 7.2232 |
| 100 | 0.11208 | 8.9219 | 8.8326 |
| 400 | 0.09710 | 10.2982 | 10.2189 |

so $1/\delta_m^* = \log m + (3\log 2 + \gamma_E + \pi/2) + O(1)$ with the $O(1)$ numerically about 0.08 and nearly constant — consistent with the $\sigma_{2m-\delta} = 1/\delta + O(\log(m+1))$ of Remark 4.12 of [1], whose $O$ is not quantified there. Since the $j$-th dimension below the resonance has $2m - \ell = j/(4m-2) + O(m^{-2})$, this predicts $b_m \approx \lceil (4m-2)/(\log m + 4.2275) \rceil$, which reproduces the certified block lengths of Theorem 6.4 to within one at every $3 \leq m \leq 60$ and exactly at most of them. The consequences — $b_m$ of order $m/\log m$, an admissible set of density zero, a counting function of order $X/\log X$ — are all *unproved*; the certified 940 admissible dimensions in $[12, 14163]$ give $940/(14163/\log 14163) \approx 0.63$. Proving $b_m = O(m/\log m)$ needs an explicit $-c\log m$ upper bound on the sum of the $u_k$, $k \neq m$, near the pole, which we have not attempted.

**What this does and does not say about "false for all dimension $n > 10$".** The sentence of [1] quoted in Section 1.1 is an expectation explicitly attributed to numerical evidence, and nothing here bears on whether it is true. What the certified data say is narrower and exact: the hypothesis of Theorem 1.1 of [1] holds at 940 of the 14152 dimensions $12 \leq n \leq 14163$, in blocks whose lengths grow more slowly than $m/2$ against windows of length about $8m$; and, as the source itself notes, when $\Theta_N \leq 0$ its construction yields no positive shifted eigenvalue. So in the remaining 13212 dimensions of that range the expectation, if correct, must be reached by some other mechanism. Whether the proportion tends to zero is the unproved density statement above.

**Mathlib's digamma.** At the revision used here (the one tagged for Lean v4.32.0), *Mathlib* defines its digamma `Complex.digamma` as `logDeriv Gamma`, in a 64-line file with the values at 0, 1 and $\frac{1}{2}$, the recurrence and meromorphy, and no series representation, reflection formula, duplication formula or numeric content. The Gauss series is therefore the only route to a value of $\Theta$ at present. The other route — eliminating $\psi$ by reflection and duplication [5, eqs. 5.5.4, 5.5.8] — does close. The reflection formula at $z = -\ell/2$ gives $\psi(-\frac{\ell}{2}) = \psi(1+\frac{\ell}{2}) + \pi\cot(\pi\ell/2)$, and the duplication formula at $z = \frac{1+\ell}{2}$ gives $\frac{1}{2}\psi(\frac{1+\ell}{2}) + \frac{1}{2}\psi(1+\frac{\ell}{2}) = \psi(1+\ell) - \log 2$; together they turn (1) into $\sigma_\ell = -\gamma_E - \psi(1+\ell) - \frac{\pi}{2}\cot(\pi\ell/2)$, whence, *using* the source's closed form $\sigma_{-1/2} = \pi/2 + 2\log 2$,

$$\Theta_N = \frac{1}{\ell} - \psi(1+\ell) - \frac{\pi}{2}\cot\frac{\pi\ell}{2} - \gamma_E - 2\log 2 - \frac{\pi}{2}.$$

(This is the only place where that closed form is invoked, and nothing else in the paper depends on it.) The route was priced and rejected: it trades one transcendental function for three, and a rigorous enclosure of $\cot(\pi\ell/2)$ at an irrational $\ell$ is harder than the series it removes.

## 8. Verification

Lemma 2.2, Proposition 3.1, Theorems 4.1, 4.3, 6.1, 6.4 and 6.6, Corollaries 4.2 and 6.5 and Propositions 6.2 and 6.3, together with Section 5, have been formally verified in Lean 4 (v4.32.0) against *Mathlib* [7, 8]; the statements are reproduced verbatim in Appendix A. The development is seven modules, 1531 lines, 66 theorems, 39 definitions and one decidability instance: `Defs` ($\psi_{\mathrm{G}}$, $\sigma$, $\ell$, $\Theta$, resonance with its decision procedure, admissibility; summability of the Gauss series for every real argument), `Bridge` (Lemma 2.2), `Series` (Proposition 3.1), `Enclose` (Section 5: interval division, the bracket on $\sqrt{n-2}$, the term and partial-sum bounds, the tail, Proposition 5.2, the two Boolean certificates and Lemma 5.4), `Range` (the sweep and Theorem 6.1, Proposition 6.3, the controls), `Mono` (Theorems 4.1 and 4.3, Corollary 4.2) and `Blocks` (the table, Theorem 6.4, Corollary 6.5, Theorem 6.6, the certificate controls of Remark 5.5 and Proposition 6.2(f)). There is no `sorry` and no axiom is declared. The whole reasoning layer — the series identity, the enclosure engine with its tail bound, the monotonicity theorem, the suffix theorem and the digamma lemma — depends only on `propext`, `Classical.choice` and `Quot.sound`, as a print of the axiom set of each of `Theta_eq`, `theta_enclosure`, `theta_pos_of_cert`, `theta_neg_of_cert`, `ThetaR_lt`, `Theta_lt_of_window`, `window_admissible`, `not_resonant_of_window` and the four `psiG` lemmas confirms. Compiled evaluation (`native_decide`) is used only to evaluate finite rational data, in 15 places: once for the $n < 150$ sweep, once for the $n \leq 400$ sweep, once for the 58-window certificate, four times inside the two-sided bound on $\Theta(80)$, once for the count 940, three times in the count of failures of the law, and four times in the three certificate controls of Remark 5.5; each of these declarations carries the corresponding `native_decide` axiom in addition to the three above, and the control `Resonant 2403` needs `propext` only. The imports are targeted (`Mathlib.Analysis.PSeries`, `Mathlib.Analysis.SpecialFunctions.Sqrt`, `Mathlib.Data.Rat.Floor`, `Mathlib.Order.Interval.Finset.Nat` and two tactic files) rather than all of *Mathlib*. As recorded at the time of the build, a clean sequential check of the seven modules took 58 s wall in all (4.7–12.6 s per module) with a peak of 3.4 GB of memory, the exact-rational sweeps outside Lean 1.7 s ($n \leq 400$) and 9.2 s (the 116 boundaries at $K = 2000$). The source of the development is currently held in a private repository: repository reference to be added at submission.

*On the reference list.* [1] is cited from its arXiv version 1 (submitted 11 August 2026; no journal reference at the time of writing), and every statement and equation number attributed to it was read from the e-print compiled from the arXiv source — twice, from two independent compilations of the same file. [2] and [3] are cited as [1] cites them (its references [11] and [36]) and were not consulted directly; in particular the wording of the conjecture is the source's quotation of it, and the content attributed to [3] is only the exponent $p_{\mathrm{JL}}$. Their bibliographic data were nevertheless verified against Crossref and zbMATH Open. [4] is the source's reference [39] and was read in full.

## References


[1] S. Chen, Y. Liu, J. Wei and W. Yang, *Nonradial stable solutions near the Joseph–Lundgren threshold*, preprint, arXiv:2608.10501v1 [math.AP], 11 August 2026.

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[3] D. D. Joseph and T. S. Lundgren, *Quasilinear Dirichlet problems driven by positive sources*, Arch. Rational Mech. Anal. **49** (1973), 241–269, doi:10.1007/BF00250508. Cited via [1]; not consulted directly.

[4] Y. Liu, K. Wang, J. Wei and K. Wu, *Qualitative properties of stable solutions to some supercritical problems*, Electron. Res. Arch. **30** (2022), no. 5, 1668–1690, doi:10.3934/era.2022084.

[5] NIST Digital Library of Mathematical Functions, Chapter 5 (Gamma function), §§5.4, 5.5, 5.7, `https://dlmf.nist.gov/5`; equations 5.4.12, 5.5.2, 5.5.4, 5.5.8 and 5.7.6 as read on 7 September 2026.

[6] The OEIS Foundation, *The On-Line Encyclopedia of Integer Sequences*, sequence A164897, `https://oeis.org/A164897`, read on 7 September 2026.

[7] L. de Moura and S. Ullrich, *The Lean 4 theorem prover and programming language*, in: Automated Deduction — CADE 28, Lecture Notes in Computer Science, Springer, 2021, 625–635, doi:10.1007/978-3-030-79876-5_37.

[8] The mathlib Community, *The Lean mathematical library*, in: Proceedings of the 9th ACM SIGPLAN International Conference on Certified Programs and Proofs (CPP 2020), ACM, 2020, 367–381, doi:10.1145/3372885.3373824.


## Appendix A. The formal statements

The Lean statements corresponding to the results above, verbatim (statements only; proofs, docstrings and attributes omitted), grouped by module. Throughout, `gaussTerm x k` is the $k$-th term of the Gauss series, `psiG` is $\psi_{\mathrm{G}}$, `sigma`, `ell`, `Theta` and `ThetaR` are $\sigma$, $\ell$, $\Theta$, $\Theta_{\mathbb{R}}$, `Resonant` and `Admissible` are Definition 2.4, `w n k` and `u n k` are $w_k$ and $u_k$ of (4)–(5), `sqrtLo`/`sqrtHi` are $a/b$, `denLo`/`denHi` the bracket on $E_k - 3s$, `sumLo`/`sumHi` are $\Sigma_K^{\mp}$, `lamq` is $\lambda$, `tailB` is $T_K(b)$, `blockBottoms` is the column $\beta$ of the table and `blockLen t` is $b_{t+3}$. Lean's $x/0 = 0$ is in force.

**`Defs.lean`**

```
noncomputable def gaussTerm (x : ℝ) (k : ℕ) : ℝ := 1 / ((k : ℝ) + 1) - 1 / ((k : ℝ) + x)
theorem summable_gaussTerm (x : ℝ) : Summable (gaussTerm x)
noncomputable def psiG (x : ℝ) : ℝ := ∑' k : ℕ, gaussTerm x k
noncomputable def sigma (ν : ℝ) : ℝ :=
  -Real.log 2 - psiG (-ν / 2) / 2 - psiG ((1 + ν) / 2) / 2
noncomputable def ell (n : ℕ) : ℝ := 1 + Real.sqrt ((n : ℝ) - 2)
noncomputable def Theta (n : ℕ) : ℝ := 1 / ell n + sigma (ell n) - sigma (-(1 / 2))
def Resonant (n : ℕ) : Prop := ∃ r : ℕ, r % 2 = 1 ∧ n = r ^ 2 + 2
def Admissible (n : ℕ) : Prop := Resonant n ∨ (¬ Resonant n ∧ 0 < Theta n)
```

**`Bridge.lean`** (Lemma 2.2)

```
theorem psiG_one : psiG 1 = 0
theorem psiG_add_one (x : ℝ) : psiG (x + 1) = psiG x + 1 / x
theorem psiG_two : psiG 2 = 1
theorem psiG_natCast_add_one (m : ℕ) :
    psiG ((m : ℝ) + 1) = ∑ k ∈ Finset.range m, 1 / ((k : ℝ) + 1)
```

**`Series.lean`** (Proposition 3.1)

```
noncomputable def w (n k : ℕ) : ℝ :=
  1 / (2 * (k : ℝ) - ell n) + 1 / (2 * (k : ℝ) + 1 + ell n) - 4 / (4 * (k : ℝ) + 1)
noncomputable def u (n k : ℕ) : ℝ :=
  (2 * ell n + 1) ^ 2 / ((4 * (k : ℝ) + 1) * (2 * (k : ℝ) - ell n) * (2 * (k : ℝ) + 1 + ell n))
theorem Theta_eq (n : ℕ) :
    Theta n = -4 + 1 / (1 + ell n) + ∑' k : ℕ, w n (k + 1)
theorem w_eq_u (n k : ℕ) (hk : 2 * (k : ℝ) - ell n ≠ 0) : w n k = u n k
theorem u_eq_sqrt (n k : ℕ) (hn : 2 ≤ n) :
    u n k = (4 * (n : ℝ) + 1 + 12 * Real.sqrt ((n : ℝ) - 2))
      / ((4 * (k : ℝ) + 1)
          * ((4 * (k : ℝ) ^ 2 + 2 * (k : ℝ) - (n : ℝ)) - 3 * Real.sqrt ((n : ℝ) - 2)))
```

**`Mono.lean`** (Theorems 4.1, 4.3, Corollary 4.2)

```
noncomputable def ThetaR (L : ℝ) : ℝ := 1 / L + sigma L - sigma (-(1 / 2))
theorem Theta_eq_ThetaR (n : ℕ) : Theta n = ThetaR (ell n)
def SameGap (m : ℕ) (L₁ L₂ : ℝ) : Prop :=
  2 * (m : ℝ) - 2 < L₁ ∧ L₁ < L₂ ∧ L₂ < 2 * (m : ℝ)
theorem ThetaR_lt {m : ℕ} {L₁ L₂ : ℝ} (hm : SameGap m L₁ L₂) (hL : 1 / 2 < L₁) :
    ThetaR L₁ < ThetaR L₂
theorem Theta_lt_of_window (j n₁ n₂ : ℕ)
    (h1 : (2 * j + 1) ^ 2 + 2 < n₁) (h2 : n₁ < n₂) (h3 : n₂ < (2 * j + 3) ^ 2 + 2) :
    Theta n₁ < Theta n₂
theorem not_resonant_of_window (j n : ℕ)
    (h1 : (2 * j + 1) ^ 2 + 2 < n) (h2 : n < (2 * j + 3) ^ 2 + 2) : ¬ Resonant n
theorem window_admissible (j n₀ : ℕ)
    (hlo : (2 * j + 1) ^ 2 + 2 < n₀) (hhi : n₀ < (2 * j + 3) ^ 2 + 2)
    (hpos : 0 < Theta n₀) (hneg : Theta (n₀ - 1) < 0) :
    ∀ n, (2 * j + 1) ^ 2 + 2 < n → n ≤ (2 * j + 3) ^ 2 + 2 → (Admissible n ↔ n₀ ≤ n)
```

**Enclose.lean** (Section 5)

```
def sqrtLo (n : ℕ) : ℚ := (Nat.sqrt ((n - 2) * 10 ^ 30) : ℚ) / 10 ^ 15
def sqrtHi (n : ℕ) : ℚ := (Nat.sqrt ((n - 2) * 10 ^ 30) + 1 : ℚ) / 10 ^ 15
theorem sqrtLo_le (n : ℕ) (hn : 2 ≤ n) : (sqrtLo n : ℝ) ≤ Real.sqrt ((n : ℝ) - 2)
theorem le_sqrtHi (n : ℕ) (hn : 2 ≤ n) : Real.sqrt ((n : ℝ) - 2) ≤ (sqrtHi n : ℝ)
def Ecf (n k : ℕ) : ℚ := 4 * (k : ℚ) ^ 2 + 2 * (k : ℚ) - (n : ℚ)
def denLo (n k : ℕ) (b : ℚ) : ℚ := Ecf n k - 3 * b
def denHi (n k : ℕ) (a : ℚ) : ℚ := Ecf n k - 3 * a
def lamq (K : ℕ) (b : ℚ) : ℚ := 1 - (1 + b) / (2 * ((K : ℚ) + 1))
def Ccf (K : ℕ) (b : ℚ) : ℚ := (2 * (1 + b) + 1) ^ 2 / (16 * lamq K b)
def tailB (K : ℕ) (b : ℚ) : ℚ := Ccf K b / (2 * (K : ℚ) ^ 2)
theorem tsum_cube_tail (K : ℕ) (hK : 1 ≤ K) :
    ∑' i : ℕ, (1 : ℝ) / ((i : ℝ) + K + 1) ^ 3 ≤ 1 / (2 * (K : ℝ) ^ 2)
theorem theta_enclosure (n K D : ℕ) (hn : 2 ≤ n) (hK : 1 ≤ K) (hD : 0 < D)
    (a b : ℚ) (ha0 : 0 ≤ a)
    (ha : (a : ℝ) ≤ Real.sqrt ((n : ℝ) - 2)) (hb : Real.sqrt ((n : ℝ) - 2) ≤ (b : ℝ))
    (hguard : ∀ k, 1 ≤ k → k ≤ K → (0 < denLo n k b ∨ denHi n k a < 0))
    (hlam : 0 < lamq K b) :
    ((-4 + 1 / (2 + b) + sumLo n a b D K : ℚ) : ℝ) ≤ Theta n ∧
      Theta n ≤ ((-4 + 1 / (2 + a) + sumHi n a b D K + tailB K b : ℚ) : ℝ)
def certPos (n K D : ℕ) : Bool :=
  guardAll n K (sqrtLo n) (sqrtHi n) && decide (0 < lamq K (sqrtHi n)) &&
    decide (0 < -4 + 1 / (2 + sqrtHi n) + sumLo n (sqrtLo n) (sqrtHi n) D K)
def certNeg (n K D : ℕ) : Bool :=
  guardAll n K (sqrtLo n) (sqrtHi n) && decide (0 < lamq K (sqrtHi n)) &&
    decide (-4 + 1 / (2 + sqrtLo n) + sumHi n (sqrtLo n) (sqrtHi n) D K
      + tailB K (sqrtHi n) < 0)
theorem theta_pos_of_cert (n K D : ℕ) (hn : 2 ≤ n) (hK : 1 ≤ K) (hD : 0 < D)
    (h : certPos n K D = true) : 0 < Theta n
theorem theta_neg_of_cert (n K D : ℕ) (hn : 2 ≤ n) (hK : 1 ≤ K) (hD : 0 < D)
    (h : certNeg n K D = true) : Theta n < 0
```

**Range.lean** (Theorem 6.1, Propositions 6.2(a)–(e), 6.3)

```
def sweepOK (K D lo hi : ℕ) (L : List ℕ) : Bool :=
  (List.range' lo (hi + 1 - lo)).all fun n =>
    if n ∈ L then (decide (Resonant n) || certPos n K D)
    else (!decide (Resonant n)) && certNeg n K D
theorem sweep_spec {K D lo hi : ℕ} {L : List ℕ} (hK : 1 ≤ K) (hD : 0 < D) (hlo : 2 ≤ lo)
    (h : sweepOK K D lo hi L = true) :
    ∀ n, lo ≤ n → n ≤ hi → (Admissible n ↔ n ∈ L)
def printedList : List ℕ := [26, 27, 49, 50, 51, 80, 81, 82, 83, 120, 121, 122, 123]
def admissibleTo400 : List ℕ :=
  [26, 27, 49, 50, 51, 80, 81, 82, 83, 120, 121, 122, 123,
   167, 168, 169, 170, 171, 223, 224, 225, 226, 227,
   286, 287, 288, 289, 290, 291, 358, 359, 360, 361, 362, 363]
theorem admissible_below_150 (n : ℕ) (h1 : 12 ≤ n) (h2 : n < 150) :
    Admissible n ↔ n ∈ printedList
theorem admissible_to_400 (n : ℕ) (h1 : 12 ≤ n) (h2 : n ≤ 400) :
    Admissible n ↔ n ∈ admissibleTo400
theorem control_26_admissible_nonresonant : ¬ Resonant 26 ∧ Admissible 26
theorem control_27_resonant : Resonant 27 ∧ Admissible 27
theorem control_25_not_admissible : ¬ Admissible 25
theorem control_79_not_admissible : ¬ Admissible 79
theorem control_123_admissible : Admissible 123
theorem control_gap_150_166 (n : ℕ) (h1 : 150 ≤ n) (h2 : n ≤ 166) : ¬ Admissible n
```

**Blocks.lean** (Theorems 6.4, 6.6, Corollary 6.5, Proposition 6.2(f), Remark 5.5)

```
def blockBottoms : List ℕ :=
  [26, 49, 80, 120, 167, 223, 286, 358, 437,
   525, 620, 724, 835, 955, 1082, 1218, 1361, 1513,
```

```
   1672, 1840, 2015, 2199, 2391, 2590, 2798, 3013, 3237,
   3468, 3708, 3955, 4211, 4474, 4746, 5026, 5313, 5609,
   5912, 6224, 6543, 6871, 7206, 7550, 7902, 8261, 8629,
   9004, 9388, 9779, 10179, 10587, 11002, 11426, 11857, 12297,
   12744, 13200, 13664, 14135]
def blockCert (K D : ℕ) : Bool :=
  (List.range 58).all fun t =>
    let j := t + 1
    let n0 := blockBottoms.getD t 0
    decide ((2 * j + 1) ^ 2 + 2 < n0) && decide (n0 < (2 * j + 3) ^ 2 + 2) &&
      certPos n0 K D && certNeg (n0 - 1) K D
theorem blockCert_ok : blockCert 2000 (10 ^ 20) = true
theorem window_block (t : ℕ) (ht : t < 58) (n : ℕ)
    (h1 : (2 * (t + 1) + 1) ^ 2 + 2 < n) (h2 : n ≤ (2 * (t + 1) + 3) ^ 2 + 2) :
    Admissible n ↔ blockBottoms.getD t 0 ≤ n
theorem block_law_fails_at_25 :
    (∀ n, 2211 < n → n ≤ 2403 → (Admissible n ↔ 2391 ≤ n)) ∧ ¬ Admissible 2390
theorem block_law_holds_at_24 :
    (∀ n, 2027 < n → n ≤ 2211 → (Admissible n ↔ 2199 ≤ n)) ∧ ¬ Admissible 2198
theorem control_2403_resonant : Resonant 2403
def blockLen (t : ℕ) : ℕ := (2 * (t + 1) + 3) ^ 2 + 2 - blockBottoms.getD t 0 + 1
theorem block_law_failure_count :
    ((List.range 58).filter fun t => decide (1 ≤ t) && decide (blockLen t ≠ (t + 6) / 2)).length
        = 31
      ∧ (∀ t < 58, 33 ≤ t → blockLen t ≠ (t + 6) / 2)
      ∧ blockLen 0 = 2 ∧ (0 + 6) / 2 = 3
theorem admissible_total_count :
    ((List.range 58).map blockLen).sum = 940
theorem cert_control_25 : certPos 25 120 (10 ^ 20) = false
theorem cert_control_26 : certNeg 26 120 (10 ^ 20) = false
theorem cert_control_27_resonant :
    certPos 27 120 (10 ^ 20) = false ∧ certNeg 27 120 (10 ^ 20) = false
theorem theta_80_two_sided : (357 : ℝ) / 10000 < Theta 80 ∧ Theta 80 < 368 / 10000
```